Q
Basic Lecture series

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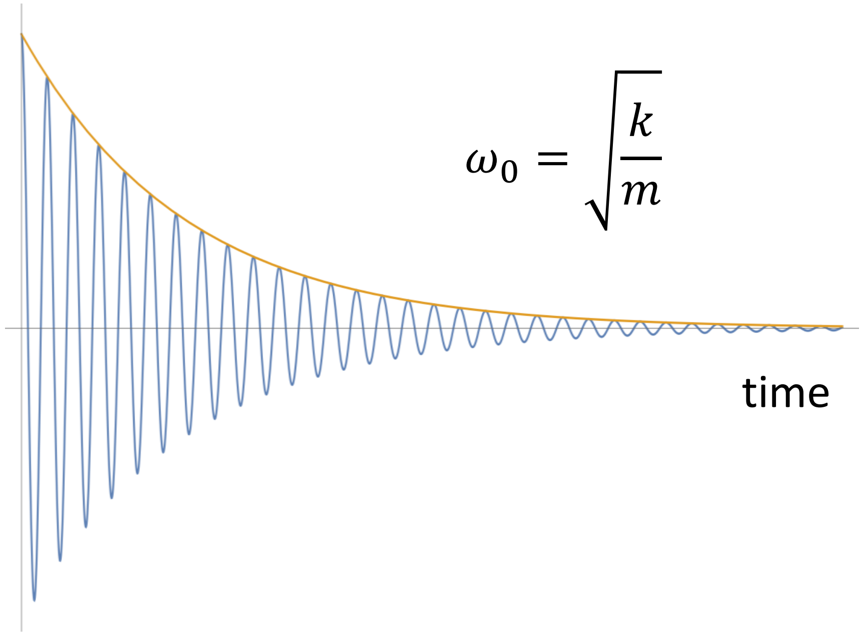
Contents

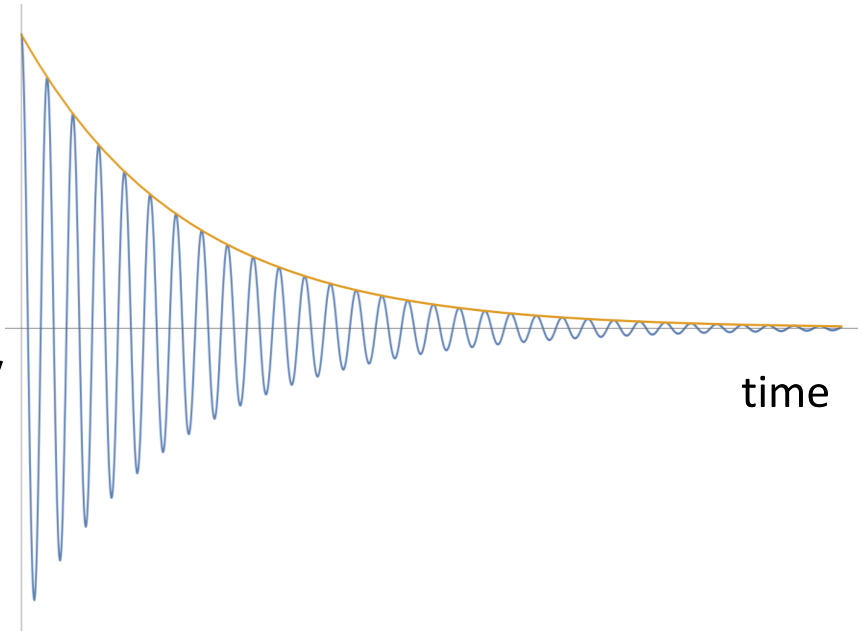
- The classical dissipative oscillator
- The fluctuation dissipation theorem
- Thermal noise spectrum
- Q
- Surface loss

The classical dissipative oscillator/viscous damping

- It is traditional to consider a dissipative oscillator with the following equation of motion
- $m\ddot{x} + b\dot{x} + kx = 0$
- i.e. a simple harmonic oscillator with a dissipative term that is proportional to the velocity
- Solution
- $x(t) = x_0\, e^{-\frac{b}{2m}t} \cos \omega_0 t + \varphi$
- We shall focus on small dissipation



Loss of energy in one cycle



- Now let us take a look at the energy loss per cycle, because we care about dissipative energy loss.
- Note that the energy is proportional to the square of the amplitude
- $\frac{\Delta \mathrm{E}}{E} = \frac{1^2 - \left(e^{-\frac{b}{2m}\frac{1}{f_0}}\right)^2}{1^2} = \frac{1 - e^{-\frac{b}{mf_0}}}{1} \simeq \frac{1 - 1 + \frac{b}{mf_0}}{1} = \frac{\mathrm{b}}{\mathrm{mf_0}}$

The fluctuation dissipation theorem

- First formulated by Harry Nyquist in 1928
  - in the context of Johnson noise (electric noise in circuits)
- Generalised to all dissipative systems in (at least local) thermal equilibrium
  - Brownian noise
  - Plank radiation law
- Via the same mechanism that movement turns into heat (dissipation), heat leads to fluctuations (in movement)

Fluctuation dissipation theorem

- Intuitively, any useful energy in a system (from mechanical or electrical, for example) turning into heat is dissipation
- Dissipation is microscopic random motion of the molecules in question
- If mechanical/electric energy can turn to heat, heat can turn into mechanical/electric energy, but as fluctuations (noise)
- This is thermal noise, and is a real limitations to high precision experiments today

The fluctuation dissipation theorem

- Given by

Generalised force, e.g. voltage

Real part of a generalised impedance, e.g. resistance

Generalised Johnson-Nyquist noise

$$\langle F^2 \rangle d\nu = 4k_B T \Re(Z(\nu)) d\nu,$$

generalised impedance definition: $F = Z(\omega)\dot{x}$

And valid only when $k_B T \gg \hbar\omega$ i.e. "high" temperatures

$$T \gg \frac{\hbar\omega}{k_B} = 4.8 \times 10^{-11} \frac{f}{1\ Hz} K$$

- Displacement version: $\langle x^2 \rangle = \frac{4k_B T}{\omega^2} \Re(Y(\omega)),$

displacement

Admittance, $Y = \frac{1}{Z}$

Thermal noise

$$\langle x^2 \rangle = \frac{4k_BT}{\omega^2}\Re(Y(\omega)),$$

displacement

Admittance, $Y = \frac{1}{Z}$

- Thermal noise is dependent on the generalised admittance
- How to calculate?
- $Y(\omega) = \frac{1}{Z(\omega)} = \frac{\dot{x}}{F}$
- For the classical dissipative oscillator, using $i\omega x = \dot{x}$ and $i\omega \dot{x} = \ddot{x}$
- $\frac{\dot{x}}{F} = \frac{1}{i\omega m + b + \frac{k}{i\omega}}$
- $\langle x^2 \rangle = \frac{4k_BT}{\omega^2}\frac{b}{b^2 + \left(\omega m - \frac{k}{\omega}\right)^2}$

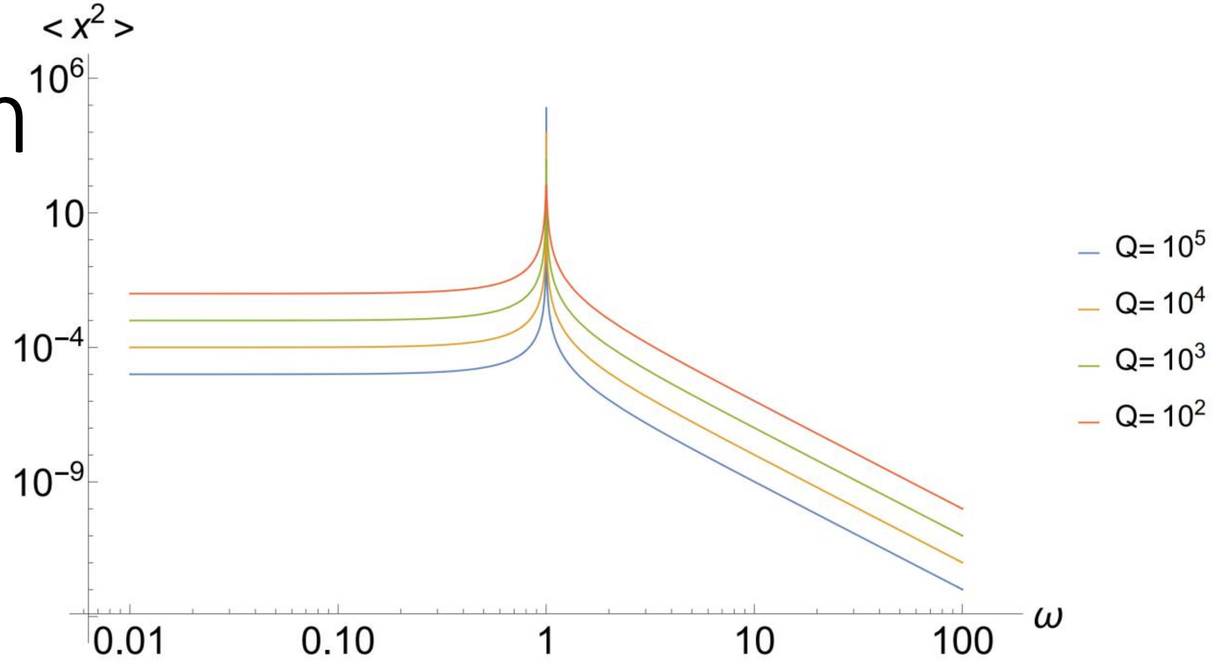
generalised impedance definition: $F = Z(\omega)\dot{x}$

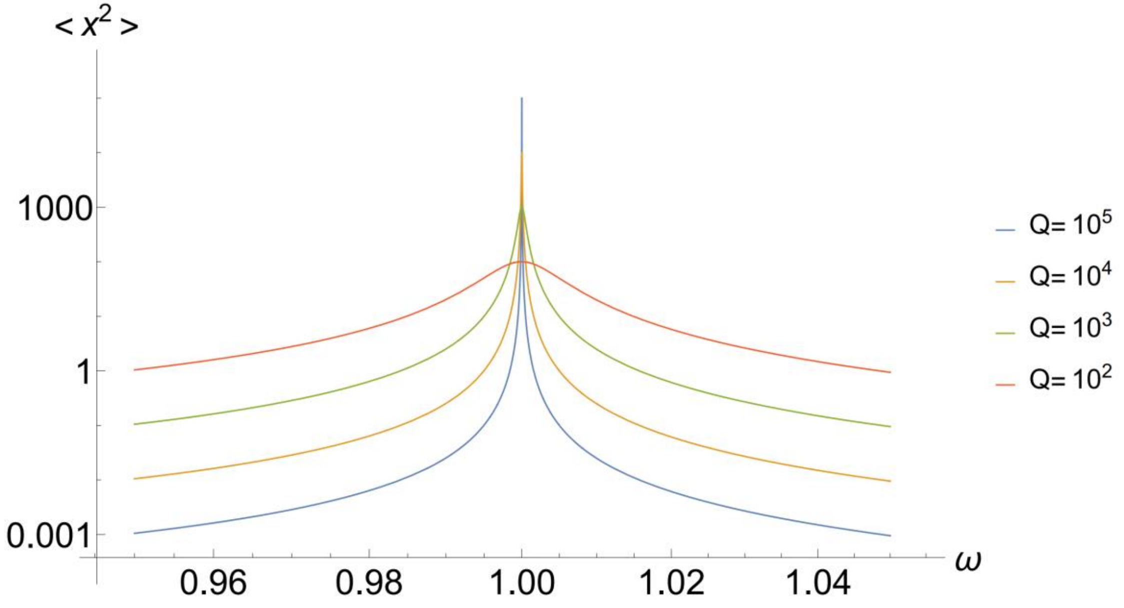
$$F = m\ddot{x} + b\dot{x} + kx$$

Actual thermal noise spectrum

Thermal noise spectrum

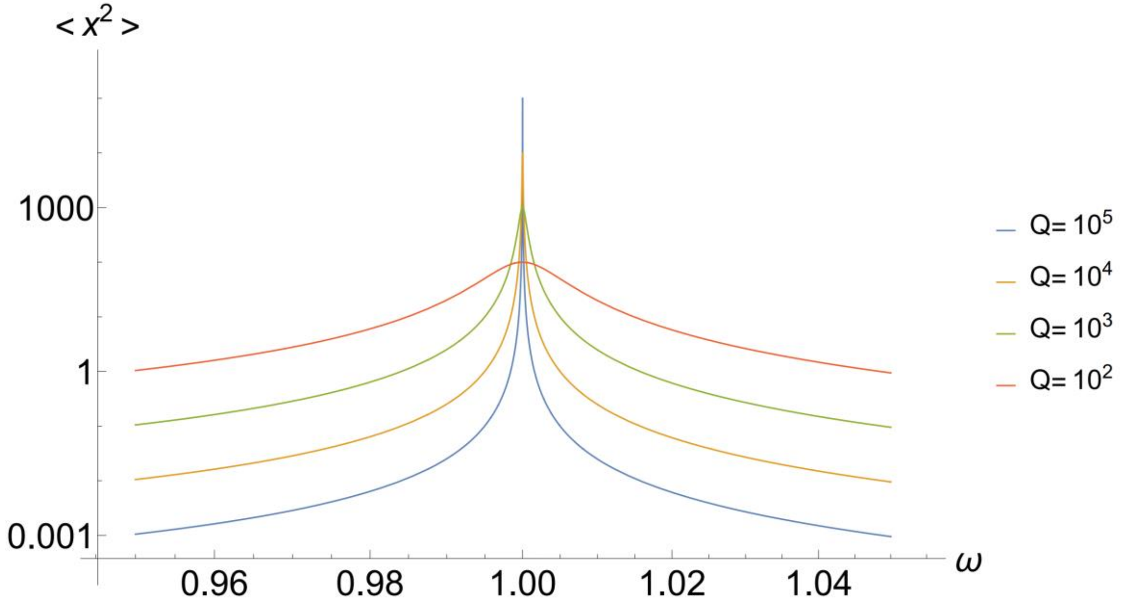
- $\langle x^2 \rangle = \frac{4k_B T}{\omega^2} \frac{b}{b^2+\left(\omega m - \frac{k}{\omega}\right)^2}$
- Graphs with different dissipative values
- $Q \simeq \frac{\omega_0 m}{b}$





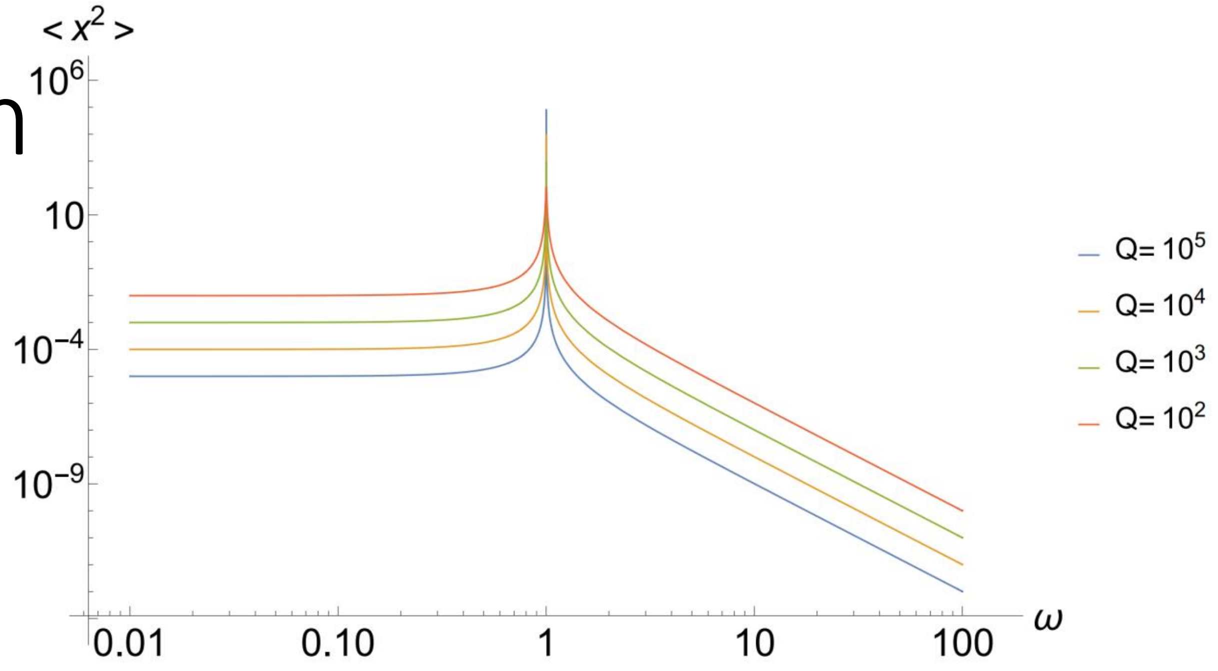
Q

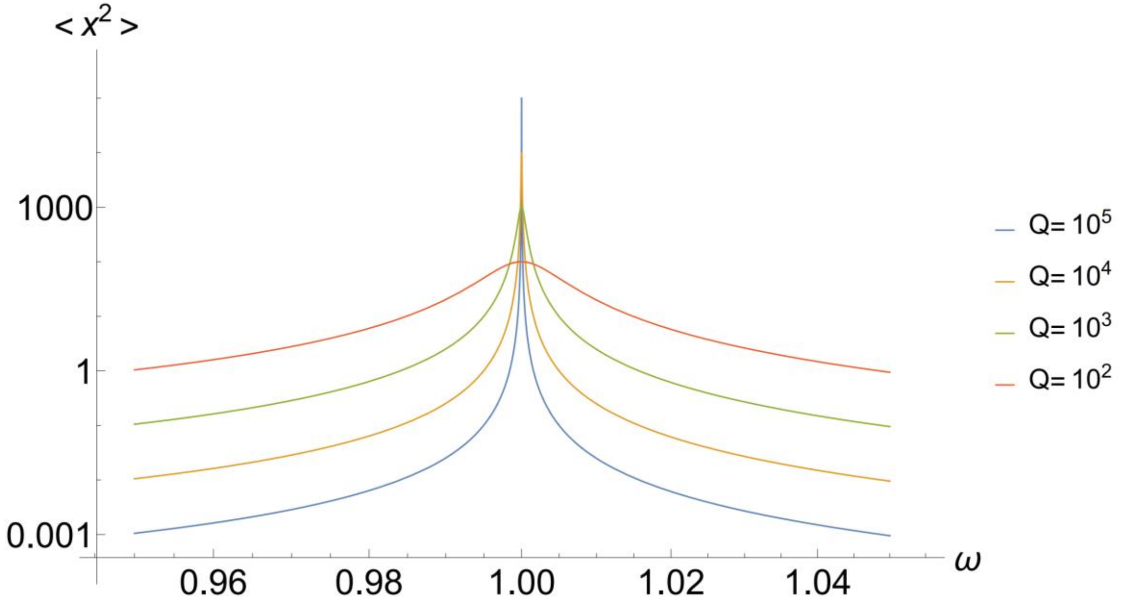
- Defined as the resonance frequency f_0 divided by the full width half maximum (of energy) Δf
- Dimensionless
- $Q = \frac{f_0}{\Delta f}$
- For the classical dissipative oscillator
- $Q \simeq \frac{\omega_0 m}{b}$



Thermal noise spectrum

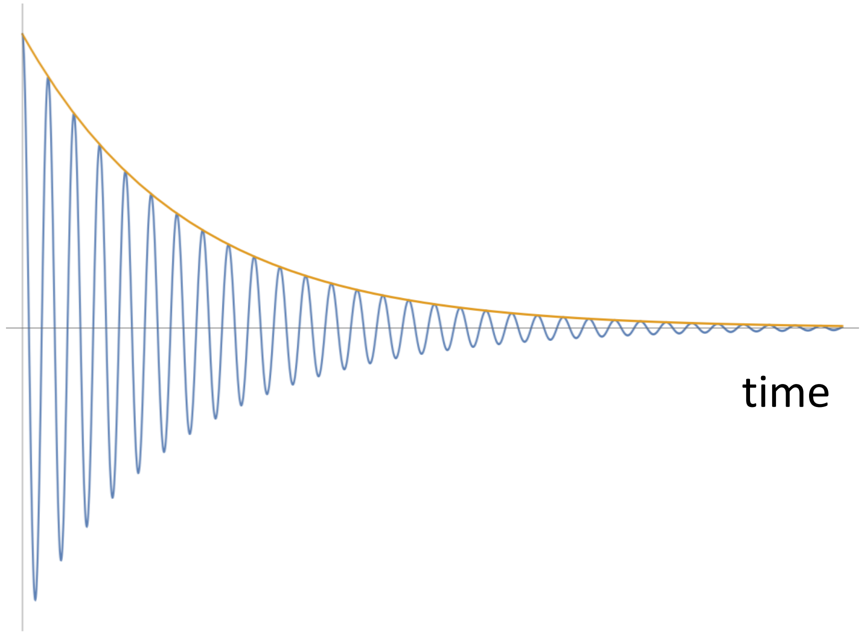
- $\langle x^2 \rangle = \frac{4k_BT}{\omega^2}\frac{b}{b^2+\left(\omega m-\frac{k}{\omega}\right)^2}$
- $Q \simeq \frac{\omega_0 m}{b}$
- For $\omega \ll \omega_0$
  - $\langle x^2 \rangle \simeq \frac{4k_BTb}{k^2} = \frac{4k_BT\omega_0 m}{k^2 Q}$
- For $\omega \approx \omega_0$
  - $\langle x^2 \rangle \simeq \frac{4k_BT}{{\omega_0}^2 b} = \frac{4k_BTQ}{{\omega_0}^3 m}$
- For $\omega \gg \omega_0$
  - $\langle x^2 \rangle \simeq \frac{4k_BTb}{\omega^4 m^2} = \frac{4k_BT\omega_0}{\omega^4 mQ}$





Another definition of Q

- Energy lost in a cycle was calculated to be
- $\frac{\Delta \mathrm{E}}{E} \simeq \frac{\mathrm{b}}{\mathrm{mf_0}}$
- $Q \simeq \frac{\omega_0 m}{b}$
- $\therefore Q = 2\pi \frac{\mathrm{E}}{\Delta E}$
- Ringdown method: $Q = \frac{\omega_0}{2}\tau$, where τ is the time constant of the ringdown envelope



Viscous damping and structural damping

- While classically viscous damping has been well studied, it doesn't apply to all cases in real life, from experiment.
- Especially how Q doesn't change with frequency.
  - $Q \simeq \frac{\omega_0 m}{b}$ for viscous damping
- Structural damping is another common form of damping that is amplitude based instead of velocity based
- $F = m\ddot{x} + k(1 + i\phi)x$
- ϕ is known as the loss factor
- Intrinsic material losses generally follow this model

Viscous damping and structural damping

$\omega_0 = \sqrt{\frac{k}{m}}$

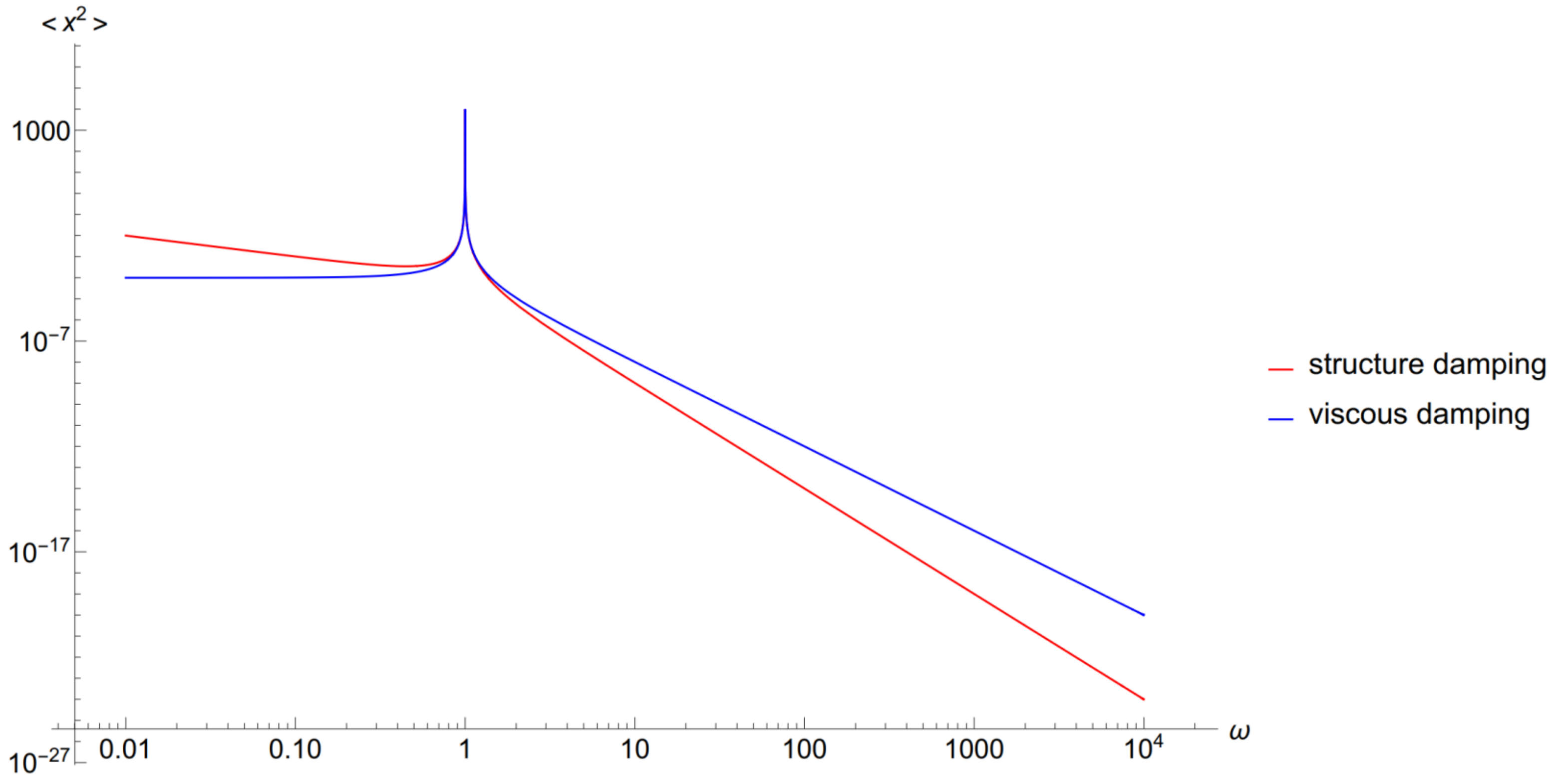
- Repeating the calculations for structural damping, we get
- $\langle x^2 \rangle = \frac{4k_BT}{\omega^2} \frac{\frac{k\phi}{\omega}}{\left(\frac{k\phi}{\omega}\right)^2 + \left(\omega m - \frac{k}{\omega}\right)^2}$
- $Q \simeq \frac{1}{\phi}$
- Note the $b \to \frac{k\phi}{\omega}$ replacement

Viscous damping: $\langle x^2 \rangle = \frac{4k_BT}{\omega^2} \frac{b}{b^2 + \left(\omega m - \frac{k}{\omega}\right)^2}$

$Q \simeq \frac{\omega_0 m}{b}$

Thermal noise spectrum (structural damping)

- For $\omega \ll \omega_0$
  - $\langle x^2 \rangle \simeq \frac{4k_BT\phi}{k\omega} = \frac{4k_BT}{k\omega Q}$
- For $\omega \approx \omega_0$
  - $\langle x^2 \rangle \simeq \frac{4k_BT}{\omega_0 k\phi} = \frac{4k_BTQ}{\omega_0 k}$
- For $\omega \gg \omega_0$
  - $\langle x^2 \rangle \simeq \frac{4k_BTk\phi}{\omega^5 m^2} = \frac{4k_BTk}{\omega^5 m^2 Q}$



Background: Quest for High Q

- Resonant-mass detectors (Weber bars)
  - Gravitational wave detectors
  - The higher the Q, the easier it will resonate, and thus the more sensitive the detectors will be
- Increase accuracy of G measurements
  - Anelastic properties of the fibre will bias the results of the popular time of swing method
- Reduce thermal noise in any precision experiment
  - Usage of fused silica for both the mirror and suspensions of aLIGO and AdVirgo
- Needed for high precision timekeeping
  - High Q means that the resonant frequency is more consistent

Various Q

- Q is very versatile and appears in many systems

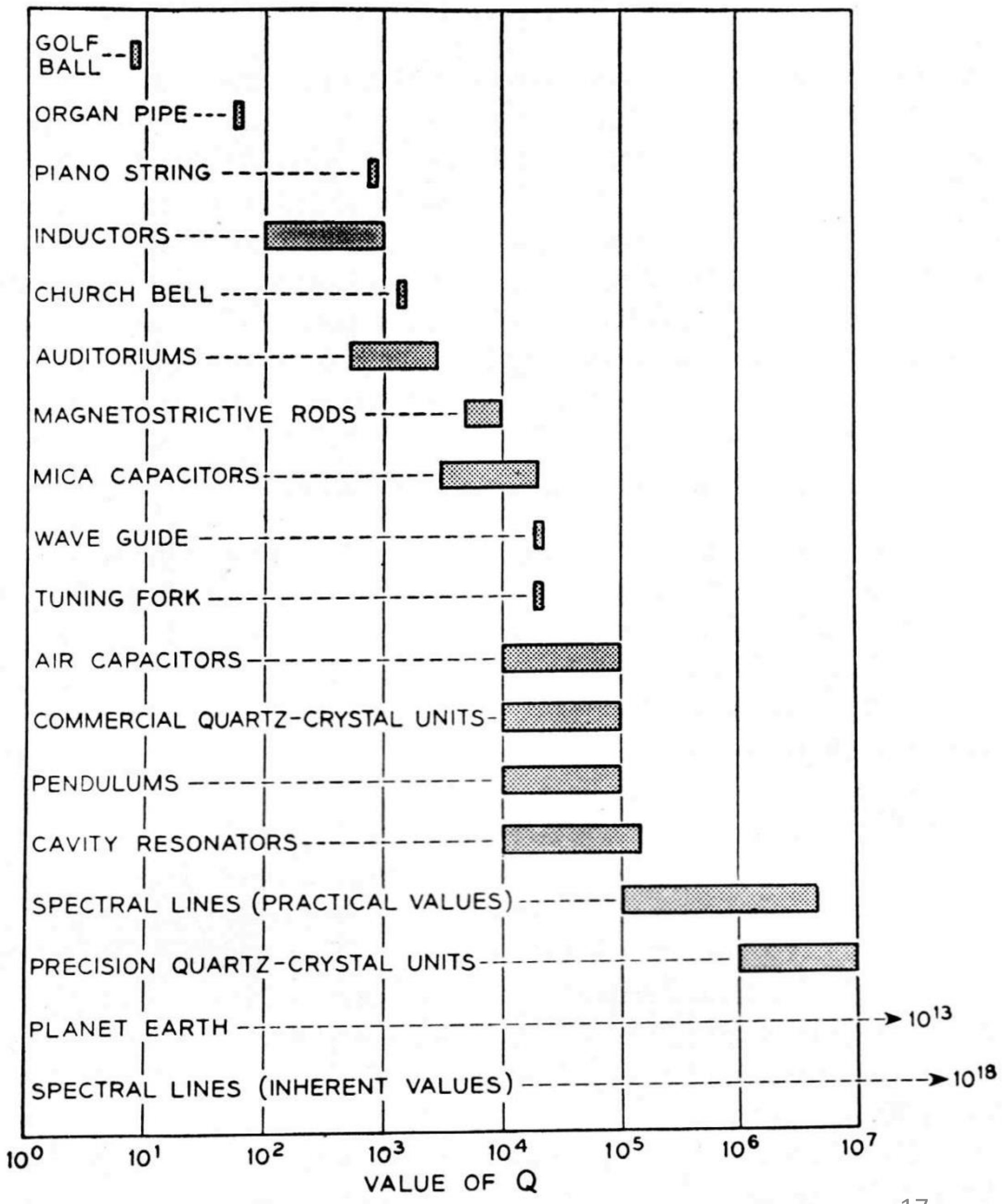
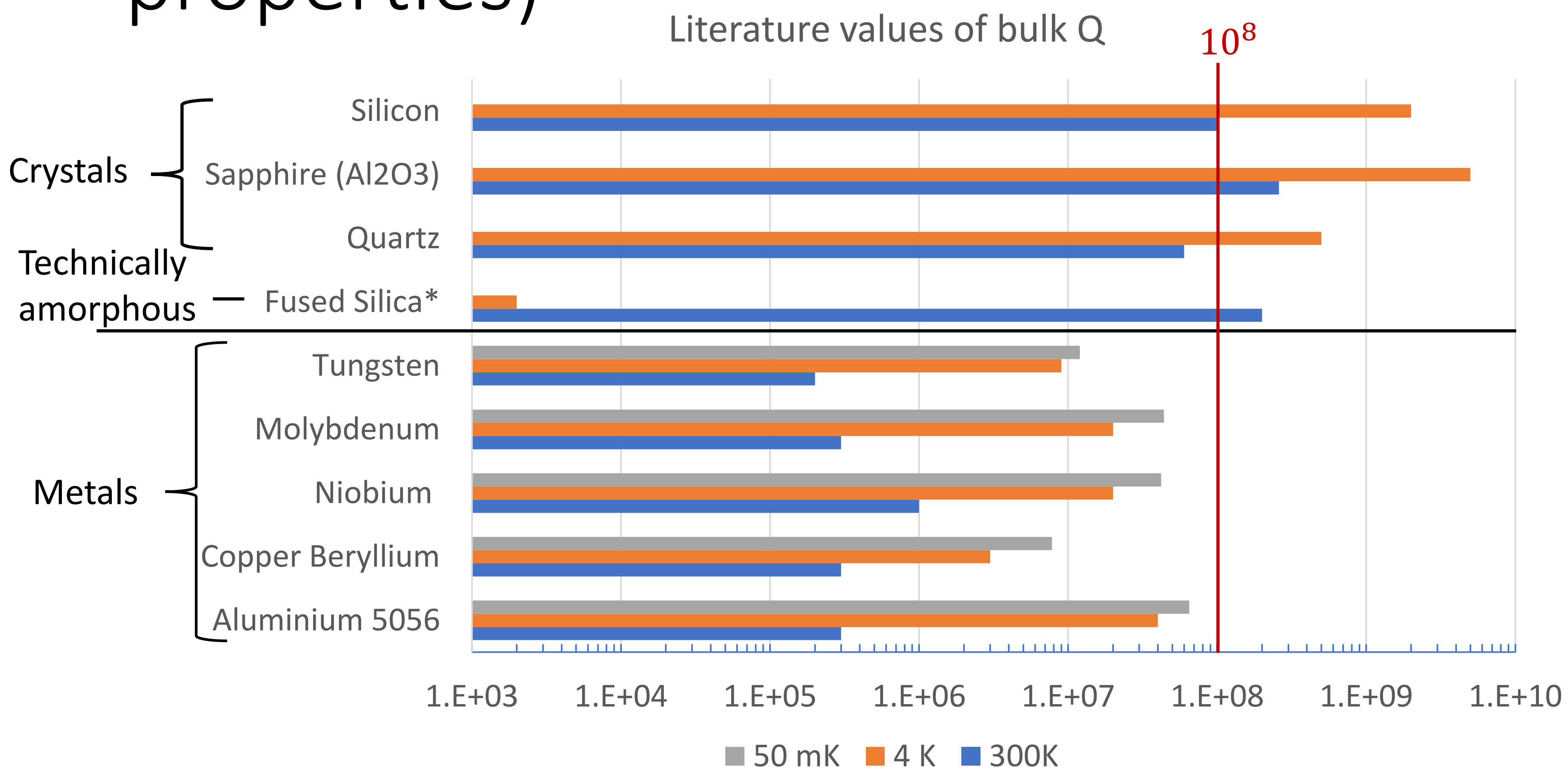


FIG. 3. Q's for various phenomena and devices.

GREEN, ESTILL I. "THE STORY OF Q." *American Scientist* 43, no. 4 (1955): 584-94. http://www.jstor.org/stable/27826701.

Reported values for bulk Q (material properties)



*Represents fibre measurements and not bulk

- Crystals have higher Q in general (lower loss)

Effective Q

- Anything that dissipates energy contributes to the effective Q
- $Q = 2\pi \frac{E}{\Delta E} = 2\pi \frac{E}{\Delta E_1 + \Delta E_2 + \Delta E_3 + \cdots}$
- $Q^{-1} = \frac{1}{2\pi} \frac{\Delta E_1 + \Delta E_2 + \Delta E_3 + \cdots}{E} = \frac{1}{2\pi}\left(\frac{\Delta E_1}{E} + \frac{\Delta E_2}{E} + \frac{\Delta E_3}{E} + \cdots\right)$
  $= {Q_1}^{-1} + {Q_2}^{-1} + {Q_3}^{-1} + \cdots$
- Note that here the various Qs are defined here with respect to the total (relevant) energy of the system

Dilution factor

- You can increase the effective Q by diluting it with a lossless potential energy
- Classical case is gravitational energy in a pendulum
- $\mathrm{Q_{eff}}^{-1} = \frac{1}{2\pi}\left(\frac{\Delta E_{strain}}{E} + \frac{\Delta E_{gravitational}}{E}\right) = \frac{1}{2\pi}\left(\frac{\Delta E_{strain}}{E}\right) = \frac{1}{2\pi}\left(\frac{\Delta E_{strain}}{E_{strain}+E_{gravitaional}}\right)$
- $Q_{eff} = 2\pi\left(\frac{E_{strain}+E_{gravitaional}}{\Delta E_{strain}}\right) = Q_{material} + 2\pi\frac{E_{gravitaional}}{\Delta E_{strain}}$
- Dilution factor $= \frac{Q_{eff}}{Q_{material}}$

Thermoelastic damping

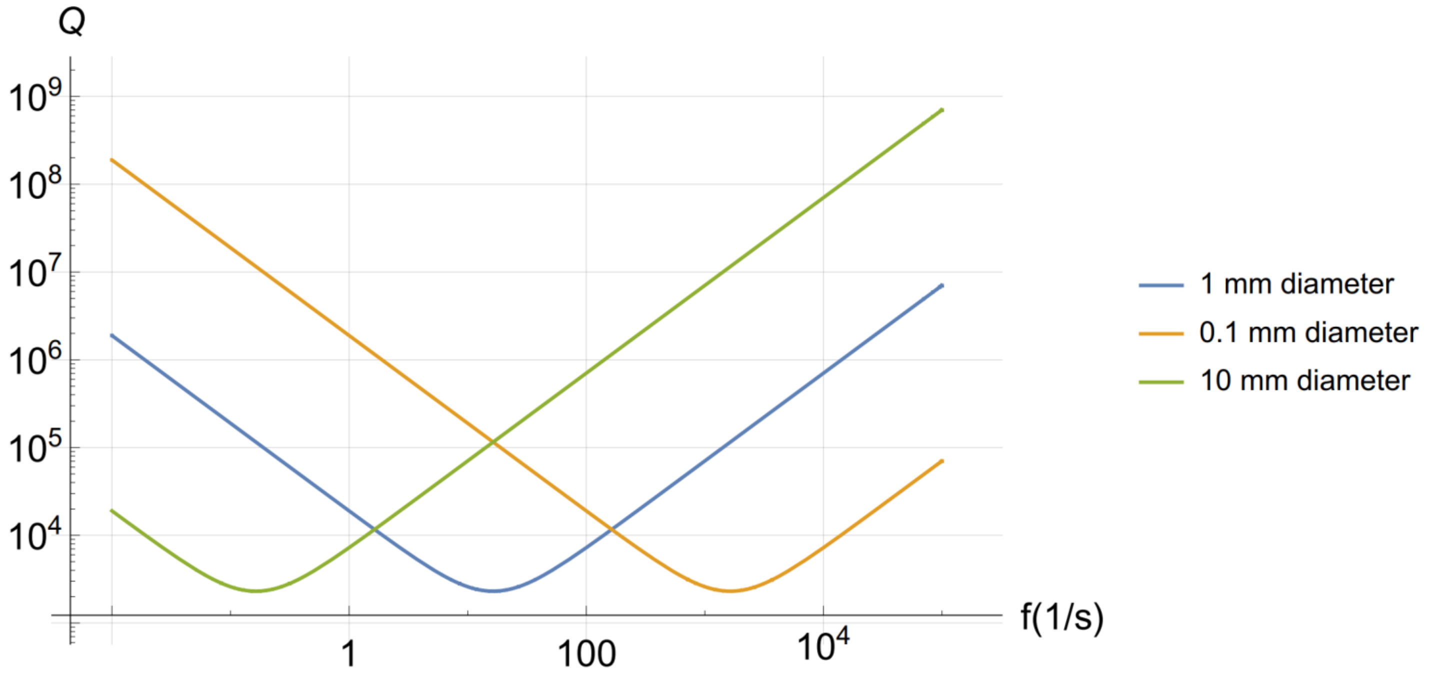
- Due to a non-zero coefficient of thermal expansion, there is temperature gradients created when a substance expands and contract
- These leads to heat flow, leading to additional energy dissipation
- Famously high for sapphire at room temperature
- Does not affect torsion pendulums, at least to first order due to lack of volume change

Thermoelastic damping

$$\phi(\omega) = \frac{\Delta_r \omega \tau_r}{1 + (\omega \tau_r)^2}$$

$$\Delta_r = \frac{Y_0 \alpha^2 T}{c}$$

$$\tau_r = \frac{c D^2}{2.16 (2\pi) \lambda}$$

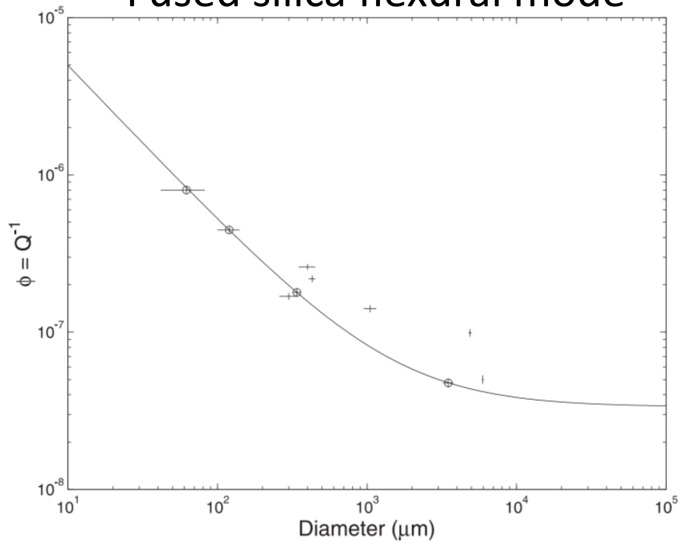


Thermoelastic Q for sapphire fibres at room temperature

Surface loss

- Literature suggests that surface loss dominates sample loss for fibres
  - Thinner fibres generally leading to lower Q
  - Emphasis on fibre handling in high Q papers
    - "The measured Q was strongly dependent on handling, with a pristine flame-polished surface yielding a Q 3-4 times higher than a surface which had been knocked several times against a copper tube"- S. Penn et al. 2001
- Surface treatments are essential in getting high Q

Fused silica flexural mode



A. Gretarsson and G. Harry 1999

Surface loss

- Why do surfaces have such low Q?

- Surfaces generally defer greatly in properties from the bulk. The surface would be polycrystalline in nature, covered with impurities, and made up of randomly orientated crystallites.
- Dislocation density would also be higher near the surface of any machined surface

Surface loss model

- Assume surface layer has a different (constant) Q from bulk
- Assume surface layer has a fixed thickness (for a specific sample)
- Assume elastic constants are identical between bulk and surface
- Assume energy stored in surface layer << energy stored in bulk ($E_{surface} \ll E_{bulk}$)
- $\frac{1}{Q_{eff}} = \frac{1}{2\pi}\frac{\Delta E}{E} = \frac{1}{2\pi}\frac{\Delta E_{bulk}+\Delta E_{surface}}{E_{bulk}+E_{surface}} \simeq \frac{1}{2\pi}\frac{\Delta E_{bulk}+\Delta E_{surface}}{E_{bulk}}$

Surface loss

$$\begin{aligned}\frac{1}{Q_{eff}} &\simeq \frac{1}{2\pi}\frac{\Delta E_{bulk}+\Delta E_{surface}}{E_{bulk}}\\ &= \frac{1}{Q_{bulk}}+\frac{1}{2\pi}\frac{\Delta E_{surface}}{E_{bulk}}\\ &= \frac{1}{Q_{bulk}}+\frac{1}{2\pi}\frac{\Delta E_{surface}}{E_{surface}}\frac{E_{surface}}{E_{bulk}}\\ &= \frac{1}{Q_{bulk}}+\frac{1}{Q_{surface}}\frac{E_{surface}}{E_{bulk}}.\end{aligned}$$

- How do we get $\frac{E_{surface}}{E_{bulk}}$?

Surface loss

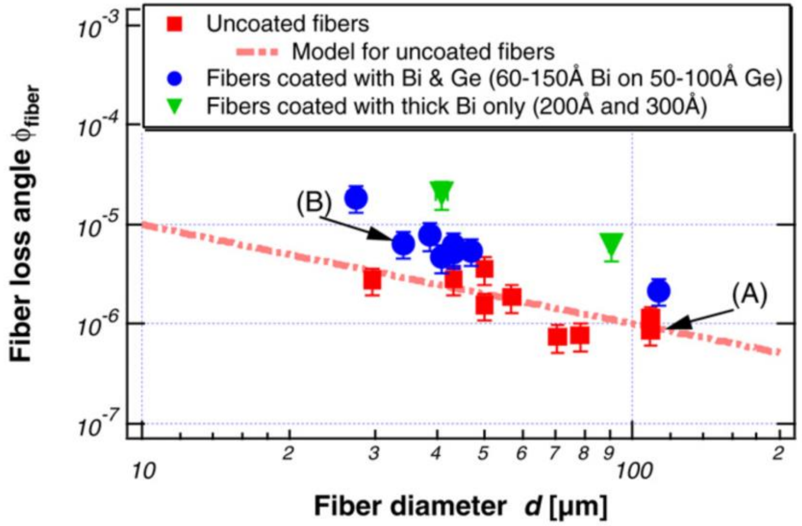
- Potential energy per unit volume due to strain is $\frac{1}{2} Y_0 \epsilon^2$
  - Y_0 is Young modulus
  - ϵ is strain
- Potential energy per unit volume due to torsion is $\frac{1}{2} \mu \gamma^2$
  - μ is shear modulus
  - γ is strain angle
- $\frac{E_{surface}}{E_{bulk}} \simeq \frac{h \int_S \gamma^2 dS}{\int_V \gamma^2 dV}$ (Assume fixed thickness h and same shear modulus for damaged surface and bulk)

$$\begin{aligned}
\frac{\int_S \gamma^2 dS}{\int_V \gamma^2 dV} &= \frac{\int_S \left(\frac{R\theta}{L}\right)^2 dS}{\int_V \left(\frac{r\theta}{L}\right)^2 dV} \\
&= \frac{\int_0^L \left(\frac{R\theta}{L}\right)^2 2\pi R dl}{\int_0^L \int_0^R \left(\frac{r\theta}{L}\right)^2 2\pi r dr dl} \\
&= \frac{\left(\frac{\theta}{L}\right)^2 2\pi R^3 L}{\left(\frac{\theta}{L}\right)^2 2\pi \frac{R^4}{4} L} \\
&= \frac{4}{R} = \frac{8}{D}.
\end{aligned}$$

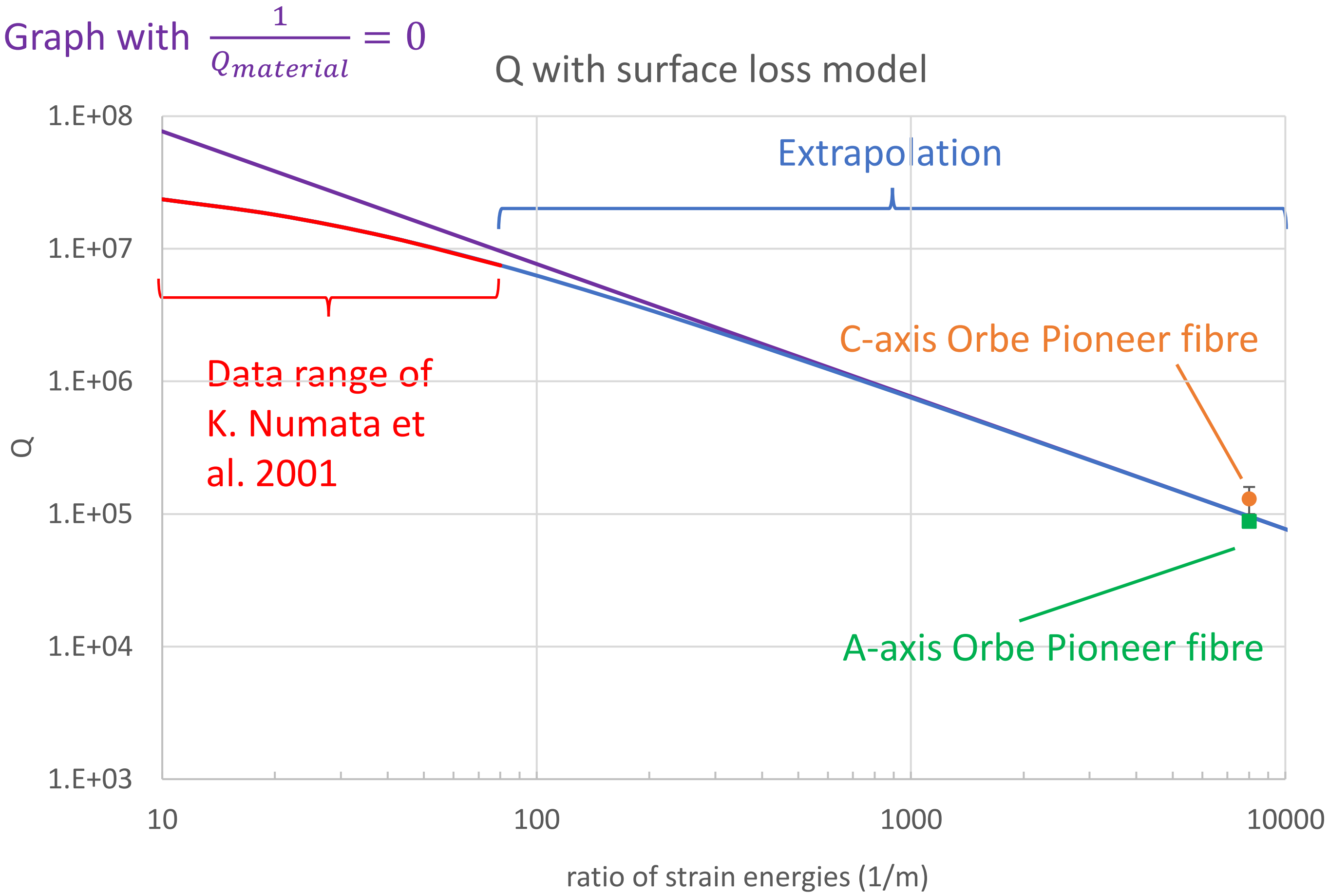
Surface loss for round torsion fibre

- $\frac{1}{Q_{eff}} = \frac{1}{Q_{bulk}} + \frac{1}{Q_{surface}} \frac{h \int_S \gamma^2 dS}{\int_V \gamma^2 dV} = \frac{1}{Q_{bulk}} + \frac{1}{Q_{surface}} h \frac{8}{D}$
- $Q_{surface}$ dominates at small diameters

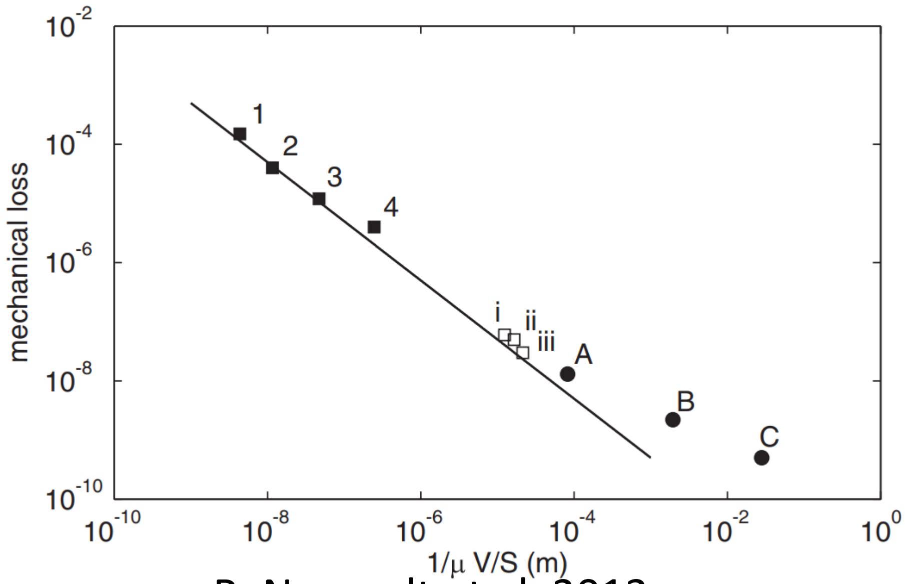
Fused silica torsion mode



K. Numata et al. 2007

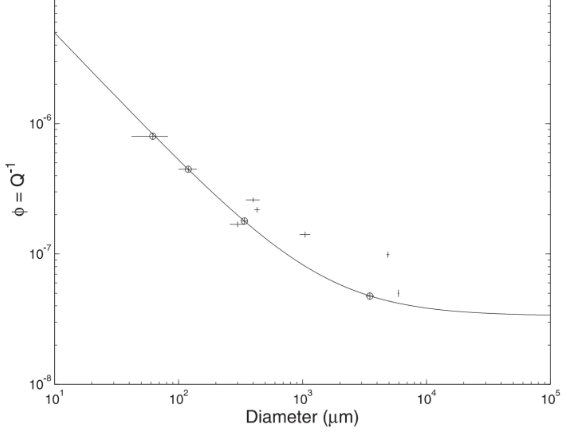


Silicon flexural mode



R. Nawrodt et al. 2013

Fused silica flexural mode



A. Gretarsson and G. Harry 1999

Surface quality

Table 3. Maximum Breaking Strength of Sapphire Bars at Room Temperature After Various Treatments

Treatment technique	Strength (10^9 Pascal)
Flame polishing, selected working area of specimen (1600°C)	7.35
Boron etching	6.86
Machine polishing, firing in oxygen (1600°C)	1.04
Annealing, machine polishing	0.78
Lapping	0.59
As manufactured (no additional treatment)	0.44

- Tensile strength is very dependent on surface quality

Systems with small dissipation, V.B. Braginsky, V.P. Mitrofanov, V.I. Panov, 1985, taken from Libowitz 1976

Suspension thermal noise in torsion pendulums (off resonance, $\omega \gg \omega_0$)

Shear modulus

Temperature

$$\langle \theta^2 \rangle \simeq \frac{4 k_B T \kappa}{\omega^5 I^2 Q} = \frac{k_B T \mu \pi D^4}{8 \omega^5 I^2 L Q}$$

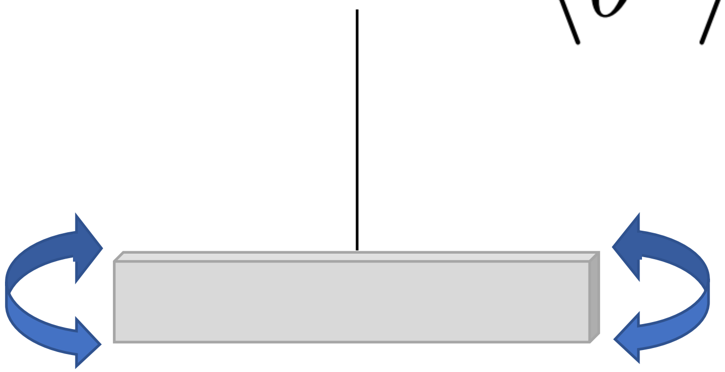
Fibre diameter

Angular frequency

Q factor

Moment of inertia of the torsion pendulum

Length of fibre



- $\langle \theta^2 \rangle df$ is the mean square of the angular displacement within the frequency interval df

Surface quality

- Some quantification here:
  - Assuming a safety factor *s*
  - And replacing *D*

$$\langle\theta^2\rangle \simeq \frac{m^2}{I^2}\frac{2k_BT\mu s^2g^2}{\omega^5 L {S_t}^2\pi Q}$$

Independent of scale, but not of shape

Safety factor

Suspension thermal noise for torsion pendulum

Length of fibre

Tensile strength

- We see that improving surface quality is key to low thermal noise

Conclusion

- Q is a dimensionless parameter that characterises dissipation of a system
- This can be used to calculate the thermal noise spectrum
- For mechanical Q, surface losses play a big role in thin materials