

Ph.D. thesis

**Study of the thermal noise caused by
inhomogeneously distributed loss**

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Chapter 1

Introduction

The aim of this research is the accurate estimation of thermal noise sensitivity limit of interferometric gravitational wave detectors. We proved experimentally that the traditional method to estimate the thermal noise breaks down when the dissipation is distributed inhomogeneously. On the other hand, our experimental results agree with the new estimation methods. The thermal noise of the detectors was evaluated using new estimation methods.

Gravitational waves are ripples of the space-time which propagate at the speed of light. The gravitational wave was derived from the general theory of relativity by Einstein in 1916 [1]. The existence of the gravitational waves was proved by the observation of the binary pulsar PR1913+16 discovered by Hulse and Taylor [2]. The observed decrease of the period of the revolution of this binary agrees well with theoretical values of orbital decay due to radiation of the gravitational wave [3]. However, the gravitational waves have not been detected directly yet. The development of the gravitational wave detectors was pioneered by Weber [4, 5] in 1960's. The construction and improvement of gravitational wave detectors are in progress in order to directly observe the gravitational wave.

The direct detection of the gravitational waves is an important subject not only in physics but also in astronomy. In physics, the detection implies the test of gravitational theories [8]. In astronomy, gravitational wave detection opens a new window of observation into the universe because the gravitational waves give information that electromagnetic waves and the neutrinos do not carry [9, 10].

The end of the twentieth century saw large interferometric gravitational wave detectors being constructed in several countries. There are four ongoing projects of interferometric gravitational wave detectors: the LIGO project [14] in the United States of America, the VIRGO project [15] of France and Italy, the GEO project [16] of Germany and United Kingdom, and the TAMA project [17] in Japan. It is expected that the research and development in these projects will realize detection of the gravitational waves in the first decade of the twenty-first century.

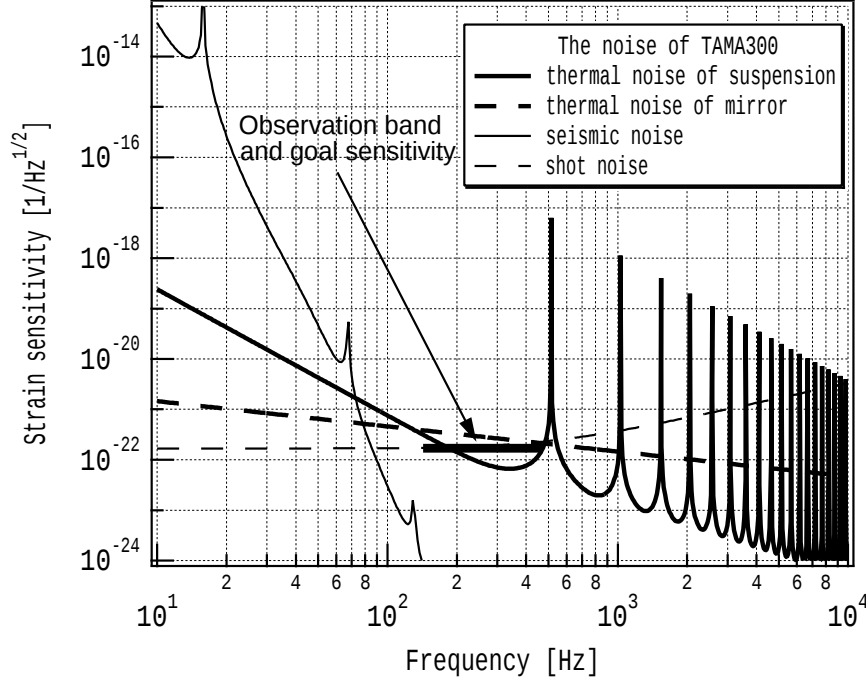


Figure 1.1: The expected sensitivity of TAMA300, the interferometric gravitational wave detector in TAMA project. The thick solid line shows the design observation band (between 150 Hz and 450 Hz) and the goal sensitivity ($h = 1.7 \times 10^{-22}/\sqrt{\text{Hz}}$) of the TAMA project. The sensitivity in the observation band is limited by the thermal noise of the mirror (thick dashed curve) and of the suspension (thick solid curves). The details of the sensitivity are described in Chapter 3.

Thermal fluctuation is one of the fundamental noise sources in the interferometric gravitational wave detectors. The thermal noise is the thermally-excited motion of the mechanical components of the interferometers. It is expected that the sensitivity of the interferometers in the observation band will ultimately be limited by the thermal noise

of its mirrors and suspensions. As an example, the expected sensitivity of TAMA300, which is the interferometer built by the TAMA project, is shown in Fig.1.1. This figure shows that the sensitivity of TAMA300 in the observation band is limited by the thermal noise. The goal sensitivity of future projects is tens or hundreds times higher than that of the current projects. Therefore, the estimation and reduction of the thermal noise is one of the most important issues in the improvement of the sensitivity of the detectors.

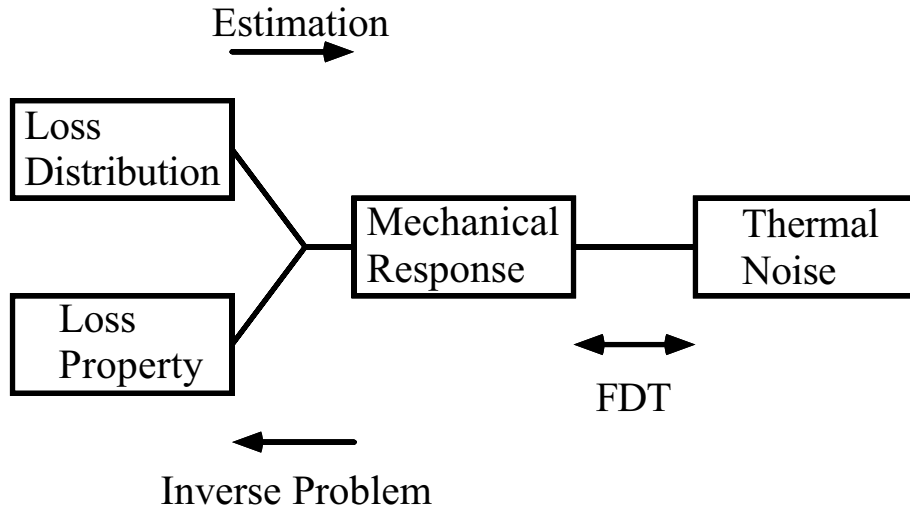


Figure 1.2: The relation between fluctuation and dissipation. FDT represents the fluctuation-dissipation theorem. This theorem describes the relation between the thermal noise and the mechanical response to the external force. Estimation is the derivation of the mechanical response from the properties (frequency dependence, etc.) and the distribution of the dissipation. Inverse problem is to get the information of the properties and of the distribution of the loss from the measurable mechanical responses.

It is extremely difficult to observe the thermal noise directly because the thermal noise is much small. Since there is the relationship between the thermal noise and dissipation, the thermal motion is evaluated from the measurement of the loss. This relation is shown in Fig.1.2. The thermal noise is related to the mechanical response to the external force by the fluctuation-dissipation theorem (FDT) [21, 22, 23, 24]. The mechanical response depends on the properties (frequency dependence, etc.) and the distribution of the loss. Estimation in Fig.1.2 is the derivation of the mechanical response from the dissipation. Inverse problem in Fig.1.2 is to get information of the properties and the distribution of the loss from the measurable mechanical responses.

The traditional method to estimate mechanical responses from the dissipation is called the normal-mode expansion method [26]. This method estimates the mechanical response from Q-values which represent the dissipation of resonant modes. This method is commonly used to evaluate the thermal noise of the interferometric gravitational wave detectors. For example, the evaluated spectrum of Fig.1.1 is obtained from the normal-mode expansion.

However, some theoretical studies [65, 68] and the models obtained from the experiments [84, 85] suggest that the mode expansion does not give a correct mechanical response when the loss is distributed inhomogeneously. Since, in general, the loss is distributed inhomogeneously, it is one of the most important issues to study in detail the thermal noise induced by the inhomogeneous dissipation. Nevertheless, the thermal noise caused by the inhomogeneous loss has been seldom and unsatisfactory investigated.

The main theme of this thesis is the study of the thermal noise caused by inhomogeneously distributed loss in order to estimate correctly the thermal noise of the gravitational wave detectors. We have developed a new estimation method replacing the mode expansion. The validity of the new estimation methods was confirmed experimentally. The thermal fluctuations of the gravitational wave detectors were evaluated using the new method. Consequently, almost main problems of the thermal noise of the inhomogeneous loss were solved in our research.

The new estimation method developed by us is called the advanced mode expansion because this method is a modification of the traditional mode expansion. This method gives the physical interpretation of the thermal noise of the inhomogeneous loss. For example, the advanced mode expansion shows the reason why the thermal noise induced by the inhomogeneous loss does not agree with the traditional mode expansion. In fact, there are other new methods, direct approaches [68, 69, 70], to estimate thermal noise. Although the results of the direct approaches are consistent with the estimation of the advanced mode expansion, the direct approaches do not give the clear physical interpretation.

In order to test the new estimation methods, the thermal motion of a leaf spring with inhomogeneous loss was measured. The results prove that the advanced mode expansion and the direct approaches are correct. On the other hand, these results are not consistent with the evaluation using the traditional mode expansion. This is the first experimental results which show the invalidity of the traditional mode expansion.

The thermal elastic vibrations of the mirrors with the inhomogeneous loss in the gravitational wave detectors were derived from the direct approach. The calculations showed that there are large discrepancies between the actual thermal noise of the mirror and the estimation of the traditional mode expansion. The theoretical estimations were checked experimentally using a mechanical model of mirrors. This measurement supported our evaluation of the thermal noise of the real mirror.

In this thesis, the details of above research are described. In Chapter 2, the physical background of the gravitational waves are explained. Chapter 3 describes the theory of the thermal noise. The traditional mode expansion is introduced here. The advance mode expansion developed by us and the physical interpretation of the thermal noise caused by the inhomogeneous loss are given in Chapter 4. The direct approaches are introduced in Chapter 5. In Chapter 6 the experimental test of the estimation methods using a leaf spring is shown. The thermal motions of the mirror with inhomogeneous loss are calculated in Chapter 7. The experimental test of this estimation of the mirror is described in Chapter 8. In Chapter 9 the requirements of the loss of the mirrors in the interferometric gravitational wave detectors and the future works of the investigation of the thermal noise of the inhomogeneous loss are summarized.

Chapter 2

Gravitational wave

The gravitational waves are ripples of the space-time. These waves are generated by catastrophic phenomena like supernova explosions, coalescence of compact binaries, and so on. The gravitational waves come from these sources without scattering and absorption because the interaction of the gravitation is extremely weak. Therefore, gravitational waves have astronomical information which electromagnetic waves and neutrinos never carry [9, 10, 11]. In order to obtain this astronomical information, gravitational wave detectors are being developed. In this chapter, properties, sources, and methods of detection of gravitational waves are introduced.

2.1 Propagation of gravitational waves

The propagation of gravitational waves is considered here [6, 7].

2.1.1 General theory of relativity

In the general theory of relativity, the gravitation is described as strain of the space-time. The proper distance, ds , between two slightly separate points in space-time is defined by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2.1)$$

where $g_{\mu\nu}$ is the metric tensor. Christoffel symbols, $\Gamma^\mu_{\nu\lambda}$, and Riemann tensor, $R^\mu_{\nu\alpha\beta}$, are expressed as

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\alpha}(g_{\alpha\nu,\lambda} + g_{\alpha\lambda,\nu} - g_{\nu\lambda,\alpha}), \quad (2.2)$$

$$R^\mu_{\nu\alpha\beta} = \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta} + \Gamma^\mu_{\gamma\alpha}\Gamma^\gamma_{\nu\beta} - \Gamma^\mu_{\gamma\beta}\Gamma^\gamma_{\nu\alpha}. \quad (2.3)$$

Ricci tensor, $R_{\mu\nu}$, Ricci scalar, R , and Einstein tensor, $G_{\mu\nu}$, are written as

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}, \quad (2.4)$$

$$R = R^\alpha_{\alpha}, \quad (2.5)$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \quad (2.6)$$

The Einstein equation, which is the fundamental equation of gravitational fields, is described as

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (2.7)$$

where $T_{\mu\nu}$ is the energy-momentum tensor which represents distributions of mass and energy in the space-time. The constants, G and c , are the gravitational constant and the speed of light, respectively.

2.1.2 Linearized Einstein equation

When weak gravitational fields are considered, the Einstein equation can be linearized. The metric tensor in the weak fields can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (2.8)$$

where $\eta_{\mu\nu}$ is the metric tensor in Minkowski space-time,

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.9)$$

The tensor, $h_{\mu\nu}$, describes the perturbation of the metric tensor. The absolute value of $h_{\mu\nu}$ is much smaller than unity, $|h_{\mu\nu}| \ll 1$. Only the first order of $h_{\mu\nu}$ is taken into account. The index raising and lowering are written in the form

$$h^\mu_{\nu} = \eta^{\mu\alpha}h_{\alpha\nu}, \quad (2.10)$$

$$h_\mu{}^\nu = \eta_{\mu\alpha}h^{\alpha\nu}. \quad (2.11)$$

In this approximation, the Einstein equation, Eq.(2.7), is rewritten as

$$h_{\mu\alpha,\nu}{}^\alpha + h_{\nu\alpha,\mu}{}^\alpha - h_{\mu\nu,\alpha}{}^\alpha - h_{,\mu\nu} - \eta_{\mu\nu}(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^\beta) = \frac{16\pi G}{c^4} T_{\mu\nu}. \quad (2.12)$$

In order to simplify Eq.(2.12), the trace reverse tensor and the Lorentz gauge condition are introduced. The trace reverse tensor, $\bar{h}_{\mu\nu}$, is defined by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad (2.13)$$

where h is the trace of $h_{\mu\nu}$. The relation between the traces of $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ is written as

$$\bar{h} = -h \quad (2.14)$$

where $\bar{h}$ is the trace of $\bar{h}_{\mu\nu}$. The Lorentz gauge condition is expressed as

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0. \quad (2.15)$$

The change of the coordinates makes an arbitrary trace reverse tensor, $\bar{h}_{\mu\nu}$, satisfy the Lorentz gauge condition, Eq.(2.15).

Using the trace reverse tensor, Eq.(2.13), which satisfies the Lorentz gauge condition, Eq.(2.15), Eq.(2.12) is simplified as

$$-\bar{h}_{\mu\nu}{}^{,\alpha}{}_\alpha = \frac{16\pi G}{c^4} T_{\mu\nu}. \quad (2.16)$$

In the vacuum, this equation is written in the form

$$-\bar{h}_{\mu\nu}{}^{,\alpha}{}_\alpha = 0. \quad (2.17)$$

Equation (2.17) shows that the perturbation of the metric satisfies the wave equation. This perturbation is called the gravitational wave.

2.1.3 Plane gravitational wave

The simplest solution of Eq.(2.17) corresponds to a plane gravitational wave,

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_\alpha x^\alpha), \quad (2.18)$$

where $A_{\mu\nu}$ is an arbitrary tensor, k^α is a wave number four-vector, and x^α represents a position in the space-time. Putting Eq.(2.18) into Eqs.(2.17) and (2.15), the important expressions of the plane gravitational wave are derived as

$$k_\alpha k^\alpha = 0, \quad (2.19)$$

$$A_{\mu\alpha} k^\alpha = 0. \quad (2.20)$$

Equation (2.19) proves that the speed of the gravitational wave is the same as that of the light. Equation (2.20) shows that the gravitational wave is the transverse wave.

2.1.4 TT gauge

The trace reverse tensor, $\bar{h}_{\mu\nu}$, satisfying the Lorentz gauge condition, Eq.(2.15), is not unique. Thus, new conditions are imposed on $\bar{h}_{\mu\nu}$ in Eq.(2.18). These new conditions are called Transverse-Traceless gauge (TT gauge),

$$A^\alpha{}_\alpha = 0, \quad (2.21)$$

$$A_{\mu\alpha}u^\alpha = 0. \quad (2.22)$$

The vector, u_α in Eq.(2.22), is an arbitrary constant timelike unit vector. The trace reverse tensor satisfying Transverse-Traceless gauge is unique. Eqs.(2.21), (2.13), and (2.14) show that $\bar{h}_{\mu\nu}$ is equal to $h_{\mu\nu}$ in the coordinates of the TT gauge.

2.1.5 Polarization

The plane gravitational wave in the coordinates of the TT gauge is considered here. It is assumed that the gravitational wave propagates along the z -axis. The wave number four-vector, k^α in Eq.(2.18), is expressed as

$$(k^0, k^1, k^2, k^3) = (k, 0, 0, k), \quad (2.23)$$

where k is the wave number of the gravitational wave. The relationship between the wave number and the angular frequency is described as

$$k = \frac{\omega}{c}. \quad (2.24)$$

The vector, u^α , in Eq.(2.22) can be rewritten as

$$u^\alpha = \delta^\alpha_0. \quad (2.25)$$

The tensor, δ^i_j in Eq.(2.25), is Kronecker's δ -symbol. Under these conditions, Eq.(2.18) is rewritten as

$$h_{\mu\nu}^{\text{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \exp[i(-\omega t + kz)], \quad (2.26)$$

where $h_{\mu\nu}^{\text{TT}}$ is $h_{\mu\nu}$ in the coordinates of the TT gauge. Equation (2.26) shows that the gravitational wave has two polarizations, the plus mode, h_+ , and the cross mode, $h_\times$.

2.1.6 Effects on free particles

The effects of the gravitational wave on a free particle are considered. A free particle obeys the geodesic equation described as

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0, \quad (2.27)$$

where τ is the proper time. It is supposed that this particle is at rest in the coordinate frame of the TT gauge initially, $t = 0$. The four-velocity of this particle at the initial moment is defined by

$$\left. \frac{dx^\mu}{d\tau} \right|_{t=0} = \delta^\mu_0. \quad (2.28)$$

Putting Eq.(2.28) into Eq.(2.27), the geodesic equation at the initial time is obtained as

$$\left. \frac{d^2 x^\mu}{d\tau^2} \right|_{t=0} = 0, \quad (2.29)$$

because Eqs.(2.2) and (2.26) show that Γ^μ_{00} vanishes in the coordinate frame of the TT gauge. Equation (2.29) proves that a particle which is at rest in the frame of the TT gauge has no acceleration. Therefore, the gravitational waves do not affect a free particle in the coordinate frame of the TT gauge.

The above discussion does not imply that the gravitational waves do not have effects on separate free particles. The deviation between two separate particles is expressed as the vector, n^μ , which connects them. Both the particles are at rest in the coordinates in the TT gauge at the initial moment. The equation of the geodesic deviation is expressed as

$$\frac{d^2 n^\mu}{d\tau^2} - R^\mu_{\alpha\beta\gamma} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} n^\gamma = 0, \quad (2.30)$$

where x^μ is a position of a particle in the space-time. Since only the first order of $h_{\mu\nu}$ are considered and the mass points move extremely much slowly than light, the approximations:

$$\tau \approx ct, \quad (2.31)$$

$$\frac{dx^\mu}{d\tau} \approx \delta^\mu_0, \quad (2.32)$$

are appropriate. Under this approximations, Eq.(2.30) is rewritten as

$$\frac{1}{c^2} \frac{d^2 n^\mu}{dt^2} + R^\mu{}_{0\gamma 0} n^\gamma = 0. \quad (2.33)$$

Calculating $R^\mu{}_{0\gamma 0}$ from Eq.(2.26), Eq.(2.33) is written in the form

$$\frac{d^2 n^\mu}{dt^2} - \frac{1}{2} \frac{\partial^2 h^\mu{}_\gamma{}^{\text{TT}}}{\partial t^2} n^\gamma = 0, \quad (2.34)$$

Since both the particles are at rest initially, the solution of Eq.(2.34) is expressed as

$$n^\mu(t) = n^\alpha(0) \left[\delta_\alpha^\mu + \frac{1}{2} h_\alpha^{\mu\text{TT}} \right]. \quad (2.35)$$

This solution implies that the separation between the free particles oscillates under the effects of a gravitational wave. The amplitude of this oscillation is proportional to the amplitude of the gravitational wave and to the distance between the free particles.

Figure 2.1 shows the time variations of the deviations between a center and free particles on a circle caused by the gravitational wave. The upper and lower parts of Fig.2.1 represent the plus and cross modes, respectively.

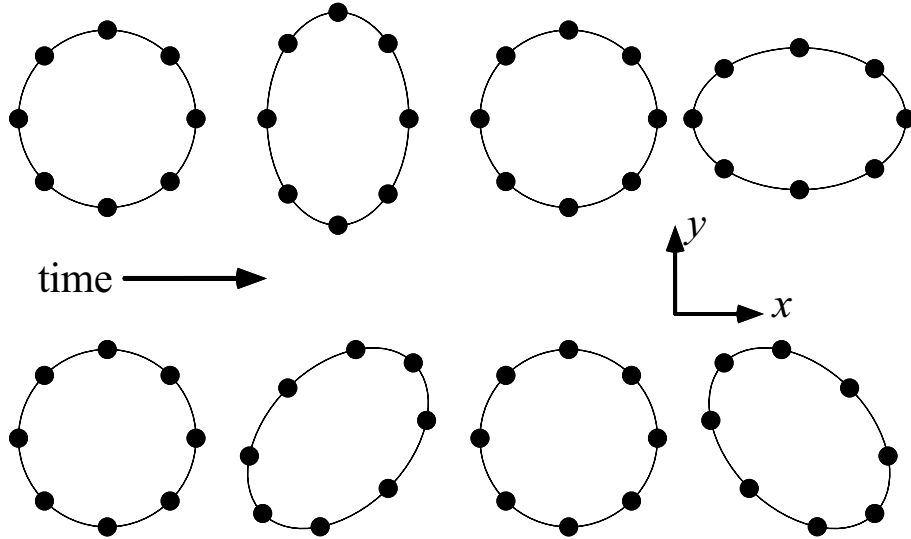


Figure 2.1: The time variations caused by the gravitational wave of the separation between a center and free particles on a circle. The dots correspond to mass points. The upper and lower parts represent the plus and cross modes, respectively.

2.1.7 Energy

The gravitational waves carry energy. The energy-momentum tensor of a gravitational wave is expressed as

$$T_{\mu\nu}^{(\text{GW})} = \frac{c^4}{32\pi G} \langle h_{jk,\mu} h^{jk}_{,\nu} \rangle. \quad (2.36)$$

This formula is the expression in the coordinate frame of the TT gauge. The sign, $\langle A \rangle$, denotes an average of A over several wavelengths.

2.2 Generation of gravitational waves

The generation of gravitational waves are discussed here [6, 9, 10, 11]. The radiation formula and sources of gravitational waves are introduced.

2.2.1 Radiation formula

When the gravitational potential and velocity of matter of a source are small, the formula of the radiation energy of the gravitational wave is expressed as

$$-\frac{dE}{dt} = \frac{G}{45c^5} \left(\frac{d^3 D_{ij}}{dt^3} \right)^2, \quad (2.37)$$

$$D_{ij} = \int \rho \left(x_i x_j - \frac{1}{3} \delta_{ij} r^2 \right) d^3 x, \quad (2.38)$$

where E and ρ are the energy and density of a source. This formula corresponds to the quadrupole radiation. There is no dipole radiation of the gravitational waves because the mass is always positive. Equations (2.37) and (2.38) prove that spherically symmetric or stationary axially symmetric systems never generate the gravitational waves.

The amplitude of the gravitational wave is described as

$$h_{ij} = -\frac{2G}{c^4} \frac{1}{r} \ddot{D}_{ij}, \quad (2.39)$$

where r is the distance from a source.

2.2.2 Sources

The sources of the gravitational waves are introduced here. There are three kinds of sources: burst sources, periodic sources, and stochastic sources.

Burst sources

The supernova explosions are a kind of burst wave sources. The amplitude¹, h , of the gravitational wave from supernovas in our galaxy and within 20 Mpc distance is expected to be about 10^{-18} and 10^{-21} , respectively. The frequency is hundreds or thousands Hz. The supernova is one of the targets of the interferometric detectors on the Earth. These gravitational waves carry information about supernova explosions and generation of neutron stars and black holes.

The coalescence of compact binaries also generates burst waves. The distance between the compact objects in a binary system becomes shorter because of the radiation of gravitational waves until the two compact objects collide. In the last several minutes before the coalescence, large burst of gravitational wave is generated. If the binary is a neutron star-neutron star binary, the frequency is expected to peak at a few thousands Hz. The amplitude of the gravitational waves, h , from these binaries within 20 Mpc distance is calculated to be about 10^{-21} . The neutron star-neutron star coalescing² is a particular important target of interferometric detectors on the Earth. Several kinds of information are derived from the gravitational waves of the neutron star-neutron star binary: the equation of state of condensed matter, the distance from the source, and so on.

If small black hole-black hole binaries (binary Black Hole MACHO) were made in the early universe, this is an important candidate of the burst wave sources for interferometric detectors on the Earth. The amplitude, h , and frequency of the gravitational wave from these binaries in our galactic halo are calculated to be 10^{-18} and 500 Hz, respectively[12, 13].

If the binary is made of two extremely large black holes³, the frequency of the gravitational wave is expected to peak at 0.01 mHz. It is expected that these large black hole

¹The parameter, h , represents h_{ij} in Eq.(2.39).

²If the gamma-ray bursts are generated in the neutron star-neutron star final coalescing, the gamma-ray burst radiates not only the gamma-ray but also the gravitational waves.

³The mass is about 10^7 times larger than that of the sun.

binaries are at centers of galaxies.

Other burst sources are the generation of large black holes and the falling matter into the large black holes. Typical frequencies are about 1 mHz and 10 mHz, respectively. It is expected that these phenomena occurs at the centers of the galaxies.

Periodic sources

The compact binaries radiate periodic gravitational waves. The generation of periodic waves by the compact binaries was proved by the observation of the slow decay of the revolution of binary pulsars [3]. Since there are many binaries in the universe, the gravitational waves from these sources can be observed as stochastic back ground. The typical frequency of these sources is about 0.01 mHz. The periodic gravitational wave from the binaries is a target of interferometric detectors in space.

If a pulsar is not axial symmetric, it can generate periodic gravitational waves. The expected frequency is twice the neutron star revolution frequency, between ten Hz and thousands Hz. This is one of the targets of resonant detectors as well as interferometric detectors.

Stochastic sources

There are two kinds of expected stochastic gravitational waves. The first kind is the random summation of the gravitational waves from burst and periodic sources. The second kind is related with the evolution of the universe. The second type is introduced here.

Before the Planck time⁴, the gravitational waves and other elementary particles were in thermal equilibrium. If these gravitational waves exist, their present spectrum is that of the black body radiation, i.e. these gravitational waves are equivalent to the cosmic microwave background radiation. The temperature of the black body radiation of the gravitational waves is estimated to be 0.9 K. If the inflation scenario is valid, these gravitational waves would vanish because of the outrageously rapid expansion of the universe. In this case, the universe would be filled with the gravitational waves caused by quantum fluctuations in the inflation.

In the history of the universe, the phase transition occurred several times: inflation,

⁴The Planck times is about 10^{-43} sec.

GUT, electroweak, and QCD. These phase transitions may have caused bubbles, cosmic strings, and fluctuations of pressure. The collisions of the bubbles, the collapses of the cosmic strings, and the fluctuations of the pressure can generate gravitational waves. Especially, the cosmic strings caused by GUT phase transition are famous candidate sources of gravitational waves.

2.3 Detection of gravitational waves

The methods of detection of the gravitational waves are described here [9, 10, 11]. There are four proposed methods: interferometric detector, resonant detector, Doppler tracking, and pulsar timing.

2.3.1 Interferometric detectors

The details of interferometric detectors are discussed here [9, 10, 11].

Principle

The Michelson interferometer is a device to measure differences between distances or transit times of light rays in two orthogonal directions. Since the gravitational waves affect this difference, the Michelson interferometers can be used as gravitational wave detectors.

The schematic view of a interferometric detector is shown in Fig.2.2. The beam splitter divides the light, two beams are reflected by the end mirrors, the reflected beams are recombined at the beam splitter, the combined light goes into the photo detector or back to the light source. The intensity at the photo detector depends on the difference between the phases of the two light beams⁵. The difference between the phases is proportional to the difference between the traveling times in the two arms of the interferometer.

The difference between the phases of the two beams at the beam splitter is estimated for an interferometer subject to a gravitational wave. It is supposed that the mirrors and the beam splitter are free. In actual interferometers, these optical components are suspended. Suspended components act in the horizontal plane as free masses at frequencies

⁵The details of the principles of the interferometer are discussed in Chapter 6.

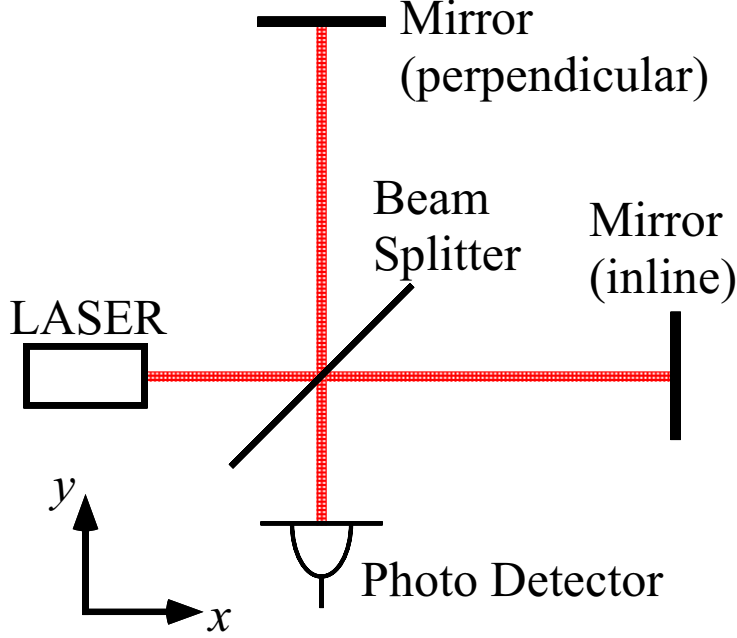


Figure 2.2: Schematic view of a Michelson interferometric gravitational wave detector. The beam splitter divides the light. These beams are reflected by the mirrors. The reflected beams are combined at the beam splitter. The combined light goes into the photo detector or back to the laser. The intensity at the photo detector depends on the difference between the traveling times of the two beams. The traveling times are affected by the gravitational wave.

higher than the resonant frequency of the pendulum. In order to simplify the discussion, the interferometer is considered at rest in the coordinate frame of the TT gauge. From Eq.(2.29), in this coordinate, the free optical components are not moved by the gravitational wave. The beam splitter is at the origin. The x -axis and y -axis are parallel to the two Michelson arms. The lengths of the two arms are l_1 and l_2 . The gravitational wave of the plus mode goes along the z -axis. The proper distance measured by the light is

$$ds^2 = -c^2 dt^2 + (1 + h_+)dx^2 + (1 - h_+)dy^2 + dz^2 = 0. \quad (2.40)$$

From Eq.(2.40), the round-trip of the light along the x -axis is derived. The integration of Eq.(2.40) is expressed as

$$\int_{t_1}^{t_0} \frac{cdt}{\sqrt{1 + h_+(t)}} = \int_0^{l_1} dx + \int_{l_1}^0 (-dx). \quad (2.41)$$

When time is t_1 , the light enters an arm. When time is t_0 , the light goes back to the beam splitter. Since the amplitude of the gravitational wave is small, $|h_+| \ll 1$, t_1 in the term of the integration of h_+ can be written in the form

$$t_1 \approx t_0 - \frac{2l_1}{c}. \quad (2.42)$$

Equation (2.41) is rewritten as

$$(t_0 - t_1) - \frac{1}{2} \int_{t_0 - \frac{2l_1}{c}}^{t_0} h_+(t) dt = \frac{2l_1}{c} \quad (2.43)$$

From a similar calculation along the y -axis, the expression which corresponds to Eq.(2.43) is,

$$(t_0 - t_2) + \frac{1}{2} \int_{t_0 - \frac{2l_2}{c}}^{t_0} h_+(t) dt = \frac{2l_2}{c}. \quad (2.44)$$

At t_2 , the light enters the perpendicular arm. At t_0 , the light goes back to the beam splitter. From Eqs.(2.43) and (2.44), the difference of the phases, ϕ , between the two beams is evaluated as

$$\phi(t_0) = \Omega(t_1 - t_2) = -\frac{2\Omega(l_1 - l_2)}{c} - \delta\phi_{\text{GR}}(t_0) \quad (2.45)$$

$$\delta\phi_{\text{GR}}(t_0) = \Omega \int_{t_0 - \frac{2l}{c}}^{t_0} h_+(t) dt. \quad (2.46)$$

The parameter, Ω , is the angular frequency of the light and it is supposed that $l_1 \approx l_2 \approx l$.

Equation (2.46) represents the effect of the gravitational wave. From this expression, the transfer function, $H_{\text{Michelson}}(\omega)$, which is the ratio of the Fourier component of the gravitational wave, h_+ , to the Fourier component of ϕ_{GR} is obtained as

$$H_{\text{Michelson}}(\omega) = \frac{2\Omega}{\omega} \sin\left(\frac{l\omega}{c}\right) \exp\left(-i\frac{l\omega}{c}\right). \quad (2.47)$$

Equation (2.47) shows that there is the optimal length, l_{opt} , of the arm of the interferometer. This length depends on the angular frequency, ω , of the gravitational wave,

$$l_{\text{opt}} = \frac{c\pi}{2\omega} = 75 \text{ [km]} \left(\frac{1 \text{ kHz}}{f} \right) \quad (2.48)$$

with $f = \omega/2\pi$.

Delay-line and Fabry-Perot

Since the typical frequency of the observable gravitational wave is a few thousand Hz at most, Eq.(2.48) shows that the optimal length of the arms is about 100 km at least. It is difficult and expensive to construct such a large interferometer. However, there are two methods to enhance the effective optical lengths in the short baselines, Delay-line and Fabry-Perot Michelson interferometers. The schematic view of these interferometers is shown in Fig.2.3. The left and right sides of Fig.2.3 show the Delay-line and Fabry-Perot interferometers, respectively. In the Delay-line interferometer, the long optical paths are folded between two mirrors. In the Fabry-Perot interferometer, Fabry-Perot cavities are placed in both the arms. The photons are stores in the cavities for a time when the light is resonant in the cavities. This represents that the optical path length is effectively enhanced.

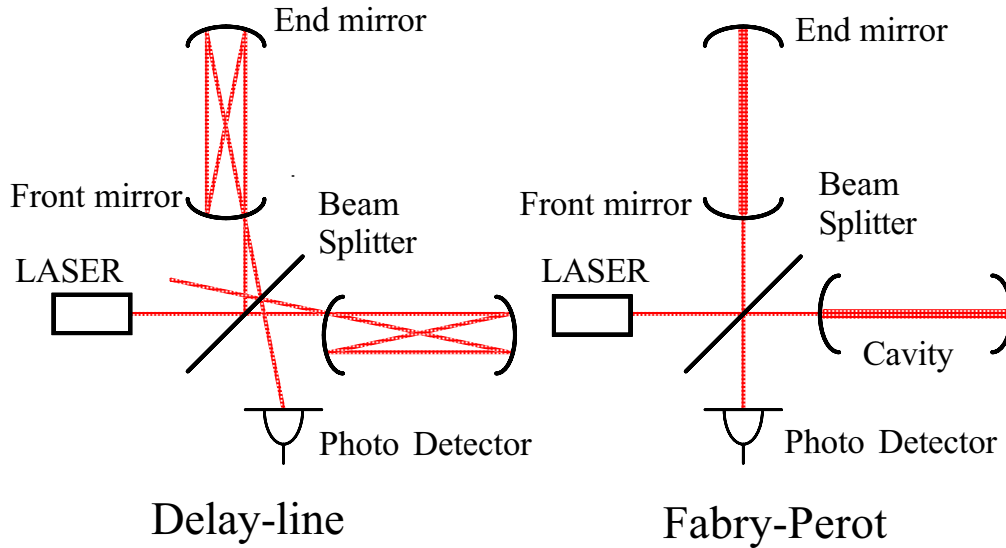


Figure 2.3: The schematic view of the Delay-line and Fabry-Perot Michelson interferometers. The left and right sides of this figure show the Delay-line and Fabry-Perot interferometers, respectively.

Current projects

Several construction projects of large interferometric gravitational wave detectors⁶ are in progress. They are summarized in Table.2.1.

Table 2.1: Current projects for large interferometric detectors

project	country	site	type	length
LIGO	U.S.A.	Hanford, Livingston	Fabry-Perot	4 km
VIRGO	Italy, France	Pisa	Fabry-Perot	3 km
GEO	Germany, U.K.	Hannover	Delay-line	600 m
TAMA	Japan	Tokyo	Fabry-Perot	300 m

In the United State of America, the LIGO project [14] is in progress. The interferometers are constructed at Hanford in Washington and Livingston in Louisiana. The distance between these sites is about 3000 km. The comparison between the outputs of the interferometers of each site is indispensable to decrease false alarms.

In the VIRGO project [15] of Italy and France, the interferometer is located at Pisa in Italy. The main focus of the VIRGO project is the detection of the low frequency gravitational waves. Since the sensitivity in the low frequency region is limited by seismic motions, an excellent seismic isolation systems, the super attenuators, was developed by the VIRGO project.

Germany and United Kingdom proceed with the GEO project [16]. The site is at Hannover in Germany. This is the only project adopting the Delay-line interferometer with dual-recycling.

TAMA [17] is the Japanese project. The interferometer is located at the Mitaka campus of National Astronomical Observatory in Tokyo. The objectives of this project are the observation of gravitational waves from near galaxies and the establishment of the technology for future projects. The interferometer of TAMA is the only interferometric detector presently operating with significant sensitivity.

⁶The interferometers in these projects are on the Earth. The target frequency is about 100 Hz. The LISA project [18] to assemble a interferometer in space are planning by NASA and ESA. The target frequency is about 1 mHz.

Noise sources

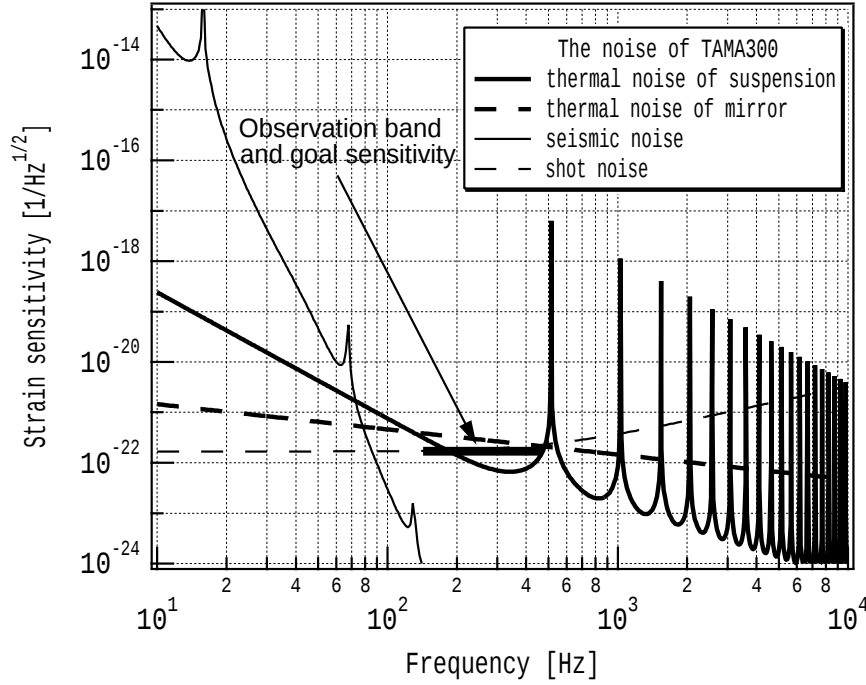


Figure 2.4: The expected noise budget of the TAMA300. The thick solid and dashed curves show the thermal noise of the suspensions and the mirrors, respectively. The thin solid curve is the seismic noise. The thin dashed curve is the shot noise level. The thick solid line shows the observation band of TAMA (between 150 Hz and 450 Hz) and the goal of the sensitivity ($h = 1.7 \times 10^{-22}/\sqrt{\text{Hz}}$).

Several factors contribute to the limit of the interferometric gravitational wave detectors. As a example of the noise budgets of interferometers, Fig. 2.4 shows the estimation of the spectral noise of TAMA300, which is the interferometer of the TAMA project. There are three fundamental noise sources: seismic noise, thermal noise, and shot noise⁷.

The sensitivity in the low frequency region is limited by the seismic motions. The measured power spectrum of the seismic motion is described as [77, 78]

$$G_{\text{seismic}}(f) = \frac{10^{-7}}{f^2} [\text{m}/\sqrt{\text{Hz}}]. \quad (2.49)$$

⁷There are many other noise sources: the noise caused by the laser source, the residual gas, and so on.

The values in Eq.(2.49) were measured near suburbs. The seismic motions in rural areas and especially mines are one hundred times smaller than that near suburbs [78]. The search of silent sites is an important issue in construction of gravitational wave detectors. Even in the quietest location on the Earth, seismic isolation systems are necessary to further suppress the seismic noise. The mirrors in the interferometer are suspended to make the mirrors act as free masses and this contributes some seismic isolation. However, the isolation power of the mirror suspension system is not high enough to allow detection of gravitational waves. Sophisticated isolation systems should be added to enhance the seismic isolation ratio.

The sensitivity in the middle frequency range is limited by the suspension and mirror thermal noise. The thermal motions of the suspension systems cause the fluctuations of the centers of the mirrors. The surfaces of the mirrors are deformed by the thermally excited elastic vibrations of the mirrors themselves. The thermal noise is the main theme in this thesis. The details are described in the following chapters.

In higher frequency region, the noise is dominated by the shot noise. The shot noise is generated by the quantum fluctuation of the number of photons stored in each beam. The limit of the measurement of the phase difference, $\delta\phi_{\text{shot}}[\text{rad}/\sqrt{\text{Hz}}]$, in a Michelson interferometer due to the shot noise can be expressed as

$$\delta\phi_{\text{shot}} = \sqrt{\frac{2\hbar\Omega}{\eta P}}, \quad (2.50)$$

where $\hbar$ is the reduced Planck constant. The parameters, Ω and P , are the angular frequency and the stored power of light, respectively. Here, η , is the quantum efficiency of the photo detector. Equation (2.50) shows that increasing of the stored power of the light reduces the shot noise. The power recycling is a useful technique to enhance the stored power of the light. A recycling mirror is inserted between the source of the light and the interferometer. This mirror reflects the light going back to the source. The reflected light goes into the interferometer again. In short, the power of the light going back to the source is re-used to enhance the power effectively.

2.3.2 Resonant detectors

Resonant motions of elastic bodies are excited by gravitational waves. The information of the gravitational waves passing through the elastic bodies is derived from monitoring

the vibrations of the elastic bodies. This is the principle of resonant detectors [4, 5].

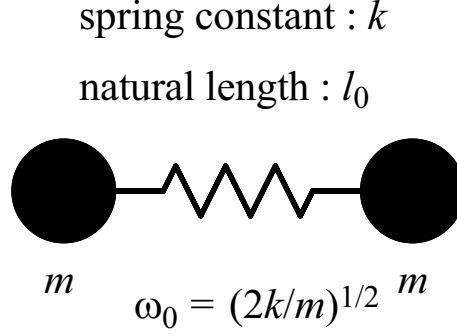


Figure 2.5: A simple model of a resonant detector. Two masses, m , are attached to the ends of a spring. The parameters, k and l_0 , are the spring constant and the natural length of this spring, respectively. The value, $\omega_0 \equiv \sqrt{2k/m}$, is the angular resonant frequency of this oscillator.

A simple model of a resonant detector is shown in Fig.2.5. Two masses, m , are attached to both the ends of a spring. The parameters, k and l_0 , are the spring constant and the natural length of this spring, respectively. The value, $\omega_0 \equiv \sqrt{2k/m}$, is the angular resonant frequency of this oscillator. In order to simplify the discussion, it is assumed that this detector is at rest in the coordinate frame of the TT gauge. The x -axis is parallel to the spring. The gravitational wave of the plus mode propagates along the z -axis. Since only the first order of h_+ is considered, the equation of the motion of the resonant detector is written in the form

$$m \frac{d^2 \xi}{dt^2} + m \frac{\omega_0}{Q} \frac{d\xi}{dt} + m \omega_0^2 \xi = \frac{1}{2} m l_0 \frac{\partial^2 h_+}{\partial t^2}, \quad (2.51)$$

where ξ is the difference between the length of the spring and the natural length, l_0 . The second term in the left-hand side of Eq.(2.51) corresponds to friction force. The parameter, Q , is the Q-value which represents the strength of dissipation⁸. Equation (2.51) is equal to the equation of the motion of a harmonic oscillator on which an external force is applied. This force corresponds to the restoring force due to the change of the proper length of the spring caused by the gravitational wave. From Eq.(2.51), the resonant detector vibrates when the frequency of the gravitational wave is the same as the resonant frequency of the detector. Thus, the observation band of the resonant detector, unlike that of interferometric detectors, is intrinsically narrow.

⁸The details of Q-values are discussed in Chapter 3.

An actual detector is a bulk made of metal. The typical mass, size, and resonant frequency are about 1 ton, a few meters, and 1 kHz, respectively. The elastic vibration of the detector is monitored using a transducer. The sensitivity is limited by the thermal elastic vibration, the noise of the transducer, and so on. In order to suppress the thermal excitation, the detectors are made of low loss material and cooled to cryogenic temperature. Low noise transducers are being continuously developed to improve the sensitivity.

2.3.3 Doppler tracking

The frequency of the electromagnetic wave traveling free space is shifted by gravitational waves. Thus, the precise measurement of the frequency standards of electromagnetic wave emitted by far away spacecraft may show the arrival of the gravitational waves. This method is called the Doppler tracking because the Doppler effect caused by the gravitational waves is detected. Usually, the electromagnetic signals going back and forth between the earth and a spacecraft are used as a probe.

Several space crafts, Viking, Voyager, ULYSSES, and so on, are used for the Doppler tracking. The observation band of this method is between 0.1 mHz and 10 mHz. The sensitivity of strain, h , is about 10^{-15} . The main noise source is plasma in space. The methods for the suppression of the noise caused by the plasma are adopting the higher frequency or two frequencies of the electromagnetic waves.

2.3.4 Pulsar timing

Pulsars are extremely precise clocks. The arrival times of signals from pulsars are affected by gravitational waves. In the pulsar timing, the information of the gravitational waves is derived from the analysis of the pulses from the pulsars. The observation band of the pulsar timing is between 10^{-8} Hz and 10^{-6} Hz. The upper limit of the spectrum of the gravitational waves obtained from the pulsar timing is the strict constraint on the scenario of the galaxies formation by the cosmic strings.

Chapter 3

Thermal noise

Thermal noise is one of fundamental noise sources in precision measurement, such as gravitational experiments. Especially, it is expected that the sensitivity of the interferometric gravitational wave detectors will be limited by the thermal noise of its mechanical components. Therefore, it is important to study the thermal noise for the improvement of the sensitivity. In this chapter, the fundamental theorem and the estimation method of the thermal noise are introduced. Moreover, the evaluation of the thermal noise of the interferometric gravitational wave detectors are discussed.

3.1 Thermal fluctuation

The thermal noise is one of the most serious problems of precise measurement. The thermal motions are fluctuations of generalized coordinates of systems due to the energy stochastically flowing to and from the heat bath. A famous example of the thermal fluctuations is the Brownian motion of small particles from pollen. This thermal motion is caused by random collisions of molecules; this physical interpretation was given by Einstein [19]. The thermal noise of resistances in electric circuits is also a well known example. The fluctuation of the voltage between both the ends of a resistance is due to thermal motions of electrons. The relationship between the fluctuation of the voltage and the resistance was found by Nyquist [20]. In the gravitation experiments, the thermal motion of mechanical oscillators is one of the limits of the measurement.

In precision experiments, generally, the observed frequency range is near a resonant frequency of mechanical systems. Since most of the energy of the thermal fluctuation is concentrated near the resonance, it is sufficient to evaluate the root mean square (rms) of the amplitude of the thermal fluctuation in these precise experiments. This value is derived from the principle of equipartition. The result is described as

$$\sqrt{\overline{x^2}} = \sqrt{\frac{k_B T}{m \omega_0^2}}, \quad (3.1)$$

where $\sqrt{\overline{x^2}}$ is the rms of the displacement of the oscillator, k_B is the Boltzmann constant¹, T is the temperature, and m and ω_0 are the mass and the resonant angular frequency of the mechanical oscillator, respectively.

Interferometric gravitational wave detectors are different. It is required that the sensitivity of the interferometric gravitational wave detectors is extremely high in broad frequency band. Therefore, the power spectrum density of the thermal noise in the broad band must be considered in the development of the detectors. In the following sections, the method of the evaluation of the spectrum of the thermal noise is introduced.

3.2 Fluctuation-dissipation theorem

The fluctuation-dissipation theorem (FDT) is one of the most important theorems of non-equilibrium statistical mechanics. The FDT predicts the relationship between the spectrum of the thermal noise and the dissipation of systems. Since the thermal fluctuations and the dissipation are governed by the same interaction between a system and the heat bath, there must be a relation between both phenomena. This theorem was established by Callen et.al [21, 22, 23, 24]. The formulae given by Einstein [19] and Nyquist [20] are the application of the FDT to small particles in liquid or air and voltage of resistances, respectively.

This theorem implies that the thermal noise can be estimated from the dissipation of systems. In most cases, the direct measurement of thermal noise is difficult because it is extremely small. Thus, loss of mechanical oscillators is measured instead of investigation of the thermal noise in the development of detectors. In this section, this important theorem of the thermal fluctuation is introduced.

¹ $k_B = 1.38 \times 10^{-23}[\text{J/K}]$.

3.2.1 FDT in a one-dimensional system

To simplify the discussion, a one-dimensional system is considered. The one-dimensional system has only one generalized coordinate, X . The generalized force, F , represents the interaction between the system and externals.

The information of the dissipation of this system is included in the response of X to F . This response is derived from the equation of motion of the system. In order to introduce the generalized force into this equation of the motion, the new term,

$$V = -F(t)X, \quad (3.2)$$

is added to the Hamiltonian² of this system [25].

Several values are defined to represent responses of the physical systems. The impedance, Z , is defined as

$$Z(\omega) \equiv \frac{\tilde{F}(\omega)}{i\omega\tilde{X}(\omega)} \quad (3.3)$$

where $\tilde{F}$ and $\tilde{X}$ are the Fourier components of the generalized force and coordinate, respectively. The real part of the impedance,

$$R(\omega) \equiv \text{Re}[Z(\omega)], \quad (3.4)$$

is called the resistance. The admittance, Y , is defined as

$$Y(\omega) \equiv \frac{1}{Z(\omega)}. \quad (3.5)$$

The real part of the admittance,

$$\sigma(\omega) \equiv \text{Re}[Y(\omega)], \quad (3.6)$$

is called the conductance. The transfer function, $H(\omega)$, is defined by

$$H(\omega) \equiv \frac{\tilde{X}}{\tilde{F}}, \quad (3.7)$$

in this thesis.

²The total energy flowing from outside into the system is described as $F(t)X$.

The relationship between the power spectrum density of X , $G_X(f)$, and the response of the system is described as

$$G_X(f) = \frac{4k_B T}{\omega^2} \sigma(\omega). \quad (3.8)$$

This relationship [23] is called the first fluctuation-dissipation theorem. Using $H(\omega)$, Eq.(3.8) is rewritten as

$$G_X(f) = -\frac{4k_B T}{\omega} \text{Im}[H(\omega)]. \quad (3.9)$$

The imaginary part of the transfer function represents the phase lag between the generalized force and its coordinate. When a phase lag exists, the average work by the force is not zero in the stationary state. Thus, the imaginary part describes the dissipation of the system. Consequently, the loss is associated with the thermal noise by the first fluctuation-dissipation theorem.

The power spectrum density of the generalized force, $G_F(f)$, is also derived from the physical response. The expression is written as

$$G_F(f) = 4k_B T R(\omega). \quad (3.10)$$

This relation [23] is called the second fluctuation-dissipation theorem.

3.2.2 FDT in a n -dimension system

In general, physical systems have several generalized coordinates. The previous discussion is extended to a system which has n coordinates, $X_1, X_2, \dots, X_n$. The interaction between this system and externals are represents by the generalized forces, $F_1, F_2, \dots, F_n$. In order to introduce these forces into the equations of the motions, the new term,

$$V = -\sum_{i=1}^n F_i X_i, \quad (3.11)$$

is added to the Hamiltonian of this system [25]. The response of this system is obtained from these equations of the motions.

Since energy can be exchanged between different degrees of freedom, the impedance and admittance of the n -dimension system are written using matrices. The relationship

between the generalized coordinates and forces is expressed as

$$\begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \vdots \\ \tilde{F}_n \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} & \cdots & Z_{1n} \\ Z_{21} & Z_{22} & \cdots & Z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{n1} & Z_{n2} & \cdots & Z_{nn} \end{pmatrix} \begin{pmatrix} i\omega \tilde{X}_1 \\ i\omega \tilde{X}_2 \\ \vdots \\ i\omega \tilde{X}_n \end{pmatrix}. \quad (3.12)$$

where Z_{ij} is the component of the impedance matrix. The admittance matrix is the inverse matrix of the impedance matrix. Equation (3.12) is rewritten as

$$\begin{pmatrix} i\omega \tilde{X}_1 \\ i\omega \tilde{X}_2 \\ \vdots \\ i\omega \tilde{X}_n \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n1} & Y_{n2} & \cdots & Y_{nn} \end{pmatrix} \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \vdots \\ \tilde{F}_n \end{pmatrix}. \quad (3.13)$$

where Y_{ij} is the component of the admittance matrix. The real parts of the impedance and admittance matrices are the resistance and conductance matrices, respectively. The component of the resistance matrix, R_{ij} , is described as

$$R_{ij} = \text{Re}[Z_{ij}]. \quad (3.14)$$

The component of the conductance matrix, σ_{ij} , is expressed in the form

$$\sigma_{ij} = \text{Re}[Y_{ij}]. \quad (3.15)$$

The transfer function is also expressed using a matrix. Equation (3.13) is rewritten as

$$\begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \vdots \\ \tilde{X}_n \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{pmatrix} \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \vdots \\ \tilde{F}_n \end{pmatrix}. \quad (3.16)$$

The first fluctuation-dissipation theorem of the n -dimension system [24] is expressed as

$$G_{X_i X_j}(f) = \frac{4k_B T}{\omega^2} \sigma_{ij}(\omega). \quad (3.17)$$

The functions, $G_{X_i X_j}(f)$, is the cross spectrum density. It is the Fourier component of the cross correlation function between X_i and X_j . When i is equal to j , the cross spectrum

density is identical with the power spectrum density of X_i . Using the transfer function matrix, Eq.(3.17) is rewritten as

$$G_{X_i X_j}(f) = -\frac{4k_B T}{\omega} \text{Im}[H_{ij}(\omega)]. \quad (3.18)$$

This formula shows that $G_{X_i X_j}$ is derived from H_{ij} which is the ratio of $\tilde{X}_i$ to $\tilde{F}_j$ when the other generalized forces, F_k ($k \neq i$), are zero.

The second fluctuation-dissipation theorem [24] is written as

$$G_{F_i F_j}(f) = 4k_B T R_{ij}(\omega). \quad (3.19)$$

This expression implies that $G_{F_i F_j}$ is derived from R_{ij} which is the real part of the ratio of $\tilde{F}_i$ to $i\omega\tilde{X}_j$ when the other generalized coordinates, X_k ($k \neq i$), are zero.

3.2.3 FDT for a linearly combined coordinate

It is often useful to define a special coordinate, which is not a natural mode of a system, but corresponds to an easily measurable quantity, like for example the surface of a mirror as seen by a laser beam profile. A new coordinate, X_{com} , of the n -dimension system can be defined as

$$X_{\text{com}} = \sum_{i=1}^n P_i X_i, \quad (3.20)$$

where P_i are arbitrary real constants. The power spectrum density of X_{com} is given by the application of the fluctuation-dissipation theorem to the transfer function, H_{com} ; this transfer function is the ratio of $\tilde{X}_{\text{com}}$ to the Fourier component of the generalized force, $\tilde{F}_{\text{com}}$. This proposition is proved here. In order to evaluate H_{com} , F_{com} is introduced into the equations of the motions. The new term,

$$V = -F_{\text{com}} X_{\text{com}} = -\sum_{i=1}^n F_{\text{com}} P_i X_i = -\sum_{i=1}^n (P_i F_{\text{com}}) X_i, \quad (3.21)$$

is added to the Hamiltonian. The comparison between Eqs.(3.11) and (3.21) suggests that the relation between X_i and F_{com} is given from substituting $P_i F_{\text{com}}$ for F_i in Eq.(3.16);

$$\begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \\ \vdots \\ \tilde{X}_n \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{pmatrix} \begin{pmatrix} P_1 \tilde{F}_{\text{com}} \\ P_2 \tilde{F}_{\text{com}} \\ \vdots \\ P_n \tilde{F}_{\text{com}} \end{pmatrix}. \quad (3.22)$$

From Eqs.(3.20) and (3.22), the transfer function, H_{com} , is described as

$$H_{\text{com}} = \frac{\tilde{X}_{\text{com}}}{\tilde{F}_{\text{com}}} = \sum_{i=1}^n \frac{P_i \tilde{X}_i}{\tilde{F}_{\text{com}}} = \sum_{i=1}^n \sum_{j=1}^n \frac{P_i H_{ij} P_j \tilde{F}_{\text{com}}}{\tilde{F}_{\text{com}}} = \sum_{i=1}^n \sum_{j=1}^n P_i P_j H_{ij}. \quad (3.23)$$

From Eqs.(3.18) and (3.20), the power spectrum density of the new coordinate, $G_{X_{\text{com}}}$, is written as

$$G_{X_{\text{com}}}(f) = \sum_{i=1}^n \sum_{j=1}^n P_i P_j G_{X_i X_j}(f) = -\frac{4k_B T}{\omega} \text{Im} \left[\sum_{i=1}^n \sum_{j=1}^n P_i P_j H_{ij}(\omega) \right] \quad (3.24)$$

From Eqs.(3.23) and (3.24), the relationship between $G_{X_{\text{com}}}$ and H_{com} is written in the form

$$G_{X_{\text{com}}} = -\frac{4k_B T}{\omega} \text{Im}[H_{\text{com}}(\omega)]. \quad (3.25)$$

This is formally the same expression as the first fluctuation-dissipation theorem, Eq.(3.9), of a one-dimensional system. As a result, G_{com} is obtained from the transfer function, H_{com} ; this transfer function corresponds to the ratio of $\tilde{X}_{\text{com}}$ to $\tilde{F}_{\text{com}}$ when the forces, $F_i = P_i F_{\text{com}} (i = 1 \sim n)$, are applied on the system. In this case, X_i is a function of the discrete parameter, i . Even though X was a function of the continuous parameter, $\mathbf{r}$, the same result would be obtained. The spectrum of the thermal fluctuation of the new coordinate, which is defined by

$$X_{\text{com}} = \int P(\mathbf{r}) X(\mathbf{r}) dV, \quad (3.26)$$

is derived from Eq.(3.25). In this case, the transfer function, H_{com} , represents the response of the system when the force, $F_{\text{com}} P(\mathbf{r})$, is applied.

From the similar consideration, the fluctuation of the new generalized force defined by

$$F'_{\text{com}} = \sum_{i=1}^n P'_i F_i, \quad (3.27)$$

is obtained. The values, P'_i , are arbitrary real constants. The impedance, Z_{com} , is the ratio of $\tilde{F}'_{\text{com}}$ to the $i\omega \tilde{X}'_{\text{com}}$ when each generalized coordinate, X_i , is equal to $P'_i X'_{\text{com}}$. The parameter, X'_{com} , is the generalized coordinate which corresponds to F'_{com} . The power spectrum density of F'_{com} is derived from applying the second fluctuation-dissipation theorem, Eq.(3.10), to Z_{com} .

3.3 Thermal noise of the harmonic oscillator

As a simple example, the thermal noise of a harmonic oscillator is discussed here [26]. This formula is the basis of the normal-mode expansion used frequently to evaluate the thermal noise of interferometric gravitational wave detectors.

3.3.1 Spectrum of thermal noise

The equation of the motion of a harmonic oscillator without dissipation is expressed as

$$m\ddot{x} + m\omega_0^2 x = F(t). \quad (3.28)$$

where m, ω_0, x , and F are the mass, the angular resonant frequency, the displacement, and the generalized force, respectively. The equation of the motion of the harmonic oscillator with loss in the frequency domain is written in the form

$$-m\omega^2 \tilde{x} + m\omega_0^2 [1 + i\phi(\omega)] \tilde{x} = \tilde{F}, \quad (3.29)$$

where $\phi(\omega)$ is the loss angle which represents the frequency dependence of the dissipation. From Eq.(3.29), the transfer function, H , is expressed as

$$H(\omega) = \frac{1}{-m\omega^2 + m\omega_0^2 [1 + i\phi(\omega)]}. \quad (3.30)$$

Substituting Eq.(3.30) for $H(\omega)$ in Eq.(3.9), the power spectrum density of the harmonic oscillator is obtained,

$$G_x(f) = \frac{4k_B T}{m\omega} \frac{\omega_0^2 \phi(\omega)}{(\omega^2 - \omega_0^2)^2 + \omega_0^4 \phi^2(\omega)}. \quad (3.31)$$

3.3.2 Viscous damping and structure damping

Using Eq.(3.31), the thermal noise caused by two well known kinds of dissipation are considered here. These two kinds of idealized dissipation are the viscous damping and the structure damping. The details of the properties of the dissipation in these two models are introduced in the next section.

When the loss is expressed as viscous damping, a resistance force proportional to the velocity is applied to the oscillator. The equation of the motion is described as

$$m\ddot{x} + m\Gamma\dot{x} + m\omega_0^2 x = F(t), \quad (3.32)$$

where Γ is a constant. In the frequency domain, this equation can be rewritten as

$$-m\omega^2 \tilde{x} + im\omega\Gamma \tilde{x} + m\omega_0^2 \tilde{x} = \tilde{F}. \quad (3.33)$$

From Eqs.(3.29) and (3.33), the expression of the loss angle is derived,

$$\phi(\omega) = \frac{\Gamma\omega}{\omega_0^2}. \quad (3.34)$$

Commonly, Eq.(3.34) is rewritten as

$$\phi(\omega) = \frac{\omega}{\omega_0 Q}, \quad (3.35)$$

where Q is the Q-value. The Q-value is frequently used as an indicator which represents the dissipation. The definition of the Q-value is in Eq.(3.44). From Eq.(3.34) and (3.35), the Q-value of the viscous damping is written in the form

$$Q = \frac{\omega_0}{\Gamma}. \quad (3.36)$$

As long as Γ is independent of the frequency, the Q-value is proportional to the resonant frequency. Substituting Eq.(3.35) for $\phi(\omega)$ in Eq.(3.31), the power spectrum of the thermal noise of the viscous damping is obtained,

$$G_x(f) = \frac{4k_B T}{mQ} \frac{\omega_0}{(\omega^2 - \omega_0^2)^2 + \omega_0^2 \omega^2 / Q^2}. \quad (3.37)$$

The typical behavior of the spectrum of the thermal noise of the viscous damping is shown in Fig3.1. Below the resonant frequency ($\omega \ll \omega_0$), the power spectrum is described as

$$G_x(f) = \frac{4k_B T}{m\omega_0^3 Q} = \text{constant}. \quad (3.38)$$

Above the resonant frequency ($\omega \gg \omega_0$), the thermal noise approximates to

$$G_x(f) = \frac{4k_B T \omega_0}{mQ} \frac{1}{\omega^4} \propto f^{-4}. \quad (3.39)$$

If the dissipation of the oscillator is described by the structure damping, the loss angle is written as

$$\phi(\omega) = \frac{1}{Q}, \quad (3.40)$$

where Q is a measured Q-value of the oscillator. The definition of Q is in Eq.(3.44). In structure damping model, the Q-value is independent of the resonant frequency. Putting Eq.(3.40) into Eq.(3.31), the spectrum of the thermal noise of the structure damping is given by:

$$G_x(f) = \frac{4k_B T}{mQ\omega} \frac{\omega_0^2}{(\omega^2 - \omega_0^2)^2 + \omega_0^4/Q^2}. \quad (3.41)$$

The typical behavior of the spectrum of the thermal noise of structure damping is shown in Fig3.1. When the frequency is lower than the resonant frequency ($\omega \ll \omega_0$), the approximate expression of the thermal noise is written in the form

$$G_x(f) = \frac{4k_B T}{m\omega_0^2 Q} \frac{1}{\omega} \propto f^{-1}. \quad (3.42)$$

On the contrary, when the frequency becomes higher than the resonant frequency ($\omega \gg \omega_0$), the thermal noise approximate to

$$G_x(f) = \frac{4k_B T \omega_0^2}{mQ} \frac{1}{\omega^5} \propto f^{-5}. \quad (3.43)$$

3.3.3 Q-value

The Q-value in Eqs.(3.37) and (3.41) is an important parameter for the estimation of the thermal noise. The definition and the methods to measure Q-values are introduced here. The definition of the Q-value is expressed as

$$Q = \frac{1}{\phi(\omega_0)}. \quad (3.44)$$

This definition implies that the Q-value is associated with the loss at the resonant frequency.

In order to estimate the thermal noise of the harmonic oscillator using Eq.(3.31), it is necessary to evaluate the loss angle, $\phi(\omega)$. Since the measurement of the loss angle

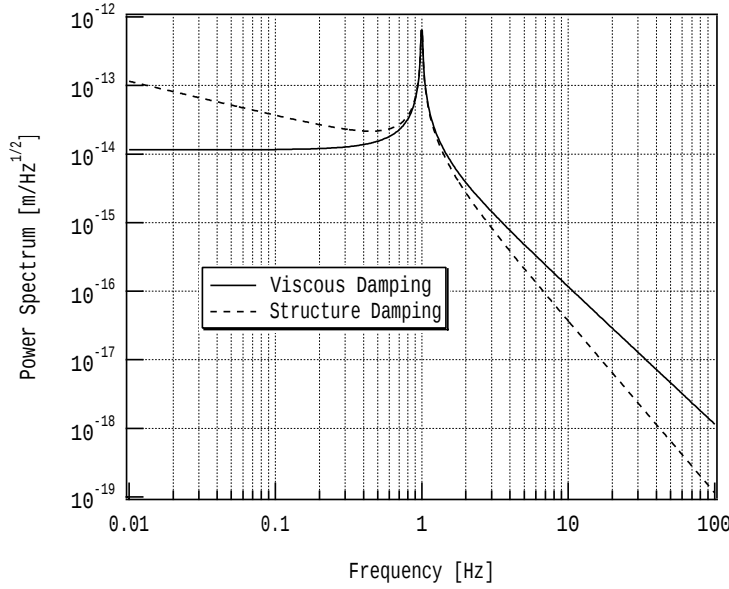


Figure 3.1: Examples of the power spectrum density of the thermal noise ($\sqrt{G_x}$) of a harmonic oscillator; the mass, m , is 1kg, the resonant frequency, $\omega_0/2\pi$, is 1Hz, and Q is 5×10^5 . The solid and dotted lines show the spectrum of the thermal noise of the viscous damping and the structure damping, respectively.

far from the resonance and in wide frequency range is difficult, the loss angle is derived commonly from the dependence of the Q -value on the resonant frequency. This dependence is obtained from the measurement in several modes. Therefore, the measurement of Q -values is indispensable for the evaluation of the thermal noise.

The most common method is the measurement of the decay time of the resonant motion. The second most common method is the measurement of the width of the resonant peak in the transfer function.

The measurement of the decay time is discussed. In order to measure the decay time of the excited resonant motion, the force,

$$F(t) = \begin{cases} F_0 \cos(\omega_0 t + \delta) & (t < 0) \\ 0 & (t > 0) \end{cases}, \quad (3.45)$$

is applied on the harmonic oscillator. The values, F_0 and δ , are arbitrary constants. The relation between the displacement of the harmonic oscillator, $x(t)$, and $F(t)$ in the frequency domain is expressed as

$$\tilde{x}(\omega) = H(\omega)\tilde{F}(\omega), \quad (3.46)$$

where $H(\omega)$ is the transfer function given by Eq.(3.30). In the time domain, this relation is rewritten in the form

$$x(t) = \int h(t-t')F(t')dt', \quad (3.47)$$

where $h(t)$ is the impulse response which is the inverse Fourier component of the transfer function, $H(\omega)$. The impulse response, $h(t)$, is written as³

$$h(t) = \frac{1}{m\omega_0} \exp\left(-\frac{\omega_0}{2Q}t\right) \sin(\omega_0 t). \quad (3.48)$$

Putting Eqs.(3.45) and (3.48) into Eq.(3.47), the decay of the resonant motion is written as

$$x(t) = \frac{F_0 Q}{m\omega_0^2} \exp\left(-\frac{\omega_0}{2Q}t\right) \sin(\omega_0 t + \delta) \quad (t > 0). \quad (3.49)$$

The decay time of the amplitude is related with ω_0/Q . Thus, the Q-value is derived directly from the measured decay time and the resonant frequency. Since the Q-value is proportional to the decay time, this method is appropriate when the Q-value is high.

The measurement of the half width, $\Delta\omega_0$, of the resonant peak of the transfer function is considered. The half width is defined by the solutions, $\Delta\omega_0$, of the equation:

$$\left| H\left(\omega_0 \pm \frac{\Delta\omega_0}{2}\right) \right|^2 = \frac{|H(\omega_0)|^2}{2}. \quad (3.50)$$

From Eqs.(3.30) and (3.50), the half width is described as⁴

$$\Delta\omega_0 = \frac{\omega_0}{Q}. \quad (3.51)$$

Thus, the Q-value are derived from the measured half width of the resonant peak. Since the half width is inverse proportional to the Q-value, this method is appropriate when the Q-value is low.

³When the impulse response is calculated, it is supposed that the Q-value is much larger than unity, $Q \gg 1$. In addition, since $x(t)$ and $F(t)$ are the real functions, the imaginary part of the transfer function is a odd function, i.e. the loss angle is a odd function.

⁴This is an approximate expression when the Q-value is larger than the unity, $Q \gg 1$.

3.4 Dissipation

From the fluctuation-dissipation theorem, the thermal noise is related to the dissipation. Several kinds of the dissipation [26] are introduced here. Sources of the loss are classified into the two categories. In the first category, the sources of the loss are external. In the second category, the sources of the dissipation are inside the oscillator itself (internal losses).

3.4.1 External losses

Typical examples of the external losses are the residual gas damping and the eddy current damping. Both these loss mechanisms are expressed by the viscous damping model.

Residual gas damping

This dissipation is caused in evacuated systems by the residual gas molecules hitting moving parts of an oscillator. Since most precision experiments are performed at sufficiently low pressure, the mean free paths of molecules of the residual gas are larger than the typical dimensions of an oscillator. Under such low pressure, the dissipation caused by the momentum transfer between the oscillator and molecules is larger than that caused by the true viscosity of the gas.

The Q-value limited by the momentum transfer [26, 27, 28] is expressed as

$$Q_{\text{gas}} = Ch \frac{\rho \omega_0}{n \sqrt{m_{\text{mol}} k_B T}}, \quad (3.52)$$

where C is a dimensionless parameter which depends on the shape of the oscillator. In most case, C is of the order of unity. The parameters, h , ρ , and ω_0 are the size, density, and angular resonant frequency of the oscillator, respectively. The values, n and m_{mol} , are the numerical density and mass of the gas molecules. An effective mass is commonly used when the residual gas is a mixture of several different gases, like air. Using the equation of state of the ideal gas, Eq.(3.52) is rewritten as

$$Q_{\text{gas}} = Ch \rho \omega_0 \sqrt{\frac{k_B T}{m_{\text{mol}} p}} \frac{1}{p}, \quad (3.53)$$

where p is the pressure. If the gas is air at room temperature, this expression is described as

$$Q_{\text{gas}} = 9 \times 10^5 \left(\frac{h}{0.25\text{m}} \right) \left(\frac{\rho}{2\text{g/cm}^3} \right) \left(\frac{f_0}{1\text{Hz}} \right) \left(\frac{1\text{Pa}}{p} \right) \quad (3.54)$$

where f_0 is the resonant frequency of the oscillator ($f_0 = \omega_0/2\pi$).

If an oil diffusion pump is used, a typical pressure is about 10^{-6} Pa. This suggests that Q_{gas} is larger than 10^{12} in most case. Q-values of oscillators are seldom larger than 10^{12} . Thus, the residual gas damping can be made negligible by operating in vacuum unless the size of the oscillator is extremely small or its resonant frequency is extremely low.

Eddy current damping

Eddy current is induced in a conductor moving in a magnetic field. Since the eddy current causes Joule heating, the kinetic energy of the conductor in the magnetic field is dissipated. This phenomenon is called the eddy current damping. The resistance force, F_{eddy} , caused by the eddy current is written as [29]

$$F_{\text{eddy}} = A\sigma B \frac{\partial B}{\partial x} \dot{x}, \quad (3.55)$$

where A is a constant which depends on the shape of the conductor, σ is the electric conductivity on the surface of the material, B and $\partial B/\partial x$ are the magnetic field and its gradient, and $\dot{x}$ is the velocity of the conductor. From Eqs.(3.32), (3.36), and (3.55), the Q-value of the eddy current damping is described as [29]

$$Q_{\text{eddy}} = \frac{m\omega_0}{A\sigma B \frac{\partial B}{\partial x}}. \quad (3.56)$$

This formula shows how to avoid the eddy current damping: magnetic shields and oscillators made from insulating materials. In general, these devices suppress the eddy current damping sufficiently. Therefore, the eddy current damping is usually negligible.

In some case, the eddy current damping is introduced intentionally to damp excitations of metal oscillators [29]. In the experiments discussed in Chapter 6 and 8, strong permanent magnets⁵ are used in order to realize selective and controlled losses.

⁵The strength of the magnetic field is about 1 T on the surface of these magnets.

3.4.2 Internal losses

Internal losses are discussed here. In most cases, the residual gas and the eddy current damping are reduced sufficiently. Thus, the dissipation is dominated by the internal losses in the material of the oscillator. In order to describe the internal losses, the complex spring constant model [26, 30] is frequently used. This model is an extension of the Hooke's law. This extended Hooke's law is expressed as

$$\tilde{F}_{\text{restoring}} = -k[1 + i\phi(\omega)]\tilde{x}, \quad (3.57)$$

where $F_{\text{restoring}}$ is the restoring force, k is the real spring constant, and ϕ is the loss angle. The equation of the motion of the harmonic oscillator derived from the Hooke's law is the same as Eq.(3.29)⁶. The Hooke's law shows that the phase of the strain of the spring lags behind that of the restoring force. The material property causing the phase lag is called anelasticity.

Two kinds of the internal dissipation are introduced here, the structure damping and the thermoelastic damping. Some investigations [31, 32, 33] suggest that loss in wires can be expressed as the sum of these two damping mechanisms.

Structure damping

In the structure damping model, ϕ is independent of the frequency. The loss angle, ϕ , is expressed as Eq.(3.40). Various experiments [34, 35, 36, 37, 38] show that the structure damping is observed in many kinds of materials. However, the mechanism of this damping is not well understood.

In addition, this model has mathematical difficulties. It is not possible that the loss angle is a constant in all the frequency range. The reason is that the loss angle must be an odd function because $\tilde{F}_{\text{restoring}}$ and $\tilde{x}$ in Eq.(3.57) are the Fourier components of the real functions. The loss angle must go to zero in the low and high frequency limits. If the loss angle was a constant in these limits, the displacement would be divergent when a step-function force is applied [36]. Therefore, the structure damping model should be seen only as valid for the dissipation which has weak dependence on the frequency.

⁶The spring constant, k , is replaced by $m\omega_0^2$.

Thermoelastic damping

The thermoelastic damping is caused by an inhomogeneous strain of a elastic body. Since the thermal expansion coefficient is not zero, the strain changes the temperature in the elastic body. If the strain is inhomogeneous, a gradient of temperature occurs. Heat flows to cancel this gradient of temperature. The elastic energy is dissipated owing to this flow of heat. The details of the process of the thermoelastic damping is fully understood [39]. Moreover, the experiments [31, 32, 33] support this theory.

The loss angle of the thermoelastic damping [26, 39] is expressed as

$$\phi(\omega) = \Delta \frac{\omega\tau}{1 + (\omega\tau)^2}, \quad (3.58)$$

where Δ represents the strength of the loss and τ corresponds to the relaxation time of the gradient of temperature. The frequency, f_0 , which corresponds to the relaxation time, τ , is defined by

$$f_0 = \frac{1}{2\pi\tau}. \quad (3.59)$$

When the frequency is f_0 , the loss angle is maximum. The maximum value is $\Delta/2$. Below this frequency, the loss angle is proportional to the frequency. Above this frequency, the loss angle is inversely proportional to the frequency.

The parameter, Δ , in a wire and a rectangular ribbon is written as

$$\Delta = \frac{E\alpha^2 T}{C\rho}, \quad (3.60)$$

where E is the Young's modulus, α is the linear coefficient of thermal expansion, T is the temperature, C is the specific heat, and ρ is the density. The value, Δ , depends on the properties of the material and on the temperature⁷. However, Δ is independent of the dimensions of the elastic body. On the contrary, the relaxation time, τ , of the gradient of the temperature depends on the dimensions of the oscillators. The characteristic frequency, f_0 in Eq.(3.59), in the wire is written as

$$f_0 = \frac{1}{2\pi\tau} = 2.16 \frac{D}{d^2}, \quad (3.61)$$

where d is the diameter of the wire, D is the thermal diffusion coefficient; D is the ratio of the thermal conductivity, κ , to $C\rho$. In the rectangular ribbon, f_0 is described as

$$f_0 = \frac{1}{2\pi\tau} = \frac{\pi}{2} \frac{D}{t^2}, \quad (3.62)$$

⁷At room temperature, Δ is about 10^{-3} in ordinary materials.

where t is the thickness of the ribbon.

The thermal noise of a cylindrical mirror caused by the thermoelastic damping was estimated recently [40, 41]. The loss angle is the same as Eq.(3.58). The strength of the loss, Δ , is expressed as

$$\Delta = \frac{E\alpha^2 T}{C\rho} \frac{1+\sigma}{1-\sigma}, \quad (3.63)$$

where σ is the Poisson's ratio. The characteristic frequency, f_0 , is written in the form

$$f_0 = \frac{1}{2\pi\tau} = \frac{2D}{\pi r_0^2}, \quad (3.64)$$

where r_0 is the beam radius on the surface of the mirror⁸.

3.5 Mode expansion

The thermal noise is obtained from the application of the fluctuation-dissipation theorem to the imaginary part of the transfer function, $H(\omega)$. However the measurement of the imaginary part of the transfer function in a broad frequency range is generally difficult [37, 42, 62, 63, 64]. This is because the imaginary part of the transfer function is in general much smaller than the real part. Thus, in order to evaluate $\text{Im}[H(\omega)]$ the normal-mode expansion method is frequently used. In this method, the imaginary part is derived from the measured Q-values. The outline of the derivation of the thermal noise from the normal-mode expansion is described here.

The observable physical quantity, X , of a system is defined as

$$X(t) = \int \mathbf{u}(\mathbf{r}, t) \cdot \mathbf{P}(\mathbf{r}) dV, \quad (3.65)$$

where $\mathbf{u}$ is the displacement of the system and $\mathbf{P}$ is a weighting function defining the physical quantity being observed. For example, when thermal fluctuations of the internal modes of the mirrors in interferometers are considered, $\mathbf{P}$ is the beam profile of laser. The power spectrum density of X , G_X , is obtained from Eq.(3.25). In order to calculate the

⁸The definition of the beam radius in [40, 41] is different from that in this thesis. In [40, 41], the beam profile is written as $(1/\pi r_0^2) \exp(-r^2/r_0^2)$. In this thesis, the profile is described as $(2/\pi r_0^2) \exp(-2r^2/r_0^2)$.

transfer function in Eq.(3.25), the generalized force, $F(t)\mathbf{P}(\mathbf{r})$, is applied on the system. Thus, the equation of motion of the system without dissipation is described as

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathcal{L}[\mathbf{u}] = F(t)\mathbf{P}(\mathbf{r}), \quad (3.66)$$

where ρ is the density, $\mathcal{L}$ is a linear operator. The first and second terms of the left-hand side of Eq.(3.66) represent the inertia and the restoring force of the small elements of the oscillator, respectively.

The solution of Eq.(3.66) is the superposition of basis functions,

$$\mathbf{u}(\mathbf{r}, t) = \sum_n \mathbf{w}_n(\mathbf{r}) q_n(t). \quad (3.67)$$

The basis function, $\mathbf{w}_n$, is the solution of the eigenvalue problem written as

$$\mathcal{L}[\mathbf{w}_n(\mathbf{r})] = -\rho \omega_n^2 \mathbf{w}_n(\mathbf{r}), \quad (3.68)$$

where ω_n and $\mathbf{w}_n(\mathbf{r})$ correspond to the angular resonant frequency and the displacement of the n -th resonant mode of the system, respectively. The displacement, $\mathbf{w}_n$, is the component of an orthogonal complete system, and is normalized to satisfy the condition written as

$$\int \mathbf{w}_n(\mathbf{r}) \cdot \mathbf{P}(\mathbf{r}) dV = 1. \quad (3.69)$$

The formula of the orthonormality is written as

$$\int \rho(\mathbf{r}) \mathbf{w}_l(\mathbf{r}) \cdot \mathbf{w}_n(\mathbf{r}) dV = m_n \delta_{ln}. \quad (3.70)$$

The parameter, m_n , is called the effective mass of the modes and δ_{ln} is the Kronecker's δ -symbol.

The function, $q_n(t)$ in Eq.(3.67), represents the time development of the n -th mode. The equation of motion of q_n is derived. Equation (3.67) is substituted for $\mathbf{u}$ in Eq.(3.66). Equation (3.66) is multiplied by $\mathbf{w}_n$ and then integrated using Eqs.(3.69) and (3.70). The result is written in the form

$$m_n \ddot{q}_n(t) + m_n \omega_n^2 q_n(t) = F(t). \quad (3.71)$$

Consequently, the time development of the n -th mode is the same as that of a harmonic oscillator on which the force, $F(t)$, is applied. The angular resonant frequency and the mass of this harmonic oscillator are equal to ω_n and m_n , respectively.

Putting Eq.(3.67) into Eq.(3.65), we obtain the relationship between X and q_n using Eq.(3.69),

$$X(t) = \sum_n q_n(t). \quad (3.72)$$

This formula shows that the observable coordinate, X , can be simply described as the superposition of the motions of the harmonic oscillators, q_n . Moreover, the kinetic energy, E_{kinetic} , of the total system is expressed as

$$E_{\text{kinetic}} = \int \frac{1}{2} \rho |\dot{\mathbf{u}}(\mathbf{r}, t)|^2 dV = \sum_n \frac{1}{2} m_n |\dot{q}_n(t)|^2. \quad (3.73)$$

The total kinetic energy is the sum overall the harmonic oscillators, $m_n |\dot{q}_n|^2/2$.

The transfer function, H_X , from the generalized force, F , to the observable coordinate, X , is evaluated. Equation (3.71) is rewritten in the frequency domain as

$$-m_n \omega^2 \tilde{q}_n + m_n \omega_n^2 \tilde{q}_n = \tilde{F}. \quad (3.74)$$

These equations do not include dissipation. To describe dissipation, the loss angles, $\phi_n(\omega)$, are introduced. Equation (3.74) is rewritten as

$$-m_n \omega^2 \tilde{q}_n + m_n \omega_n^2 [1 + i\phi_n(\omega)] \tilde{q}_n = \tilde{F}. \quad (3.75)$$

From Eqs.(3.72) and (3.75), H_X is described as

$$H_X(\omega) = \frac{\tilde{X}}{\tilde{F}} = \frac{\sum_n \tilde{q}_n}{\tilde{F}} = \sum_n \frac{1}{-m_n \omega^2 + m_n \omega_n^2 [1 + i\phi_n(\omega)]}. \quad (3.76)$$

The transfer function, H_X , of the system is the sum overall the harmonic oscillators with dissipation.

The power spectrum density of X , G_X , is derived from Eqs.(3.25) and (3.76). The formula is written as

$$G_X(f) = \sum_n \frac{4k_B T}{\omega} \frac{\omega_n^2 \phi_n(\omega)}{m_n [(\omega^2 - \omega_n^2)^2 + \omega_n^4 \phi_n^2(\omega)]}. \quad (3.77)$$

Therefore, the thermal motion of the system is the sum of the harmonic oscillators in the normal-mode expansion.

From Eq.(3.77), the thermal noise is derived from the angular resonant frequency, ω_n , and the effective mass, m_n , and the loss angle, ϕ_n , of each mode. The angular resonant

frequency and the displacement of the mode, $\mathbf{w}_n$, are obtained from the eigenvalue problem, Eq.(3.68). From $\mathbf{w}_n$ and Eq.(3.70), the effective mass is calculated. If $\mathbf{w}_n$ is not normalized as Eq.(3.69), Eq.(3.70) is rewritten as

$$m_n = \frac{\int \rho(\mathbf{r}) |\mathbf{w}_n(\mathbf{r})|^2 dV}{\left| \int \mathbf{w}_n(\mathbf{r}) \cdot \mathbf{P}(\mathbf{r}) dV \right|^2}. \quad (3.78)$$

The loss angle is derived from the experiments. Since the measurement of the loss angle in a wide frequency range is commonly difficult, the loss angle is evaluated from the dependence of the Q-value on the resonant frequency. The relationship between ϕ_n and Q_n is expressed as

$$Q_n = \frac{1}{\phi_n(\omega_n)}. \quad (3.79)$$

The structure damping model, Eq.(3.40), is frequently used when the thermal noise of interferometers is estimated. Consequently, the thermal noise is evaluated from the calculation of m_n and ω_n and the measurement of Q-values using the normal-mode expansion.

3.6 Thermal noise of the interferometer

The thermal noise of interferometric gravitational wave detectors are estimated from the normal-mode expansion here. These results are compared with the goals of sensitivity of current and future projects.

There are two kinds of thermal noise affecting interferometers. The first one is the thermal noise of suspensions. This noise generates fluctuations of the centers of mass of mirrors caused by thermal vibrations of the suspensions. The second one is the thermal noise of internal modes of the mirrors. This noise corresponds to surface deformation of the mirrors caused by the thermally excited elastic vibrations of the mirrors themselves.

3.6.1 Suspension

A suspension is a complex system. However, when the thermal noise of the suspension is evaluated, only the part which directly suspends a mirror is relevant. Since the observation band of detectors is higher than the resonant frequencies of the suspension, the

thermal fluctuations of parts which are far from the mirror are only marginally transferred to the mirror [43].

The mirror is usually suspended by wires. To simplify the consideration⁹, it is supposed that the mirror is suspended by a single wire [26, 44]. The upper end of the wire is fixed. The mirror is treated as a mass point. The x -axis is set on the wire. The origin is the fixed point of the wire. The x -coordinate of the mirror position is l . From Eq.(3.65), the observed coordinate, X , is defined as

$$X = \int u(x, t) P(x) dx, \quad (3.80)$$

where u is the transverse displacement of the wire and P is a suitable weighting function. Since the motion of the center of the mirror is discussed, the weighting function, P , is expressed as

$$P(x) = \delta(x - l), \quad (3.81)$$

where $\delta(x)$ is the δ -function.

The eigenvalue problem is derived from the wave equation of the wire. The expression is written as

$$\frac{T}{A} \frac{\partial^2 w_n}{\partial x^2} = -\rho \omega_n^2 w_n, \quad (3.82)$$

where T, A , and ρ are the tension, cross section and density of the wire, respectively¹⁰. In this case, the tension is equal to the product of the mass of the mirror, M , and the acceleration of gravity, g . Since the upper end is fixed, the boundary condition is described as

$$w_n(0) = 0. \quad (3.83)$$

Another boundary condition is the equation of motion of the mirror. This boundary condition is expressed as

$$-T \left. \frac{\partial w_n}{\partial x} \right|_{x=l} = -M \omega_n^2 w_n(l). \quad (3.84)$$

⁹The mirror is frequently suspended by two loop wires. The discussion in this subsection is appropriate for such a suspension system. However, in this system, m in Eq.(3.88) corresponds to the total mass of the two loop wire.

¹⁰In this equation, the elasticity of the wire is neglected. The effect of the elasticity was investigated in [45].

From Eqs.(3.82), (3.83), and (3.84), ω_n is the solution of the equation described as

$$\cos\left(\frac{\omega_n l}{v}\right) - \left(\frac{\omega_n v}{g}\right) \sin\left(\frac{\omega_n l}{v}\right) = 0, \quad (3.85)$$

where v is the velocity of the transverse wave of the wire. This speed is written in the form

$$v = \sqrt{\frac{T}{\rho A}} = \sqrt{\frac{Mg}{\rho A}}. \quad (3.86)$$

Since the mass of the mirror, M , is much larger than the mass of the wire, m ($=\rho A l$), the lowest order solution of Eq.(3.85) is expressed as

$$\omega_0 \approx \sqrt{\frac{g}{l}}, \quad (3.87)$$

which is the resonant frequency of the pendulum. The other solutions are written as

$$\omega_n \approx n\pi\omega_0\sqrt{\frac{M}{m}} \quad (3.88)$$

These are the resonant frequencies of the standing wave of the wire. These modes are called the violin modes. The solution of the eigenvalue problem, Eq.(3.82), is described as

$$w_n(x) = \sin\left(\frac{\omega_n x}{v}\right). \quad (3.89)$$

Putting Eqs.(3.81) and (3.89) into Eq.(3.78), the effective mass is obtained. Since the point mass is on the lower end of the wire, Eq.(3.78) is rewritten as

$$m_n = \frac{1}{|w_n(l)|^2} \int_0^l \rho A |w_n(x)|^2 dx + M. \quad (3.90)$$

Using Eq.(3.85), the formula of the effective mass is reduced as

$$m_n = \frac{M}{2} \left[1 + \frac{1}{\cos^2(\omega_n l/v)} \left(\frac{\omega_n}{\omega_0} \right)^2 \right] \quad (3.91)$$

$$\approx \begin{cases} M & (n = 0) \\ \frac{M}{2} \left(\frac{\omega_n}{\omega_0} \right)^2 & (n \neq 0) \end{cases}. \quad (3.92)$$

Commonly, the structure damping model is adopted to express the dissipation. The theoretical discussions [46, 47] suggest that the Q-value of the pendulum mode, Q_0 , is twice as large as those of the violin modes, Q_n . The loss angle is written in the form

$$\phi_n(\omega) = \begin{cases} \frac{1}{Q_0} & (n = 0) \\ \frac{2}{Q_0} & (n \neq 0) \end{cases}. \quad (3.93)$$

The Q-values of the violin modes, Q_n , are frequently measured because the decay time of the pendulum mode can be extremely long¹¹. Inserting Eqs.(3.87), (3.88), (3.92), and (3.93) into Eq.(3.77), we obtain the expression of the thermal noise of the suspension.

The approximate expression of the thermal noise of the suspension is derived here. Most of the observation band of gravitational wave detectors is between the resonant frequencies of the pendulum mode (about 1 Hz) and the first violin mode (several hundreds Hz). From Eq.(3.92), the effective masses of the violin modes are much larger than that of the pendulum mode. Consequently, in the observation band, the thermal noise of the suspension is dominated by the contribution of the pendulum mode. Thus, only the thermal fluctuation of the pendulum mode is estimated. Since the observation band is higher than f_0 , the approximate formula is written as

$$\begin{aligned} \sqrt{G_{\text{suspension}}(f)} &= 2.6 \times 10^{-19} \text{m}/\sqrt{\text{Hz}} \left(\frac{f_0}{1\text{Hz}} \right) \left(\frac{1\text{kg}}{m} \right)^{1/2} \left(\frac{10^5}{Q_0} \right)^{1/2} \\ &\times \left(\frac{T}{300\text{K}} \right)^{1/2} \left(\frac{100\text{Hz}}{f} \right)^{5/2}. \end{aligned} \quad (3.94)$$

3.6.2 Mirror

When the thermal noise of the internal modes of a mirror in interferometric gravitational wave detectors is calculated, the mirror is treated as a elastic cylinder. It is a good approximation to consider the mirror as an inertial system and that all the surfaces are free because the mirror is suspended by wires. Cylindrical coordinates are employed with the origin at the center of the mirror and the z -axis along the cylindrical axis. The observable coordinate, X , is expressed as

$$X = \int_{\text{surface}} u_z(\mathbf{r}) P(\mathbf{r}) dS, \quad (3.95)$$

¹¹If the Q-value of the pendulum mode is as low as 10^5 , the decay time is about a day.

where u_z is the z -component of the displacement vector, $\mathbf{u}$. The weighting function, P , is the Gaussian profile of the beam. The optical axis usually is aligned with the cylindrical axis. The weighting function is written as

$$P(\mathbf{r}) = \frac{2}{\pi r_0^2} \exp\left(-\frac{2r^2}{r_0^2}\right), \quad (3.96)$$

where r_0 is the beam radius.

The eigenvalue problem is derived from the equation of motion of an elastic body [48]. If the material is isotropic¹², the eigenvalue problem is written as

$$\frac{E}{2(1+\sigma)}\Delta\mathbf{w}_n + \frac{E}{2(1+\sigma)(1-2\sigma)}\text{grad div}\mathbf{w}_n = -\rho\omega_n^2\mathbf{w}_n, \quad (3.97)$$

where E is the Young's modulus, σ is the Poisson ratio, and ρ is the density of the mirror. The boundary condition is that there is no stress on all the surfaces of the mirror. There are two methods to solve this eigenvalue problem. The first one is the method proposed by Hutchinson [49]. This is a very accurate semi-analytical algorithm to simulate resonances of an isotropic elastic cylinder. The relative errors between the measured and calculated resonant frequencies are 0.6% at most [52]. Hutchinson's method is frequently used to estimate the thermal noise of mirrors [50, 51, 52]. The second method is the finite element method which is a numerical method. With this method, it is possible to calculate the mechanical responses of various shapes of elastic systems even though the material is anisotropic¹³ and non uniform. The relative errors between the measured and calculated resonant frequencies are also 0.6% [53].

The effective masses are derived from the solution of the eigenvalue problem, Eq.(3.97). Equation (3.78) is rewritten as

$$m_n = \frac{\int_{\text{volume}} \rho(\mathbf{r})|\mathbf{w}_n(\mathbf{r})|^2 dV}{\left|\int_{\text{surface}} w_{n,z}(\mathbf{r})P(\mathbf{r})dS\right|^2}, \quad (3.98)$$

where $w_{n,z}$ is the z -component of $\mathbf{w}_n$.

Frequently, it is supposed that the loss is described by the structure damping and that the Q-values of all the modes are the same. The loss angle is written as

$$\phi_n(\omega) = \frac{1}{Q}. \quad (3.99)$$

¹²The fused silica which is the material adopted in the current projects is isotropic.

¹³The sapphire which is a candidate material used in the future projects is anisotropic.

Recent experiments suggest that Q-values governed by the intrinsic loss of the material of the mirror are independent of modes [53, 54].

Putting calculated ω_n and Eqs.(3.98) and (3.99) into Eq.(3.77), the thermal noise of the mirror is evaluated. Since the observation band is lower than the resonant frequencies of the mirror, Eq.(3.77) is rewritten as

$$G_{\text{mirror}}(f) = \sum_n \frac{4k_B T}{m_n \omega_n^2 Q} \frac{1}{\omega}. \quad (3.100)$$

Since the number of modes is infinite, only the contributions of the modes below a cut-off frequency are considered. Thus, the estimated value of the thermal noise depends on the cut-off frequency. This dependence calculated for a TAMA mirror [52] is shown in Fig.3.2. Figure.3.2 proves that the contributions from higher modes are large. The thermal noise amplitude integrated overall modes is about three times larger than that of the first mode alone. Therefore, the cut-off frequency must be sufficiently high to estimate the thermal noise accurately. As an empirical rule, when the wavelength of the cut-off frequency is smaller than the beam radius, the contributions of the higher modes than the cut-off frequency become negligible.

The dependence of the thermal noise on the parameters of the mirror and beam size was investigated using the previous discussions [50, 51, 52]. The thermal noise is almost independent of the radius of the mirror when the mirror radius is sufficient large¹⁴. The thermal noise is almost independent of the distance between the optical axis and the cylindrical axis. The thermal noise depends on the aspect ratio which is the ratio of the thickness to the diameter of the mirror. When the aspect ratio is between 0.3 and 1.0, the thermal noise is minimum. Since the aspect ratio of mirrors in all the current projects are between 0.3 and 1.0, the optimum geometry has already been selected for the thermal noise. From Eq.(3.100), the thermal noise of this optimum mirror is expressed as

$$\begin{aligned} \sqrt{G_{\text{mirror}}(f)} &= 1.2 \times 10^{-19} \text{m}/\sqrt{\text{Hz}} \left(\frac{1 \text{cm}}{r_0} \right)^{1/2} \left(\frac{7.24 \times 10^{10} \text{Pa}}{E} \right)^{1/2} \\ &\times \left(\frac{10^6}{Q} \right)^{1/2} \left(\frac{T}{300 \text{K}} \right)^{1/2} \left(\frac{100 \text{Hz}}{f} \right)^{1/2}. \end{aligned} \quad (3.101)$$

¹⁴The ordinary radius of the mirror is at least three times larger than the beam radius to reduce the diffraction losses.

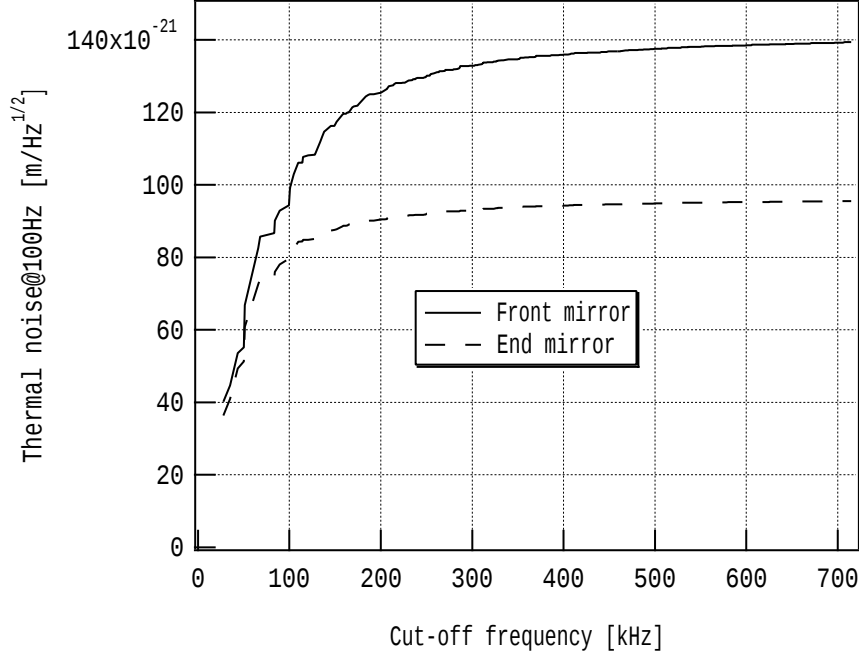


Figure 3.2: Dependence on the cut-off frequency of the estimated amplitude of the thermal noise of a mirror used in TAMA on the cut-off frequency. The material of the mirror is the fused silica. The mirror is 5 cm in radius and 6 cm in height. The resonant frequency of the first mode is 28 kHz. The beam radius is 8 mm at the front mirror (solid line) and 15 mm at the end mirror (dotted line), respectively. The Q-values are 10^6 .

3.6.3 Comparison with the goal

From the previous results of the mode expansion, the thermal noise of the TAMA interferometer, TAMA300, was derived¹⁵. In addition, this estimation of the thermal noise was compared with the sensitivity goal. The estimated thermal noise of the suspensions (thick solid curve) and the mirrors (thick dashed curve) of TAMA300 are shown in Fig.3.3. The seismic noise (thin solid curve) and the shot noise level (thin dashed curve) are also shown. The estimation of thermal noise in Fig.3.3 was derived from the measured Q-values. The measured Q-value of the first violin mode of the suspension which was similar to that of TAMA300 was 1.5×10^5 [55]. From Eq.(3.93), the Q-value

¹⁵The amplitude of the thermal noise of an interferometer is twice times larger than that of a suspension or a mirror because a Fabry-Perot Michelson interferometer has four mirrors which comprise two cavities.

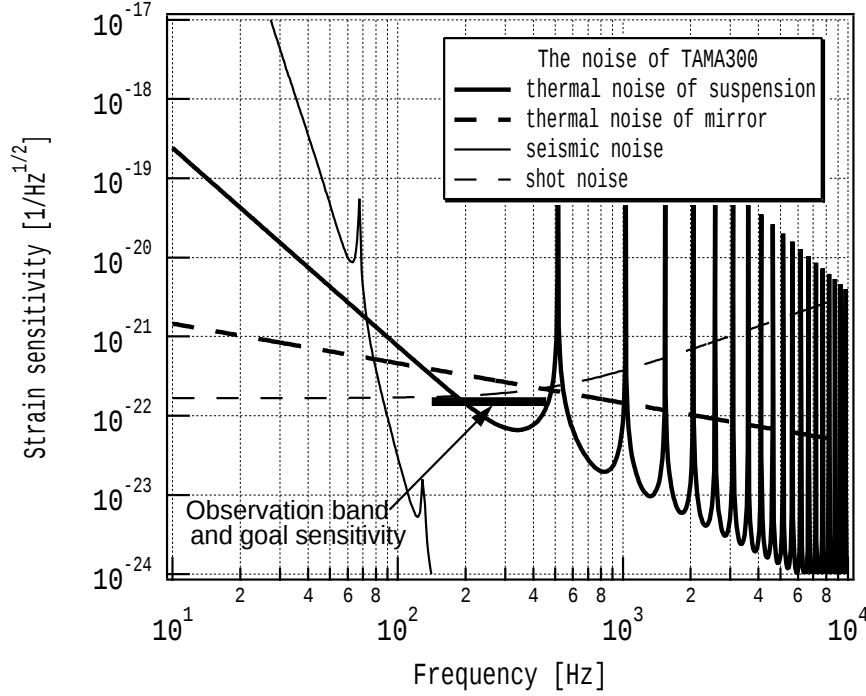


Figure 3.3: The expected noise budgets of the TAMA interferometer (TAMA300). The thick solid and dashed curves show the thermal noise of the suspensions and the mirrors, respectively. The Q-values of the pendulum mode and the internal modes are 3×10^5 and 3×10^6 . The thin solid curve is the seismic noise. The thin dashed curve is the shot noise level. The thick solid line shows the observation band (150Hz $\sim$ 450Hz) and the goal of the sensitivity ($h = 1.7 \times 10^{-22}/\sqrt{\text{Hz}}$) of TAMA.

of the pendulum mode was 3.0×10^5 . The measured intrinsic Q-values of the mirror was 3×10^6 [53, 54].

Figure 3.3 shows that the sensitivity in the observation band (150Hz $\sim$ 450Hz) will be limited by the thermal noise. Moreover, since the goal sensitivity in the observation band is $1.7 \times 10^{-22}/\sqrt{\text{Hz}}$, the thermal noise of TAMA300 is comparable with it. In the other current projects, LIGO, VIRGO, and GEO, the amplitude of the thermal noise is also comparable with the target sensitivity. On the other hand, in future projects, the goal sensitivity is ten or one hundred times better than that in the current projects. Consequently, the reduction of the thermal noise is one of the most important issues in future projects. In the Large-scale Cryogenic Gravitational wave Telescope (LCGT) [56], which is the Japanese future project, the mirrors and suspensions are cooled to cryogenic

temperatures in order to decrease the thermal noise. The research on the methods to cool the mirrors and suspensions shows that sufficiently low temperatures can be reached. At these temperatures, the thermal noise of the cooled mirrors and of the suspensions is sufficiently small to meet the goal sensitivity of LCGT [57, 58, 59, 60].

3.7 Problems of mode expansion

As shown in this chapter, the normal-mode expansion is the basis of the estimation of the thermal noise of the interferometric gravitational wave detectors. The results derived from the mode expansion affect strategies of reduction of the thermal noise. Nevertheless, there are problems in the normal-mode expansion. One of them is a main theme of this thesis. These problems are discussed here.

The main problem is that the mode expansion is not correct when the losses are distributed inhomogeneously. This is because the inhomogeneity of the losses causes correlations between fluctuations in motions of different modes [65]. The formula, Eq.(3.77), derived from the mode expansion does not include these correlations. Since losses are mainly localized, this is a serious problem. There are only a few theoretical investigations¹⁶ on the thermal noise caused by inhomogeneous loss: the details of this problem are discussed in the following chapters.

Another serious problem is in the evaluation method of the loss angles. Commonly, the loss angle is obtained from measurement of Q-values of modes. Since the Q-values represent dissipation at the resonant frequencies, it is difficult to estimate the loss angle at frequencies which are far from resonance. For example, recent theoretical researches [40, 41] showed that the thermoelastic damping of sapphire has a significant role at low frequencies. However, at the resonant frequencies, the dissipation of sapphire is dominated by other losses because the thermoelastic damping is small above the first

¹⁶Majorana and Ogawa showed that the thermal noise of a double oscillator with inhomogeneous loss is not consistent with the estimation obtained from the mode expansion [65]. Levin suggested that the thermal noise of a cylindrical mirror of which the loss is concentrated in the reflective coating is larger than the evaluation from the mode expansion [68]. Logan et al. has researched loss of a mirror with a coupling between modes [84]. Gillespie has studied effects of loss in spacers between a mirror and magnets [85].

resonance. Thus, it is impossible to derive the information of the thermoelastic damping from the measurement of the Q-values. In order to solve this problem, methods to measure the loss angle at the off-resonance are developed: measurement of anelastic relaxation [61] and measurement at anti-resonant frequencies [62, 63, 64].

Chapter 4

Advanced mode expansion

In the previous chapter, the normal-mode expansion adopted frequently to estimate thermal noise of interferometric gravitational wave detectors was introduced. Some theoretical researches [65, 68, 84, 85] show that the thermal motions of oscillators with the inhomogeneous loss do not agree with the estimation obtained from the mode expansion. Therefore, it is necessary to develop new evaluation methods replacing the mode expansion.

We have modified the traditional mode expansion to evaluate the thermal noise correctly. This new modified method is called the advanced mode expansion. Majorana and Ogawa have discussed the modification of the mode expansion using a double harmonic oscillator with inhomogeneous viscous damping [65]. This discussion is extended to the general system. In addition, we found that this new method clarifies the physical interpretation of the thermal noise of the inhomogeneous losses. For example, the advanced mode expansion shows that the reason why the thermal noise of the inhomogeneous loss is not consistent with the traditional mode expansion. Although there are other new estimation methods, direct approaches [68, 69, 70] introduced in the next chapter, these methods do not give the clear physical interpretation. The advanced mode expansion and the physical interpretation are described here.

4.1 Coupling between modes

The problem of the traditional mode expansion described in the previous chapter starts on the introduction of the dissipation term into the equation of the motion. The dissipation terms are introduced after the system is decomposed into the modes. The n -th loss angle, $\phi_n(\omega)$, is put into Eq.(3.74) which is the equation of the motion of the n -th mode. The validity of this introduction has not been checked. In the advanced mode expansion, the dissipation term is introduced before the equation of motion of the total system is decomposed.

The advanced mode expansion shows that coupling terms between the modes are obtained from the dissipation term when the loss is inhomogeneous. These coupling terms cause the discrepancy between the advanced and the traditional mode expansion. In order to investigate the coupling terms, the advanced mode expansion is applied to the system with inhomogeneous viscous damping and then with structure damping.

4.1.1 Viscous damping

The equation of motion of the system with inhomogeneous viscous damping is considered using the advance mode expansion. The friction force, which is proportional to the velocity of the elements of the system, is applied on this system. Introducing the friction force term into Eq.(3.66), the equation of the motion with the loss is obtained;

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} + \rho \Gamma(\mathbf{r}) \frac{\partial \mathbf{u}}{\partial t} - \mathcal{L}[\mathbf{u}] = F(t) \mathbf{P}(\mathbf{r}), \quad (4.1)$$

where Γ is the coefficient of the friction force and depends on the position, $\mathbf{r}$.

The solution of Eq.(4.1) is expressed as

$$\mathbf{u}(\mathbf{r}, t) = \sum_n \mathbf{w}_n(\mathbf{r}) q_n(t). \quad (4.2)$$

This equation is the same as Eq.(3.67). Since the loss is small, the dissipation term is treated as perturbation. Thus, the basis functions of the equation with loss are the same as those without loss. The basis function, $\mathbf{w}_n$, is the solution of the eigenvalue problem written as

$$\mathcal{L}[\mathbf{w}_n(\mathbf{r})] = -\rho \omega_n^2 \mathbf{w}_n(\mathbf{r}), \quad (4.3)$$

which is the same as Eq.(3.68). These basis functions satisfy Eq.(3.69) which are the normalized condition.

The equations of motion of q_n is derived. Eq.(4.2) is put into Eq.(4.1). Eq.(4.1) multiplied by $\mathbf{w}_n$ is integrated overall the volume. From Eq.(3.70), in the frequency domain, the result is written in the form

$$-m_n\omega^2\tilde{q}_n + m_n\omega_n^2\tilde{q}_n + \sum_k i\alpha_{nk}(\omega)\tilde{q}_k = \tilde{F}, \quad (4.4)$$

where α_{nk} is defined by

$$\alpha_{nk} = \omega \int \rho(\mathbf{r})\Gamma(\mathbf{r})\mathbf{w}_n(\mathbf{r}) \cdot \mathbf{w}_k(\mathbf{r}) = \alpha_{kn}. \quad (4.5)$$

The term which includes the coefficient, α_{nn} , represents the loss of the n -th mode. The relationship between α_{nn} and the loss angle, ϕ_n , is represented as¹

$$\phi_n(\omega) = \frac{\alpha_{nn}(\omega)}{m_n\omega_n^2}. \quad (4.6)$$

The Q-value of the n -th mode is written as

$$Q_n = \frac{1}{\phi_n(\omega_n)} = \frac{m_n\omega_n^2}{\alpha_{nn}(\omega_n)}. \quad (4.7)$$

Eq.(4.4) is rewritten as

$$-m_n\omega^2\tilde{q}_n + m_n\omega_n^2[1 + i\phi_n(\omega)]\tilde{q}_n + \sum_{k \neq n} i\alpha_{nk}(\omega)\tilde{q}_k = \tilde{F}. \quad (4.8)$$

The last term in the left-hand side of Eq.(4.8) does not exist in the formula obtained from the traditional mode expansion, Eq.(3.75). This is the difference between the advanced and traditional mode expansion. This part includes the coordinates of the other modes, $q_k(k \neq n)$. Therefore, it corresponds to couplings between the modes.

The traditional mode expansion is an appropriate method, when and only when all the coupling coefficients, $\alpha_{kn}(k \neq n)$, vanish. The comparison between Eqs.(3.70) and (4.5) shows that all the couplings vanish when and only when Γ is independent of position:

$$\Gamma(\mathbf{r}) = \Gamma. \quad (4.9)$$

¹From Eqs.(4.5) and (4.6), the kinetic energy is dissipated in the system with viscous damping.

This expression implies that the traditional mode expansion is valid when the viscous damping is homogeneous. Consequently, the inhomogeneity of dissipation causes the couplings between the modes.

When Γ is independent of the position, Eq.(4.5) is rewritten as

$$\alpha_{nk} = \omega m_n \Gamma \delta_{nk}, \quad (4.10)$$

where δ_{ij} is Kronecker's delta symbol. Eq.(4.6) is simplified as

$$\phi_n(\omega) = \frac{\Gamma \omega}{\omega_n^2} \quad (4.11)$$

The Q-value of n -th mode is expressed as

$$Q_n = \frac{1}{\phi_n(\omega_n)} = \frac{\omega_n}{\Gamma}. \quad (4.12)$$

The loss angle and Q-value of each mode depend on the resonant frequency of the mode.

4.1.2 Structure damping

The advanced mode expansion is applied to the elastic body with inhomogeneous structure damping. The equation of motion of an isotropic elastic body without loss [48] is written in the form

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial \sigma_{ij}}{\partial x_j} = F(t) P_i(\mathbf{r}), \quad (4.13)$$

where u_i and P_i are the i -th components of the displacement, $\mathbf{u}$, and the weighting function, $\mathbf{P}$, respectively. The stress tensor, σ_{ij} , is defined by

$$\sigma_{ij} = \frac{E_0}{1 + \sigma} \left(u_{ij} + \frac{\sigma}{1 - 2\sigma} u_{ll} \delta_{ij} \right), \quad (4.14)$$

where E_0 is Young's modulus, σ is Poisson ratio, and u_{ij} is the strain tensor. The value, u_{ll} , represents the trace of the strain tensor. The strain tensor is described as

$$u_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (4.15)$$

The solutions of Eq.(4.13) is decomposed as

$$u_i(\mathbf{r}, t) = \sum_n w_{n,i}(\mathbf{r}) q_n(t), \quad (4.16)$$

where $w_{n,i}$ is the i -th component of the displacement, $\mathbf{w}_n$, of the n -th resonant mode. Again, since the loss is small, the dissipation term is treated as perturbation and the basis functions of the equation with loss are the same as those without loss. The displacement of the n -th mode, $\mathbf{w}_n$, is a solution of the eigenvalue problem written as

$$\frac{\partial \sigma_{n,ij}}{\partial x_j} = -\rho \omega_n^2 w_{n,i}, \quad (4.17)$$

where $\sigma_{n,ij}$ is the stress tensor of the n -th resonant mode. The displacement of n -th mode, $\mathbf{w}_n$, satisfy Eq.(3.69) which are the normalized condition.

The dissipation term is introduced into Eq.(4.13). The structure damping is described using the complex Young's modulus. In the frequency domain, Eq.(4.14) is rewritten as

$$\tilde{\sigma}_{ij} = \frac{E_0[1 + i\phi(\omega, \mathbf{r})]}{1 + \sigma} \left(\tilde{u}_{ij} + \frac{\sigma}{1 - 2\sigma} \tilde{u}_{ll} \delta_{ij} \right), \quad (4.18)$$

where ϕ is the loss angle². This is a position dependent extension of the complex Hooke's law, Eq.(3.57). Since the loss is inhomogeneous, the loss angle depends on the position, $\mathbf{r}$. Putting Eq.(4.18) into Eq.(4.13) in the frequency domain, the dissipation term is introduced in the equation of motion:

$$-\rho \omega^2 \tilde{u}_i - \frac{\partial \tilde{\sigma}'_{ij}}{\partial x_j} - i \frac{\partial \phi(\omega, \mathbf{r}) \tilde{\sigma}'_{ij}}{\partial x_j} = \tilde{F} P_i(\mathbf{r}). \quad (4.19)$$

The value, $\tilde{\sigma}'_{ij}$, is the real part of the $\tilde{\sigma}_{ij}$ in Eq.(4.18):

$$\tilde{\sigma}'_{ij} = \frac{E_0}{1 + \sigma} \left(\tilde{u}_{ij} + \frac{\sigma}{1 - 2\sigma} \tilde{u}_{ll} \delta_{ij} \right). \quad (4.20)$$

The equation of the motion of q_n is derived. Eq.(4.16) in the frequency domain is put into Eq.(4.19). Eq.(4.19) multiplied by $\mathbf{w}_n$ is integrated in all the volume. From Eq.(3.70), the result is written in the form

$$-m_n \omega^2 \tilde{q}_n + m_n \omega_n^2 \tilde{q}_n + \sum_k i \alpha_{nk}(\omega) \tilde{q}_k = \tilde{F}, \quad (4.21)$$

where α_{nk} is defined by³

$$\alpha_{nk} = \int \frac{E_0 \phi(\omega, \mathbf{r})}{1 + \sigma} \left[w_{n,ij} w_{k,ij} + \frac{\sigma}{1 - 2\sigma} w_{n,ll} w_{k,ll} \right] = \alpha_{kn}, \quad (4.22)$$

²Since the loss is the structure damping, the loss angle is independent of the frequency. However, the discussion in this section is appropriate also when the loss angle depends on the frequency.

³This definition is appropriate when the surface is fixed ($u_i = 0$) or there is no surface stress ($\sigma_{ij} n_j = 0$, where n_j is the j -th component of the normal unit vector of the surface.).

where $w_{n,ij}$ is the strain tensor of the n -th mode. The expression, $w_{n,ll}$, corresponds to the trace of $w_{n,ij}$. The term which includes the coefficient, α_{nn} , represents the loss of the n -th mode. The relationship between α_{nn} and the loss angle, ϕ_n , is represented as⁴

$$\phi_n(\omega) = \frac{\alpha_{nn}(\omega)}{m_n \omega_n^2}. \quad (4.23)$$

The Q-value of the n -th mode is written as

$$Q_n = \frac{1}{\phi_n(\omega_n)} = \frac{m_n \omega_n^2}{\alpha_{nn}(\omega_n)}. \quad (4.24)$$

Eq.(4.21) is rewritten as

$$-m_n \omega^2 \tilde{q}_n + m_n \omega_n^2 [1 + i\phi_n(\omega)] \tilde{q}_n + \sum_{k \neq n} i\alpha_{nk}(\omega) \tilde{q}_k = \tilde{F}, \quad (4.25)$$

The last term in the left-hand side of Eq.(4.25) does not exist in the formula derived from the traditional mode expansion, Eq.(3.75). Again, this part includes the coordinate of the other modes, $q_k (k \neq n)$, corresponding to couplings between modes.

The traditional mode expansion is an appropriate method, when and only when all the coupling coefficients, $\alpha_{kn} (k \neq n)$, vanish. If the third term of the left-hand side in Eq.(4.19) is proportional to the second term, the coupling terms vanish. The third term is rewritten as

$$-i \frac{\partial \phi(\mathbf{r}) \tilde{\sigma}'_{ij}}{\partial x_j} = -i \phi(\omega, \mathbf{r}) \frac{\partial \tilde{\sigma}'_{ij}}{\partial x_j} - i \tilde{\sigma}'_{ij} \frac{\partial \phi(\omega, \mathbf{r})}{\partial x_j}. \quad (4.26)$$

The first term in Eq.(4.26) is proportional to the second term in Eq.(4.19). Thus, when and only when the second term in Eq.(4.26) vanishes, the traditional mode expansion is correct. This condition is satisfied when the loss angle is independent of the position:

$$\phi(\omega, \mathbf{r}) = \phi(\omega). \quad (4.27)$$

This expression implies that the traditional mode expansion is valid when the structure damping is homogeneous. Therefore, the inhomogeneity of the structure damping causes the couplings between the modes.

⁴From Eq.(4.22), a half of α_{nn} is equivalent to the integration of the product of the loss angle, $\phi(\mathbf{r})$, and the elastic energy density. Thus, the elastic energy is dissipated in the system with the structure damping.

When ϕ is independent of the position, Eq.(4.22) is rewritten as⁵

$$\alpha_{nk} = m_n \omega_n^2 \phi \delta_{nk}. \quad (4.28)$$

Eq.(4.23) is simplified as

$$\phi_n(\omega) = \phi(\omega). \quad (4.29)$$

The Q-value of the n -th mode is expressed as

$$Q_n = \frac{1}{\phi_n(\omega_n)} = \frac{1}{\phi(\omega_n)}. \quad (4.30)$$

The loss angle of the n -th mode is the same as ϕ .

4.1.3 Summary of coupling

The advanced mode expansion is applied on the system with the inhomogeneous viscous or structure damping. From both the cases, the similar conclusions are derived. The discussion of the advanced mode expansion is summarized and some comments are added.

In the traditional mode expansion, the dissipation terms are put into the equations of the motions after the system is decomposed. The loss angle is introduced into the equation of the motion of each mode. These equations are expressed as

$$-m_n \omega^2 \tilde{q}_n + m_n \omega_n^2 [1 + i\phi_n(\omega)] \tilde{q}_n = \tilde{F}. \quad (4.31)$$

In the advanced mode expansion, the equation of the motion with the dissipation term is decomposed. The equation of the motion of each mode is given by

$$-m_n \omega^2 \tilde{q}_n + m_n \omega_n^2 [1 + i\phi_n(\omega)] \tilde{q}_n + \sum_{k \neq n} i\alpha_{nk}(\omega) \tilde{q}_k = \tilde{F}. \quad (4.32)$$

The third term in the left-hand side of Eq.(4.32) is the discrepancy between the traditional and advanced mode expansion. Since this term include the coordinates of the other modes, $q_k (k \neq n)$, this term introduces the couplings between the modes. Eqs.(4.5) and (4.22) show that the coupling coefficient, α_{nk} , depends on the properties and distribution of the loss.

⁵Because the third term of the left-side hand in Eq.(4.19) is proportional to the second term.

When the loss is homogeneous viscous or structure damping, the coupling terms vanish because the coupling coefficients, α_{nk} ($n \neq k$), is proportional to the inner product of the n -th and k -th modes in this case. This implies that the inhomogeneity of the losses causes the couplings between the modes. Therefore, the traditional mode expansion is not correct when the losses are not homogeneous⁶.

The reason why the inhomogeneity of the losses causes the couplings is discussed here. The decay of a resonant mode is considered. When the loss is homogeneous in the elastic body, the phase of the decay motion does not depend on the position. The shape of displacement of the system does not change while the resonant motion decays. On the other hand, if the dissipation is inhomogeneous, the phase of the motion near the concentrated loss lags with respect to other parts [67]. The shape of the displacement becomes different from that of the resonant mode while the resonant motion decays. This inhomogeneous loss generates excitation of other modes.

In the advanced mode expansion, the dissipation term is treated as perturbation. From the perturbation theory in the quantum mechanics [66], this treatment is appropriate when the condition expressed as

$$|\alpha_{nk}|^2 \ll \frac{m_n m_k}{4} |\omega_n^2 - \omega_k^2|^2 \quad (4.33)$$

is satisfied. From Eqs.(4.33) and (4.40), if the difference between the resonant frequencies is sufficiently larger than the half widths of the resonant peaks, the loss can be treated as perturbation in most cases.

4.2 Properties of coupling

The properties of the coupling are discussed here before the effects of the coupling terms on the thermal noise are investigated. The maximum of the absolute value and the sign of the coupling terms are considered.

⁶In some cases, there are the couplings between the modes even though the loss is distributed homogeneously. The thermoelastic damping is an example. The thermoelastic damping is expressed by the imaginary part of E and σ in Eq.(4.18) [40, 48]. Even though the imaginary parts does not depend on the position, the couplings exist because the imaginary part of $\tilde{\sigma}_{ij}$ is not proportional to the real part. These cases are not considered in this thesis.

4.2.1 Maximum of the absolute value

The discrepancy between the advanced and traditional mode expansion is proportional to the absolute value of the coupling coefficients, $\alpha_{nk} (n \neq k)$. However, the maximums of the absolute values of the couplings exist. This maximum value is evaluated here.

From Eqs.(4.5) and (4.22), α_{nk} is expressed as

$$\alpha_{nk} = \sum_i \int f_{n,i}(\mathbf{r}) \phi(\mathbf{r}) f_{k,i}(\mathbf{r}) dV, \quad (4.34)$$

where $f_{n,i}$ is derived from the n -th basis function and $\phi (\geq 0)$ represents the distribution of the loss.

The Cauchy-Schwarz's inequality is written as

$$\int |F(\mathbf{r})|^2 dV \int |G(\mathbf{r})|^2 dV \geq \left| \int F(\mathbf{r}) G(\mathbf{r}) dV \right|^2, \quad (4.35)$$

where F and G are arbitrary real functions. When and only when F is expressed as

$$F = cG, \quad (4.36)$$

where c is an arbitrary constant, the right and left hands in Eq.(4.35) are equal. Another expression of the Cauchy-Schwarz's inequality is describe as

$$\sum_n |a_n|^2 \sum_n |b_n|^2 \geq \left| \sum_n a_n b_n \right|^2, \quad (4.37)$$

where a_n, b_n are arbitrary real progressions. When and only when a_n is expressed as

$$a_n = c b_n, \quad (4.38)$$

where c is an arbitrary constant, the right and left hands in Eq.(4.37) are equal.

Substituting $f_{n,i} \sqrt{\phi}$ and $f_{k,i} \sqrt{\phi}$ for F and G in Eq.(4.35) respectively, the following expression is obtained,

$$\int f_{n,i}(\mathbf{r}) \phi(\mathbf{r}) f_{n,i}(\mathbf{r}) dV \int f_{k,i}(\mathbf{r}) \phi(\mathbf{r}) f_{k,i}(\mathbf{r}) dV \geq \left| \int f_{n,i}(\mathbf{r}) \phi(\mathbf{r}) f_{k,i}(\mathbf{r}) dV \right|^2. \quad (4.39)$$

From Eqs.(4.34), (4.37), (4.39), (4.6), and (4.23), the inequality of α_{nk} is derived,

$$\begin{aligned}
 |\alpha_{nk}| &= \left| \sum_i \int f_{n,i}(\mathbf{r}) \phi(\mathbf{r}) f_{k,i}(\mathbf{r}) dV \right| \\
 &\leq \sum_i \left| \int f_{n,i}(\mathbf{r}) \phi(\mathbf{r}) f_{k,i}(\mathbf{r}) dV \right| \\
 &\leq \sum_i \sqrt{\int f_{n,i}(\mathbf{r}) \phi(\mathbf{r}) f_{n,i}(\mathbf{r}) dV} \sqrt{\int f_{k,i}(\mathbf{r}) \phi(\mathbf{r}) f_{k,i}(\mathbf{r}) dV} \\
 &\leq \sqrt{\sum_i \int f_{n,i}(\mathbf{r}) \phi(\mathbf{r}) f_{n,i}(\mathbf{r}) dV} \sqrt{\sum_i \int f_{k,i}(\mathbf{r}) \phi(\mathbf{r}) f_{k,i}(\mathbf{r}) dV} \\
 &= \sqrt{\alpha_{nn} \alpha_{kk}} \\
 &= \sqrt{m_n \omega_n^2 \phi_n(\omega) m_k \omega_k^2 \phi_k(\omega)}. \tag{4.40}
 \end{aligned}$$

From Eqs.(4.36) and (4.39), the absolute value of α_{nk} ($n \neq k$) is near the maximum when $\mathbf{w}_n$ is similar to $c\mathbf{w}_k$, where c is an arbitrary constant, in the localized loss. This condition is realized when the typical size of the volume in which the dissipation is concentrated is smaller than the wavelength of the n -th and k -th mode. When the loss is localized in the extremely small region, the absolute values of a lot of coupling coefficients are near the maximums. Therefore, the discrepancy between the actual thermal noise and the estimation of the traditional mode expansion is near the maximum when the dissipation is concentrated in a small volume.

4.2.2 Sign

From Eqs.(4.5) and (4.22), the sign of the coupling coefficients, α_{nk} ($n \neq k$), depend on the distribution of the dissipation. Moreover, the sign of α_{nk} depends on the observed area. The observed area is expressed by the weighting function, $\mathbf{P}(\mathbf{r})$, in Eqs.(3.65) and (3.69). The basis function, $\mathbf{w}_n$, in Eqs.(4.5) and (4.22) is normalized so that $\mathbf{w}_n$ satisfies Eq.(3.69). The sign of the displacement, $\mathbf{w}_n$, depends on the observed area. Therefore, the sign of the coupling coefficient depends on the observation area and on the distribution of the loss.

4.3 Thermal noise of the system with coupling

The formula of the thermal noise in the advanced mode expansion is discussed here. It is shown that the coupling terms correspond to the correlations between the fluctuations in the motions of the modes.

4.3.1 Formula of two mode system

To simplify the discussion, a system which has only two modes is considered. From Eq.(4.32), the equations of motions of the two modes are written in the form

$$-m_1\omega^2\tilde{q}_1 + m_1\omega_1^2[1 + i\phi_1(\omega)]\tilde{q}_1 + i\alpha_{12}\tilde{q}_2 = \tilde{F} \quad (4.41)$$

$$-m_2\omega^2\tilde{q}_2 + m_2\omega_2^2[1 + i\phi_2(\omega)]\tilde{q}_2 + i\alpha_{21}\tilde{q}_1 = \tilde{F} \quad (4.42)$$

From Eqs.(3.72), (4.41), and (4.42), the total system of the transfer function, $H_X(\omega)$, is obtained as

$$\begin{aligned} H_X(\omega) &= \frac{\tilde{X}}{\tilde{F}} = \frac{\tilde{q}_1 + \tilde{q}_2}{\tilde{F}} \\ &= \frac{-m_1\omega^2 + m_1\omega_1^2(1 + i\phi_1) - m_2\omega^2 + m_2\omega_2^2(1 + i\phi_2) - 2i\alpha_{12}}{[-m_1\omega^2 + m_1\omega_1^2(1 + i\phi_1)][-m_2\omega^2 + m_2\omega_2^2(1 + i\phi_2)] + \alpha_{12}^2}. \end{aligned} \quad (4.43)$$

If the two modes are well separated in frequency, i.e. the difference between the resonant frequencies of the two modes is larger than their half widths, α_{12}^2 in the denominator of Eq.(4.43) is negligible because of the upper limit of the coupling coefficient, Eq.(4.40). Thus, Eq.(4.43) is rewritten as

$$\begin{aligned} H_X(\omega) &= \frac{1}{-m_1\omega^2 + m_1\omega_1^2(1 + i\phi_1)} + \frac{1}{-m_2\omega^2 + m_2\omega_2^2(1 + i\phi_2)} \\ &\quad - \frac{2i\alpha_{12}}{[-m_1\omega^2 + m_1\omega_1^2(1 + i\phi_1)][-m_2\omega^2 + m_2\omega_2^2(1 + i\phi_2)]}. \end{aligned} \quad (4.44)$$

The first and second terms correspond to the transfer function calculated from the traditional mode expansion, Eq.(3.76). The last term is derived from the coupling term.

Putting Eq.(4.44) into Eq.(3.9), the power spectrum of the fluctuation of X is obtained:

$$\begin{aligned} G_X(f) &= \sum_{n=1}^2 \frac{4k_B T}{m_n \omega} \frac{\omega_n^2 \phi_n(\omega)}{(\omega^2 - \omega_n^2)^2 + \omega_n^4 \phi_n^2(\omega)} \\ &\quad + \frac{4k_B T}{m_1 m_2 \omega} \frac{2\alpha_{12} [(-\omega^2 + \omega_1^2)(-\omega^2 + \omega_2^2) - \omega_1^2 \phi_1(\omega) \omega_2^2 \phi_2(\omega)]}{[(\omega^2 - \omega_1^2)^2 + \omega_1^4 \phi_1^2(\omega)][(\omega^2 - \omega_2^2)^2 + \omega_2^4 \phi_2^2(\omega)]}. \end{aligned} \quad (4.45)$$

The first term in Eq.(4.45) are the same as the formula of the traditional mode expansion, Eq.(3.77). The extra term is not taken into account in the traditional mode expansion. Since the thermal noise in the broad frequency range are considered, the frequency of interest is far from the resonant frequencies. Thus, Eq.(4.45) is rewritten as

$$G_X(f) = \sum_{n=1}^2 \frac{4k_B T}{m_n \omega} \frac{\omega_n^2 \phi_n(\omega)}{(\omega^2 - \omega_n^2)^2} + \frac{4k_B T}{m_1 m_2 \omega} \frac{2\alpha_{12}}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}. \quad (4.46)$$

4.3.2 Correlation between modes

The last term in Eq.(4.46) represents the correlation between the fluctuations of the motions of the first and second modes [65]. This interpretation is derived from the FDT here. The fluctuations and the correlation of the generalized forces, F_n , which correspond to the coordinates, q_n , of the modes are evaluated from the FDT. From Eq.(3.11), the relation between the generalized forces and the coordinates of the modes is described as

$$-m_1 \omega^2 \tilde{q}_1 + m_1 \omega_1^2 [1 + i\phi_1(\omega)] \tilde{q}_1 + i\alpha_{12} \tilde{q}_2 = \tilde{F}_1, \quad (4.47)$$

$$-m_2 \omega^2 \tilde{q}_2 + m_2 \omega_2^2 [1 + i\phi_2(\omega)] \tilde{q}_2 + i\alpha_{21} \tilde{q}_1 = \tilde{F}_2. \quad (4.48)$$

Using Eqs.(3.12), (3.14), and (3.19), the power spectrum density of F_n , G_{F_n} , and the cross spectrum density between F_1 and F_2 , $G_{F_1 F_2}$, are evaluated from Eqs.(4.47) and (4.48):

$$G_{F_n}(f) = 4k_B T \frac{m_n \omega_n^2 \phi_n(\omega)}{\omega}, \quad (4.49)$$

$$G_{F_1 F_2}(f) = 4k_B T \frac{\alpha_{12}(\omega)}{\omega}. \quad (4.50)$$

The power spectrum density, G_{F_n} , are independent of α_{12} . On the other hand, $G_{F_1 F_2}$ depends on α_{12} . Therefore, the coupling term has no effect on the amplitude of the fluctuation of the generalized forces, while the coupling simply causes correlation between the generalized forces.

Having correlation between the generalized forces which correspond the modes, correlation between the motions of the modes must also exist. Using Eqs.(3.16) and (3.18), the power spectrum density of q_n , G_{q_n} , and the cross spectrum density between q_1 and

q_2 , $G_{q_1q_2}$, are evaluated from Eqs.(4.47) and (4.48)⁷:

$$G_{q_n}(f) = \frac{4k_B T}{m_n \omega} \frac{\omega_n^2 \phi_n(\omega)}{(\omega^2 - \omega_n^2)^2}, \quad (4.51)$$

$$G_{q_1q_2}(f) = \frac{4k_B T}{m_1 m_2 \omega} \frac{\alpha_{12}}{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}. \quad (4.52)$$

These formulae are approximated expressions when the frequency is far from resonance. The fluctuations of the modes, Eq.(4.51), and the correlation, Eq.(4.52), correspond to the first and second terms in the formula of the advanced mode expansion, Eq.(4.46), respectively. The power spectrum, G_{q_n} , is independent of α_{12} . On the contrary, $G_{q_1q_2}$ depends on α_{12} . Therefore, the coupling term has no effect on the amplitudes of the fluctuations of the modes, while it causes the correlation. Although the sign of the correlation between the generalized forces, Eq.(4.50), depends on only α_{12} , the sign of the correlation between the displacements, Eq.(4.52), depends on not only α_{12} but also on the frequency. This is because the sign of the real part of the transfer function from the n -th generalized force, F_n , to the n -th coordinate, q_n , depends on the frequency.

There is a maximum for the absolute value of the correlation, Eq.(4.52), because of the upper limit of the absolute value of the coupling coefficient in Eq.(4.40). This upper limit is derived as

$$|G_{q_1q_2}| \leq \sqrt{\frac{4k_B T}{m_1 \omega} \frac{\omega_1^2 \phi_1}{(\omega^2 - \omega_1^2)^2}} \sqrt{\frac{4k_B T}{m_2 \omega} \frac{\omega_2^2 \phi_2}{(\omega^2 - \omega_2^2)^2}} = \sqrt{G_{q_1} G_{q_2}}. \quad (4.53)$$

This expression is the same as the upper limit of the correlation derived from the definition of the cross spectrum. Thus, the upper limit of the coupling corresponds to the perfect correlation.

The correlation between the modes, the second term in Eq.(4.46), represents the discrepancy between the advanced and traditional mode expansion. Thus, if this term is comparable to the sum of the fluctuation of the modes expressed in the first term in Eq.(4.46), the discrepancy between the advanced and traditional mode expansion is not negligible. If the sign of the second term in Eq.(4.46) is negative, the estimation of the advanced mode expansion can be smaller than the evaluation of the traditional mode expansion. The conditions for which the difference between the advanced and traditional mode expansion becomes large is discussed here. From Eq.(4.53), the maximum of the

⁷The term, α_{12}^2 , in the denominator is negligible because of the discussion to derive Eq.(4.44).

absolute value of the correlation, $G_{q_1 q_2}$, in Eq.(4.52), is equal to the geometric mean of the fluctuations of the first and second modes, G_{q_1} and G_{q_2} in Eq.(4.51). The first term in Eq.(4.46) is the sum of the fluctuations of the first and second modes, G_{q_1} and G_{q_2} . Thus, when the absolute value of the coupling coefficient, α_{12} , is about maximum and the thermal motions of the first and second modes, G_{q_1} and G_{q_2} , are about the same, the contribution of the correlation, the second term in Eq.(4.46), is nearly equal to the sum of the fluctuations of the modes, the first term in Eq.(4.46). On the contrary, when the thermal noise is dominated by the contribution of one mode, the contribution of the correlation is smaller than the sum of the fluctuations of the modes, i.e. the difference between the advanced and traditional mode expansion is not serious even though the absolute value of the coupling coefficient, α_{12} , is almost maximum.

4.3.3 Formula of general system

The previous discussion about the thermal noise of the two mode system is easily generalized for all systems. The formula of the transfer function, Eq.(4.44), is rewritten as

$$H_X(\omega) = \sum_n \frac{1}{-m_n \omega^2 + m_n \omega_n^2 (1 + i\phi_n)} - \sum_{k \neq n} \frac{i\alpha_{nk}}{[-m_n \omega^2 + m_n \omega_n^2 (1 + i\phi_n)][-m_k \omega^2 + m_k \omega_k^2 (1 + i\phi_k)]}. \quad (4.54)$$

Putting Eq.(4.54) into Eq.(3.9), the formula of the thermal noise is obtained as

$$G_X(f) = \sum_n \frac{4k_B T}{m_n \omega} \frac{\omega_n^2 \phi_n(\omega)}{(\omega^2 - \omega_n^2)^2 + \omega_n^4 \phi_n^2(\omega)} + \sum_{k \neq n} \frac{4k_B T}{m_n m_k \omega} \frac{\alpha_{nk} [(-\omega^2 + \omega_n^2)(-\omega^2 + \omega_k^2) - \omega_n^2 \phi_n(\omega) \omega_k^2 \phi_k(\omega)]}{[(\omega^2 - \omega_n^2)^2 + \omega_n^4 \phi_n^2(\omega)][(\omega^2 - \omega_k^2)^2 + \omega_k^4 \phi_k^2(\omega)]}. \quad (4.55)$$

At the off-resonance frequencies, this formula is rewritten as

$$G_X(f) = \sum_n \frac{4k_B T}{m_n \omega} \frac{\omega_n^2 \phi_n(\omega)}{(\omega^2 - \omega_n^2)^2} + \sum_{k \neq n} \frac{4k_B T}{m_n m_k \omega} \frac{\alpha_{nk}}{(\omega^2 - \omega_n^2)(\omega^2 - \omega_k^2)}. \quad (4.56)$$

The first and second terms of the right-hand side of Eq.(4.56) represent the sum of the fluctuations of the modes and the correlations between the fluctuations of the modes, respectively.

When the contributions of the correlations, the second term in Eq.(4.56), is about as large as the sum of the fluctuations of the modes, the first term in Eq.(4.56), the difference between the advanced and traditional mode expansion is a serious issue. From the consideration about the two mode system, when the absolute values of many correlations are large and the fluctuations of a lot of modes are about equal to each other, the contributions of the correlations are comparable to the summation of the fluctuations of the modes. On the other hand, if the thermal noise is dominated by the contribution of one mode, the correlation terms are smaller than the contribution of this mode, i.e., in this case, the discrepancy between the advanced and traditional mode expansion is negligible even though the coupling coefficients are large.

4.3.4 Thermal noise of interferometer

The previous considerations give important suggestions for the research of the thermal noise of the interferometric gravitational wave detectors. Figure 3.2 shows that the thermal noise of the mirror is composed of the contributions of many modes. Since losses localized in small volumes cause large couplings, the thermal noise of the mirror with the losses distributed inhomogeneously is much different from the evaluation of the traditional mode expansion. On the other hand, the discrepancy between the thermal noise of the suspension and the the evaluation from the traditional mode expansion is negligible even though the loss is not uniform. This is because the thermal noise of the suspension is dominated by the fluctuation of the pendulum mode in the observation band. Consequently, the inhomogeneity of the dissipation is a serious problem in the thermal noise of the mirror and does not have an important role in the thermal noise of the suspension. In this thesis, the thermal noise of the mirror with inhomogeneous losses are considered in Chapters 7 and 8.

4.3.5 Root mean square

Equation (4.55) shows that the couplings, α_{nk} , caused by the inhomogeneity of the losses affects the spectrum of the thermal noise. However, the couplings have no effect on the root mean square (rms) of the thermal noise because the integration of the cross

spectrum between q_n and q_k in Eq.(4.52) overall the frequency range vanish:

$$\overline{q_n(t)q_k(t)} = \int_0^\infty G_{q_n q_k}(f) df = \frac{k_B T}{m_n \omega_n^2} \delta_{nk}, \quad (4.57)$$

where δ_{nk} is the Kronecker's δ -symbol. The sign, $\overline{A}$, denotes a time average of A . From Eq.(4.57), the root mean square of the observable physical quantity, X , expressed as Eq.(3.72) is obtained:

$$\sqrt{\overline{X^2}} = \sqrt{\sum_{n,k} \overline{q_n q_k}} = \sqrt{\sum_n \overline{q_n^2}} = \sqrt{\sum_n \frac{k_B T}{m_n \omega_n^2}}. \quad (4.58)$$

Consequently, when the rms of X is calculated, it can be considered that the fluctuations of the modes are independent of each other in spite of the inhomogeneity of the losses, i.e. $\overline{X^2}$ is the sum of $\overline{q_n^2}$ derived from the principle of equipartition, Eq.(3.1).

4.4 Measurement of coupling

The useful information is obtained from the measurement of the coupling. Eq.(4.56) shows that the coupling between the modes causes the discrepancy between the advanced and the traditional mode expansion. This implies the necessity of the measurement of the couplings to correctly estimate thermal noise. From Eqs.(4.5) and (4.22), the coupling terms depends on the distribution of the losses. Thus, the distribution of the losses can be derived from the measured coupling. This is a solution of the inverse problem shown in Fig.1.2 to derive the distribution and the properties of the loss from the mechanical response. The methods of the measurement of the coupling are discussed here. The measurements near the resonant frequencies and at the off-resonant frequencies are considered.

4.4.1 Measurement near resonant frequencies

The correlation term is maximum near the resonant frequencies. However, the resonant mode term is also maximum. The ratio of the term of the correlation to that of the resonant mode is about ϕ_n at most. Therefore, the measurement near the resonant frequencies is not useful to obtain the coupling coefficients, α_{nk} .

As an example, the measurement of the transfer function is considered. From Eq.(4.54), at the resonant frequency, $\omega_n/2\pi$, the transfer function of the system is described as

$$H_X(\omega_n) = \frac{1}{im_n\omega_n^2\phi_n} - \sum_{k \neq n} \frac{1}{im_n\omega_n^2\phi_n} \frac{i\alpha_{nk}}{-m_k\omega_n^2 + m_k\omega_k^2(1 + i\phi_k)}. \quad (4.59)$$

The terms of the other modes are neglected. The second term in Eq.(4.59) is the object of the measurement. The ratio of the second term to the first term in Eq.(4.59) is written as

$$\sum_{k \neq n} -\frac{i\alpha_{nk}}{-m_k\omega_n^2 + m_k\omega_k^2(1 + i\phi_k)}. \quad (4.60)$$

Since ϕ_k is smaller than the unity, Eq.(4.60) approximates a pure imaginary number. The coupling terms shift the phase of the transfer function, H_X . The order of magnitude of this phase shift is about ϕ_n at most because of the upper limit of α_{nk} , Eq.(4.40). Thus, in order to estimate the coupling from the measurement of the phase of the transfer function near the resonance, the phase error of the measurement should be smaller than ϕ_n . This is a difficult experiment in general.

As another example, the measurement of the thermal noise is discussed. Eq.(4.55) shows that the spectrum of the thermal noise at resonance is almost independent of α_{nk} . However, the absolute value of the correlation term in Eq.(4.55) has a local maximum, when the frequency is $f_n \pm \Delta f_n/2$, where f_n is the resonant frequency and Δf_n is the half width of the resonant peak defined by Eq.(3.50). From Eq.(4.55), at these frequencies, the thermal noise is expressed as

$$G_X(f_n \pm \Delta f_n/2) = \frac{2k_B T}{m_1\omega_n^3\phi_n} \mp \sum_{k \neq n} \frac{4k_B T}{m_n\omega_n^3\phi_n} \frac{\alpha_{nk}}{m_k(-\omega_n^2 + \omega_k^2)}. \quad (4.61)$$

The terms of the fluctuations of the other modes can be neglected. The second term in Eq.(4.61) is the object of the measurement. The ratio of the second term to the first term in Eq.(4.61) is written as

$$\mp \sum_{k \neq n} \frac{2\alpha_{nk}}{m_k(-\omega_n^2 + \omega_k^2)}. \quad (4.62)$$

Since the upper limit of α_{nk} is described as Eq.(4.40), the order of the absolute value of Eq.(4.62) is about ϕ_n at most. Thus, in order to estimate the coupling from the measurement of the spectrum of the thermal noise near the resonance, the relative error of the measurement must be smaller than ϕ_n . Again, this is also difficult.

4.4.2 Measurement at off-resonance frequencies

The method to derive the coupling from the measurement of the thermal noise or the transfer function in a broad frequency range is discussed. The difference between the measured values and the estimation of the traditional mode expansion does not always represents the couplings. There is the possibility that this difference corresponds to the discrepancy between the real loss angle and the adopting loss angle. However, there is a method to obtain the coupling from the measured spectrum. The signs of the coupling coefficients, α_{nk} , depend on the observation point. Thus, the measurement at various points should show the information of the couplings. For example, if the sign of the difference between the observed spectrum and the estimation of the traditional mode expansion depends on the observation points, the observed difference corresponds to the couplings. On the other hand, if this sign is independent of the observation points, the observed difference indicates that the chosen loss is not correct. Consequently, the coupling at off-resonant frequencies can be derived from the measurement of the thermal noise or of the transfer function at various points and from the comparison with the calculation based on the traditional mode expansion.

Chapter 5

Direct approach

In the previous chapter, the advanced mode expansion was developed because the traditional mode expansion is wrong when the loss is distributed inhomogeneously. The discussion in the previous chapter proved that there is a large discrepancy between the calculations of the advanced and that of the traditional mode expansion when the thermal noise consists of the contributions of many modes and the dissipation is highly inhomogeneous. However, in such a case, the calculation in the advanced mode expansion is complicated because many modes must be taken into account. Fortunately, there are more suitable methods for a practical computation. These methods are called direct approaches in this thesis. The computations in the direct approaches are easier even when there are contributions from many modes. This is because in direct approaches the thermal noise is evaluated without the mode decomposition.

In this chapter, three kinds of direct approaches are introduced. These methods are proposed by Levin [68], Nakagawa [69], and Tsubono [70]. Moreover, it is showed that the results of these three methods and that of the advanced mode expansion agree with each other.

5.1 Levin's approach

In the method proposed by Levin [68], the thermal noise is derived from the dissipated energy. The fluctuation-dissipation theorem, Eq.(3.9), shows the relation between the

thermal noise and the imaginary part of the transfer function. The imaginary part of the transfer function represents the energy dissipated in the system. The calculation of the dissipated energy is simpler than that of the imaginary part of the transfer function. Thus, Levin rewrote the fluctuation-dissipation theorem using the dissipated energy. The result is described as

$$G_X(f) = \frac{2k_B T}{\pi^2 f^2} \frac{W_{\text{loss}}}{F_0^2}, \quad (5.1)$$

where G_X is the power spectrum of the fluctuation of the observed coordinate, X , defined by Eq.(3.65), and f is the frequency. The value, W_{loss} , is the average dissipated power when the oscillatory force, $F_0 \cos(2\pi f t) \mathbf{P}(\mathbf{r})$, is applied on the system.

Levin applied this formula to the mirror of the interferometer. It is supposed that the dissipation can be described by the structure damping model. This dissipation is expressed using the complex Young's modulus as Eq.(4.18):

$$E = E_0[1 + i\phi(\mathbf{r})]. \quad (5.2)$$

The dissipated power, W_{loss} , is then written in the form

$$W_{\text{loss}} = 2\pi f \int \mathcal{E}(\mathbf{r}) \phi(\mathbf{r}) dV, \quad (5.3)$$

where $\mathcal{E}$ is the elastic energy density when the strain is at its maximum. When $\mathcal{E}$ was evaluated, Levin adopted the approximation that the static pressure, $F_0 \mathbf{P}(\mathbf{r})$, is applied on the mirror. This is possible because the observation band of the gravitational wave detectors is lower than the resonant frequencies of the mirror.

Levin calculated the thermal noise of the mirror with homogeneous loss and confirmed that the result agrees with the estimation derived from the traditional mode expansion. In addition, Levin suggested that the thermal noise of a cylindrical mirror in which the losses are concentrated on the surface illuminated by the laser beam is larger than the evaluation from the traditional mode expansion [68]. Bondu, Hello, and Vinet have evaluated the thermal noise of the mirror with homogeneous loss using Levin's approach [71]. The thermal noise of the mirror caused by the thermoelastic damping was calculated based on this method [40, 41].

5.2 Nakagawa's approach

Nakagawa has shown a method to estimate thermal noise using Green functions. The generalized equation of motion of the elastic body without loss is described as¹

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left(c_{ijkl} \frac{\partial u_l}{\partial x_k} \right) = F(t) P_i(\mathbf{r}), \quad (5.4)$$

where u_i and P_i are the i -th component of the displacement, $\mathbf{u}$, and the weighting function, $\mathbf{P}$, respectively. The parameter, ρ , is the density, and c_{ijkl} is the stiffness parameter. The dissipation term is introduced into Eq.(5.4). In the frequency domain, the result is written as

$$-\rho \omega^2 \tilde{u}_i - \frac{\partial}{\partial x_j} \left\{ [c'_{ijkl}(\omega) - i c''_{ijkl}(\omega, \mathbf{r})] \frac{\partial \tilde{u}_l}{\partial x_k} \right\} = \tilde{F} P_i(\mathbf{r}), \quad (5.5)$$

where c'_{ijkl} and c''_{ijkl} correspond respectively to the real and imaginary parts of c_{ijkl} in Eq.(5.4). The parameter, c''_{ijkl} , represents the dissipation. The solution of Eq.(5.5) is expressed as

$$\tilde{u}_i(\mathbf{r}) = \int \chi_{ij}(\omega, \mathbf{r}, \mathbf{r}_1) \tilde{F} P_j(\mathbf{r}_1) dV_1, \quad (5.6)$$

where $\chi_{ij}(\omega, \mathbf{r}, \mathbf{r}_1)$ is the Green function.

From the balance between the energy brought by the pressure, $\mathbf{P}$, and the dissipated energy, an important relation is derived. This relation is expressed as

$$\text{Im}[\chi_{ij}(\omega, \mathbf{r}_1, \mathbf{r}_2)] = - \int \left[\frac{\partial \chi_{li}(\omega, \mathbf{r}, \mathbf{r}_1)}{\partial x_k} \right] c''_{klmn}(\omega, \mathbf{r}) \left[\frac{\partial \chi_{nj}(\omega, \mathbf{r}, \mathbf{r}_2)}{\partial x_m} \right]^* dV. \quad (5.7)$$

The relation between the imaginary part of the Green function and the cross spectrum density between $u_i(\mathbf{r}_1)$ and $u_j(\mathbf{r}_2)$, $G_{u_i u_j}(\omega, \mathbf{r}_1, \mathbf{r}_2)$, is written in the form² [69]

$$G_{u_i u_j}(\omega, \mathbf{r}_1, \mathbf{r}_2) = -\frac{4k_B T}{\omega} \text{Im}[\chi_{ij}(\omega, \mathbf{r}_1, \mathbf{r}_2)]. \quad (5.8)$$

The power spectrum density of the observed coordinate, X , defined by Eq.(3.65) is described as

$$G_X(f) = \int P_i(\mathbf{r}_1) G_{u_i u_i}(\omega, \mathbf{r}_1, \mathbf{r}_2) P_i(\mathbf{r}_2) dV_1 dV_2. \quad (5.9)$$

¹This equation is valid also in an anisotropic elastic body.

²This formula corresponds to Eq.(14) in [69]. Although the sign of the right-hand of Eq.(14) in [69] is positive, the correct sign is negative. Moreover, in [69], the two-sided spectrum is used. Since the one-sided spectrum is adopted in this thesis, Eq.(5.8) is Eq.(14) in [69] multiplied by two.

The thermal fluctuation of X is derived from the Green functions, Eqs.(5.7), (5.8), and (5.9),

Nakagawa suggested that the Green function in the static state, $\chi_{ij}(0, \mathbf{r}_1, \mathbf{r}_2)$, is used when the thermal noise in the lower frequency range is estimated. In addition, the thermal noise of the one-dimensional elastic system with homogeneous loss in the lower frequency range was evaluated based on his approach.

5.3 Tsubono's approach

In Tsubono's approach, the transfer function, $H(\omega)$, defined as Eq.(3.7), is derived from the transfer matrix [70]. The transfer matrix [63, 72, 73] was developed to analyze the vibration of complex systems. In this method the elements of the system are replaced by matrices. The product of these matrices represents the total system. The transfer function is derived from this product. The thermal noise is calculated from the application of the fluctuation-dissipation theorem to the transfer function evaluated from the transfer matrix method. The advantage of the transfer matrix is simplicity. The calculation only involves multiplying the matrices. The matrices are derived from the elements through a simple process. Moreover, even in the high frequency range, the calculation of the transfer matrix is simple. In Levin's and Nakagawa's approaches, the calculation is difficult in the high frequency range. However, there is a demerit of the transfer matrix method. The system must be segmented as Fig.5.1 to use the transfer matrix and it is impossible to subdivide some systems. For example, continuous oscillators which have two or three dimensions are not segmentable as Fig.5.1. Thus, the transfer matrix is not a appropriate method to estimate the thermal noise of a disk or of a bulk.

The total system is divided as Fig.5.1. There are n elements in Fig.5.1. We consider the points at both the sides of each element. The $(i - 1)$ -th and i -th points are the left and right ends of the i -th element, respectively. Every point has the state vector, $\mathbf{z}_i$. The state vector are composed of the Fourier components of the displacement of the generalized coordinates of the point and of the generalized forces applied on the point itself. For example, in the system which consists of mass points and springs, the state

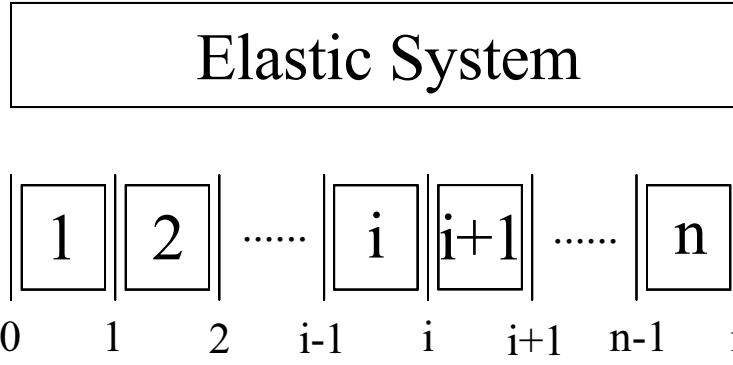


Figure 5.1: A segmented system. The total system is divided into n elements. There are the points at both the sides of the elements. The $(i - 1)$ -th and i -th points are the left and right ends of the i -th element, respectively. Every point has the state vector, $\mathbf{z}_i$.

vector is defined by

$$\mathbf{z}_i = \begin{pmatrix} X \\ F \end{pmatrix}_i, \quad (5.10)$$

where X and F are the Fourier components of the displacement of and the applied force on the i -th point, respectively. As another example, bends of beams are considered. The state vector is expressed as

$$\mathbf{z}_i = \begin{pmatrix} X \\ V \\ \Psi \\ M \end{pmatrix}_i, \quad (5.11)$$

where X, V, Ψ, M are the Fourier components of the displacement of the beam, the shear force, the rotation angle, and the moment of the i -th point, respectively. The definitions of these parameters are in Fig.5.2. In this thesis, the $(2m - 1)$ -th component of the state vector represents the generalized displacement. The $2m$ -th component is the generalized force which corresponds to the $(2m - 1)$ -th component. The sign of the displacement and force is defined as Fig.5.3.

The i -th state vector, $\mathbf{z}_i$, is connected to the $(i - 1)$ -th state vector, $\mathbf{z}_{i-1}$, by the transfer matrix, $\mathbf{T}_i$, of i -th element. This relation is defined by

$$\mathbf{z}_i = \mathbf{T}_i \mathbf{z}_{i-1}. \quad (5.12)$$

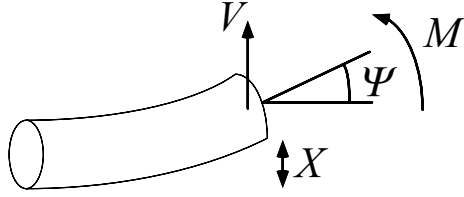


Figure 5.2: The definitions of X, V, Ψ, M . X, V, Ψ, M are the Fourier component of the displacement of the beam, the shear force, the rotation angle, and the moment of the i -th point, respectively.

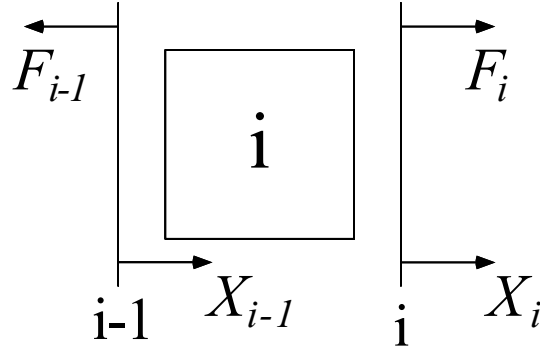


Figure 5.3: The definitions of the signs of the generalized displacement (X_i) and the generalized force (F_i). When the direction of the displacement and that of the force are the same as those of the arrows in this figure, the signs are positive.

The matrix, $\mathbf{T}_i$, is calculated from the equation of motion in the frequency domain. The effect of the dissipation is included. When the external force is not applied on the system, Eq.(5.12) can be rewritten as

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_m \end{pmatrix}_i = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1m} \\ t_{21} & t_{22} & \cdots & t_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mm} \end{pmatrix}_i \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_m \end{pmatrix}_{i-1}. \quad (5.13)$$

When the external force is applied on the system, Eq.(5.12) is expressed as

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_m \\ 1 \end{pmatrix}_i = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1m} & F_1 \\ t_{21} & t_{22} & \cdots & t_{2m} & F_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mm} & F_m \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_i \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_m \\ 1 \end{pmatrix}_{i-1}, \quad (5.14)$$

where F_i is a function of the external force.

From Eq.(5.12), the state vector at both ends of the system, $\mathbf{z}_0$ and $\mathbf{z}_n$, can be connected to each other. This relation is expressed as

$$\mathbf{z}_n = \mathbf{T}_n \cdot \mathbf{T}_{n-1} \cdots \mathbf{T}_2 \cdot \mathbf{T}_1 \mathbf{z}_0. \quad (5.15)$$

Thus, the matrix, $\mathbf{T}_{\text{total}}$, which corresponds to the total system is the product of the matrices of the elements, $\mathbf{T}_i$:

$$\mathbf{T}_{\text{total}} = \mathbf{T}_n \cdot \mathbf{T}_{n-1} \cdots \mathbf{T}_2 \cdot \mathbf{T}_1. \quad (5.16)$$

Components of the state vectors at the both ends, $\mathbf{z}_0$ and $\mathbf{z}_n$, are governed by the boundary conditions. When the end is fixed, the components which represent the displacement vanish. On the other hand, when the end is free, the components which correspond to the force are zero.

An outline of the calculation of the transfer function is introduced. The generalized force is applied on the observation point. In general, this applied force is treated as the external force. Thus, the transfer matrix of the total system has the same form as that of the matrix in Eq.(5.14)³. It is supposed that the left-hand end ($\mathbf{z}_0$) is free and that

³When the observation point is at an end of the system, the state vector at this end point includes the applied force as the boundary condition, i.e. this applied force is not treated as the external force. Thus, the transfer matrix of the total system is the same form as that of the matrix in Eq.(5.13).

the right-hand end ($\mathbf{z}_n$) is fixed. The relation between $\mathbf{z}_0$ and $\mathbf{z}_n$ is written in the form

$$\begin{pmatrix} 0 \\ \zeta_2 \\ \vdots \\ 0 \\ \zeta_{2m} \\ 1 \end{pmatrix}_n = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1\ 2m-1} & t_{1\ 2m} & F_1 \\ t_{21} & t_{22} & \cdots & t_{2\ 2m-1} & t_{2\ 2m} & F_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ t_{2m-1\ 1} & t_{2m-1\ 2} & \cdots & t_{2m-1\ 2m-1} & t_{2m-1\ 2m} & F_{2m-1} \\ t_{2m\ 1} & t_{2m\ 2} & \cdots & t_{2m\ 2m-1} & t_{2m\ 2m} & F_{2m} \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ 0 \\ \vdots \\ \zeta_{2m-1} \\ 0 \\ 1 \end{pmatrix}_0. \quad (5.17)$$

Eq.(5.17) is rewritten as

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_n = \begin{pmatrix} t_{11} & \cdots & t_{1\ 2m-1} & F_1 \\ \vdots & \ddots & \vdots & \vdots \\ t_{2m-1\ 1} & \cdots & t_{2m-1\ 2m-1} & F_{2m-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{2m-1} \\ 1 \end{pmatrix}_0. \quad (5.18)$$

Eq.(5.18) is simplified as

$$\begin{pmatrix} t_{11} & \cdots & t_{1\ 2m-1} \\ \vdots & \ddots & \vdots \\ t_{2m-1\ 1} & \cdots & t_{2m-1\ 2m-1} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{2m-1} \end{pmatrix}_0 = - \begin{pmatrix} F_1 \\ \vdots \\ F_{2m-1} \end{pmatrix}. \quad (5.19)$$

From Eq.(5.19), $\mathbf{z}_0$ is evaluated as a function of the applied force. The transfer matrix which connects $\mathbf{z}_0$ to the state vector at the observation point is estimated from the multiplication of the matrices of the elements. Therefore, the displacement of the observation point is calculated as a function of the generalized force. Substituting the transfer function obtained from these discussion for $H(\omega)$ in Eq.(3.9), the spectrum of the thermal noise is obtained.

Tsubono has investigated the thermal noise of a one-dimensional system. When the loss was homogeneous, the estimation from the transfer matrix was consistent with the results of the traditional mode expansion. On the other hand, the results prove that the traditional mode expansion fails when the loss is not uniform.

5.4 Consistency of three direct approaches

The consistency of the three direct approaches is investigated here. The thermal noise of a one-dimensional elastic system shown in Fig.5.4 was estimated using the three

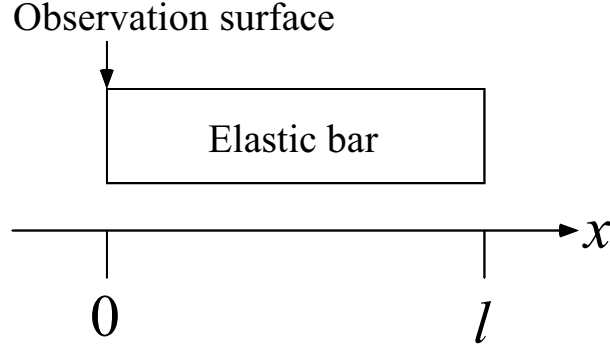


Figure 5.4: A one-dimensional elastic system. The length, cross section, Young's modulus, and density are l , A , E_0 , and ρ , respectively. This bar is free. The coordinate (x) at the left-hand and right-hand sides are zero and l , respectively. The thermal longitudinal vibration at the left-hand side ($x = 0$) are estimated based on three direct approaches.

methods and the results were compared.

The length and cross section of the elastic bar are l and A , respectively. The bar is free. The coordinate (x) at the left-hand and right-hand sides are zero and l , respectively. The thermal longitudinal vibration at the left-hand side ($x = 0$) is estimated with the three direct approaches. The frequency range of interest is lower than the first longitudinal resonant frequency. This is a simple model of the mirror in the interferometric gravitational wave detectors. The equation of the motion of this elastic bar without the dissipation is described as [48]

$$\rho \frac{\partial^2 u}{\partial t^2} = E_0 \frac{\partial^2 u}{\partial x^2}, \quad (5.20)$$

where u is the longitudinal displacement, ρ is the density, and E_0 is the Young's modulus. Since the thermal noise at the left-hand side ($x = 0$) is calculated, the generalized force, F , is applied on the left-hand end. The first boundary condition is expressed as

$$F = -E_0 A \frac{\partial u}{\partial x} \Big|_{x=0}. \quad (5.21)$$

The other end is free. The second boundary condition is written as

$$0 = -E_0 A \frac{\partial u}{\partial x} \Big|_{x=l}. \quad (5.22)$$

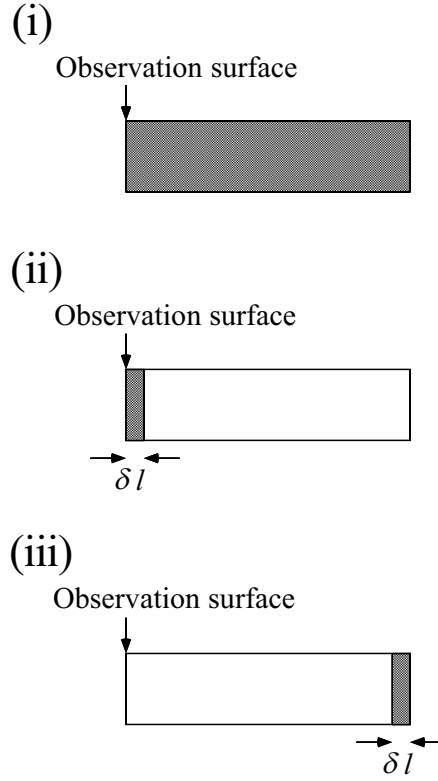


Figure 5.5: The distribution of the loss. The three types of the distribution is considered. The gray parts show the volume in which the dissipation is concentrated. The parameter, δl , is the thickness of the loss layer. The model of the loss is the structure damping model described using the complex Young's modulus, $E_0[1 + i\phi(x)]$.

It is assumed that the dissipation is expressed by the structure damping model. This model is described by the complex Young's modulus as Eq.(5.2):

$$E = E_0[1 + i\phi(x)]. \quad (5.23)$$

Three types of loss distributions are considered in Fig.5.5 . The gray parts in Fig.5.5 represents the volume in which the dissipation is concentrated. The first type is the homogeneous loss. The loss angle, $\phi_1(x)$, is written in the form

$$\phi_1(x) = \phi. \quad (5.24)$$

In the second type, the loss is concentrated on the observation surface. The loss angle,

$\phi_2(x)$, is described as

$$\phi_2(x) = \begin{cases} \phi & (0 \leq x \leq \delta l) \\ 0 & (\delta l \leq x \leq l) \end{cases}, \quad (5.25)$$

where δl is the thickness of the loss layer. The value, δl , is much smaller than l . The loss layer in the third type is at the opposite side. The loss angle, $\phi_3(x)$, is expressed as

$$\phi_3(x) = \begin{cases} 0 & (0 \leq x \leq l - \delta l) \\ \phi & (l - \delta l \leq x \leq l) \end{cases}. \quad (5.26)$$

5.4.1 Levin's approach

In Levin's approach, the thermal noise is derived using Eqs.(5.1) and (5.3). Since $\phi(x)$ is known, the problem is the calculation of the elastic energy density, $\mathcal{E}(x)$. When the oscillatory force, $F_0 \cos(2\pi ft)$, is applied on the edge of the bar, the maximum of the elastic energy density is expressed as

$$\mathcal{E}(x) = \frac{1}{2} E_0 \left(\frac{\partial u_0}{\partial x} \right)^2, \quad (5.27)$$

where u_0 is the amplitude of u . From Eqs.(5.20), (5.21), and (5.22), u_0 is written as

$$u_0 = -\frac{F_0 v}{E_0 A \sin(\omega l/v) \omega} \cos \left[\frac{\omega(x-l)}{v} \right], \quad (5.28)$$

where v is the velocity of the longitudinal wave written in the form

$$v = \sqrt{\frac{E_0}{\rho}}. \quad (5.29)$$

Putting Eq.(5.28) into Eq.(5.27), the elastic energy density is obtained:

$$\mathcal{E}(x) = \frac{F_0^2}{2E_0 A^2} \frac{\sin^2 \left[\frac{\omega(x-l)}{v} \right]}{\sin^2 \left(\frac{\omega l}{v} \right)}. \quad (5.30)$$

Since the thermal noise in the lower frequency range is evaluated, $\omega l/v$ is much smaller than unity. Thus, Eq.(5.30) is rewritten as

$$\mathcal{E}(x) \approx \frac{F_0^2}{2E_0 A^2} \left(\frac{x-l}{l} \right)^2. \quad (5.31)$$

From Eqs.(5.1), (5.3), (5.31), and $\phi(x)$, the thermal noise is evaluated.

5.4.2 Nakagawa's approach

From Eqs.(5.7), (5.8), and (5.9), the thermal longitudinal vibration of a one-dimensional elastic system is described as

$$G(x, f) = \frac{4k_B T}{\omega} \int \left[\frac{\partial \chi_{11}(x, x_1)}{\partial x} \right] c''_{1111}(\omega, x_1) \left[\frac{\partial \chi_{11}(x, x_1)}{\partial x} \right]^* dx_1, \quad (5.32)$$

where f is the frequency corresponds to the angular frequency, ω . Here, c''_{1111} is equal to the imaginary part of the complex Young's modulus,

$$c''_{1111}(\omega, x) = E_0 \phi(\omega, x). \quad (5.33)$$

The Green function, χ_{11} , is expressed as⁴ [69]

$$\chi_{11}(\omega, x_1, x_2) = - \frac{\cos \left[\frac{\omega(l-|x_1-x_2|)}{v} \right] + \cos \left[\frac{\omega(l-x_1-x_2)}{v} \right]}{2\rho\sqrt{A}v\omega \sin \left(\frac{\omega l}{v} \right)}. \quad (5.34)$$

Since the thermal noise in the lower frequency range is investigated, $\omega l/v$ is much smaller than unity. Thus, Eq.(5.34) is rewritten as

$$\chi_{11}(\omega, x_1, x_2) = - \frac{2 - \frac{1}{2} \left[\frac{\omega(l-|x_1-x_2|)}{v} \right]^2 - \frac{1}{2} \left[\frac{\omega(l-x_1-x_2)}{v} \right]^2}{2\rho\sqrt{A}\omega^2 l}. \quad (5.35)$$

From Eqs.(5.32), (5.33), (5.35), and the loss angle, $\phi(x)$, the thermal noise at the end of bar, $G(0, f)$, is derived.

5.4.3 Tsubono's approach

The generalized force is applied on the end of the bar. Since the boundary condition includes the effect of this force, this applied force is not treated as an external force. Thus, the transfer matrix is the same type as that of the matrix in Eq.(5.13). The basis of the calculation is the transfer matrix of the elastic bar with the homogeneous structure damping defined by [72, 73, 63]

$$\mathbf{T}(l, \phi) = \begin{pmatrix} \cos kl & \frac{\sin kl}{kE_0(1+i\phi)A} \\ -kE_0(1+i\phi)A \sin kl & \cos kl \end{pmatrix}, \quad (5.36)$$

⁴ $\bar{\rho}$ in [69] is $\rho\sqrt{A}$.

where l is the length of the bar, and ϕ is the loss angle of the complex Young's modulus, Eq.(5.23). The complex wave number, k , is described as

$$k^2 = \frac{\rho\omega^2}{E_0(1 + i\phi)}. \quad (5.37)$$

The overall transfer matrix of the first type of the distribution, Eq.(5.24), is described as

$$\mathbf{T}_{\text{total}} = \mathbf{T}(l, \phi). \quad (5.38)$$

The overall transfer matrix of the second type, Eq.(5.25), is defined by

$$\mathbf{T}_{\text{total}} = \mathbf{T}(\delta l, \phi)\mathbf{T}(l - \delta l, 0). \quad (5.39)$$

The overall transfer matrix of the third type, Eq.(5.26), is written as

$$\mathbf{T}_{\text{total}} = \mathbf{T}(l - \delta l, 0)\mathbf{T}(\delta l, \phi). \quad (5.40)$$

An end in this case is the observation point. The other end is free. The state vectors of both the ends are connected with each other by the overall transfer matrix, $\mathbf{T}_{\text{total}}$. This relation is expressed as

$$\begin{pmatrix} X \\ F \end{pmatrix}_{\text{obs}} = \mathbf{T}_{\text{total}} \begin{pmatrix} X \\ 0 \end{pmatrix}_{\text{free}}, \quad (5.41)$$

where obs and free represent the observation point and the free end, respectively. The parameters, X and F , are the Fourier components of the displacement of the end points and the applied force, respectively. The transfer matrix, $\mathbf{T}_{\text{total}}$, is written as

$$\mathbf{T}_{\text{total}} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \quad (5.42)$$

From Eq.(5.42), the transfer function, $H(\omega)$, is derived as

$$H(\omega) = \left. \frac{X}{F} \right|_{\text{obs}} = \frac{t_{11}}{t_{21}}. \quad (5.43)$$

The thermal noise spectrum is calculated from the application of the fluctuation-dissipation theorem, Eq.(3.9), to this transfer function, Eq.(5.43). Since the thermal noise of interest is in the lower frequency range, only the first order of $\omega l/v$ is considered.

5.4.4 Results

The thermal noise of the elastic bar shown in Fig.5.4 is calculated using the three direct approaches. These results agree with each other. Therefore, the consistency of the three direct approach is proved.

The results derived from the three direct approaches are summarized. The thermal noise, G_1 , caused by the loss of the first type, Eq.(5.24), is obtained as

$$G_1(f) = \frac{2k_B T l \phi}{3\pi E_0 A} \frac{1}{f}. \quad (5.44)$$

The thermal motion, G_2 , of the second type, Eq.(5.25), is derived as

$$G_2(f) = \frac{2k_B T (\delta l) \phi}{\pi E_0 A} \frac{1}{f}. \quad (5.45)$$

The thermal noise, G_3 , of the third distribution, Eq.(5.26), is expressed as

$$G_3(f) = \frac{2k_B T (\delta l)^3 \phi}{3\pi E_0 A l^2} \frac{1}{f}. \quad (5.46)$$

These results give useful hints about the general properties of thermal noise. The calculations of the second and third distributions in Fig.5.5 from the traditional mode expansion are the same because the Q-values of both the cases are the same. However, all direct approaches predict that G_3 is $(\delta l/l)^2/3$ times smaller than G_2 . This result shows that the traditional mode expansion is not valid when the loss is not homogeneous. Moreover, these results suggest that the thermal noise of the mirror is larger when the loss is concentrated near the observation point. This conclusion agrees with Levin's considerations [68]. The details of the thermal noise of the mirror with inhomogeneous losses are discussed in Chapter 7.

5.5 Consistency with advanced mode expansion

The consistency between the direct approach and the advanced mode expansion is considered. The estimation of the direct approaches was compared with that of the advanced mode expansion. The experiments of the thermal noise of the leaf spring and a prototype mirror with inhomogeneous eddy current damping are described in Chapter 6 and Chapter 8, respectively. The thermal noise of these oscillators was evaluated from the

direct approach (Tsubono's approach) and from the advanced mode expansion. These calculated results of the leaf spring and the prototype of the mirror are shown in Figs.6.7 and 8.8, respectively. These figures prove that the estimation of the direct approach is at least in these cases consistent with that of the advanced mode expansion.

Chapter 6

Experimental test of the estimation

The discussions in Chapter 4 prove theoretically that the traditional mode expansion fails when the loss is distributed inhomogeneously. On the contrary, it is expected that new methods, the advanced mode expansion in Chapter 4 and the direct approaches in Chapter 5, are valid even when the dissipation is not uniform. However, there have been few experimental test of the evaluation methods of the thermal noise caused by the inhomogeneous loss. In order to check the failure of the traditional mode expansion and the validity of the new methods, the thermal noise of a simple oscillator with inhomogeneous loss was measured. The results of this experiment showed that the new methods are correct and that the traditional mode expansion fails. This is the first experimental evidence of the failure of the traditional mode expansion. The details of this experiment [74, 75] are described in this chapter.

6.1 Outline of the experiments

To test the estimation methods, the thermal motion of a metal leaf spring with inhomogeneous loss was measured. This loss was intentionally introduced using eddy current generated by strong permanent magnets. To confirm that the measured motion was the thermal fluctuation, the thermal noise was evaluated from the measured mechanical response of the leaf spring using the fluctuation-dissipation theorem.

An advantage of the leaf spring geometry was that there is a large difference between

the noise estimation from the traditional mode expansion and new methods because of the large loss inhomogeneity. The dissipation induced by the eddy current was localized near the magnet and much larger than the original loss of the leaf spring. The resultant distribution of loss was highly inhomogeneous. Another advantage of this setup was that the thermal noise could be evaluated without ambiguity because the properties and distribution of the loss caused by the eddy current are well known. Moreover, it was possible to measure the thermal noise and the imaginary part of the transfer function because this oscillator was light and soft and had a large source of the dissipation.

6.2 Leaf spring

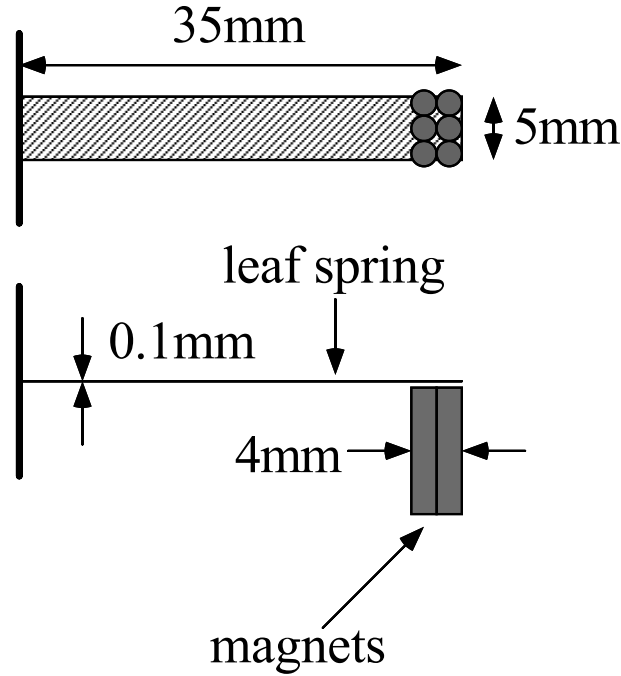


Figure 6.1: Dimensions and configuration of the leaf spring and the magnets.

The leaf spring was a plate made of aluminum alloy (Al97/Mg3: GOODFELLOW CAMBRIDGE Ltd.) with a size of 35 mm $\times$ 5 mm $\times$ 0.1 mm, as shown in Fig.6.1. In order to realize the desired inhomogeneous loss, neodymium permanent magnets were set near the surface of the plate, as shown in Fig.6.1; they produced an eddy current in a limited

area of the plate. Six magnets were used; each one was 2 mm in diameter and 10 mm in length. The gap between the leaf spring and the magnets was about 0.5 mm. The power spectrum of the thermal noise of the leaf spring between the first mode ($\omega_1/2\pi=60$ Hz) and the second mode ($\omega_2/2\pi=360$ Hz) was measured.

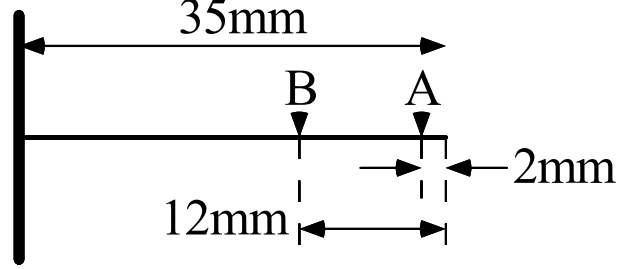


Figure 6.2: Positions of points A and B. These are the damped and observation positions.

The thermal fluctuation caused by inhomogeneous losses depends on both the positions of the damped area and the observation point. In order to investigate this dependence, the two points, A and B, shown in Fig.6.2 were selected as the center of the damped area and the observation point. Thus, there were four configurations, which are named A_dA_o , A_dB_o , B_dA_o , $B dB_o$. These configurations are shown in Fig.6.3. For example, in A_dB_o , the center of the damped area is A and the observation point is B.

6.3 Estimation of thermal noise of the leaf spring

In order to compare with the measured values, the spectrum of the thermal noise was evaluated from the traditional and from advanced mode expansion. The direct approach was also used to evaluate the thermal motion. The results showed large discrepancy between the estimation from the traditional and advanced mode expansion. This implies that this leaf spring was appropriate for this experiment to check the estimation method of the thermal noise. Moreover, it was confirmed the calculated results from the advanced mode expansion are the same as that of the direct approach. The reason why the large difference between the estimation of the advanced and traditional mode expansion exists was also considered.

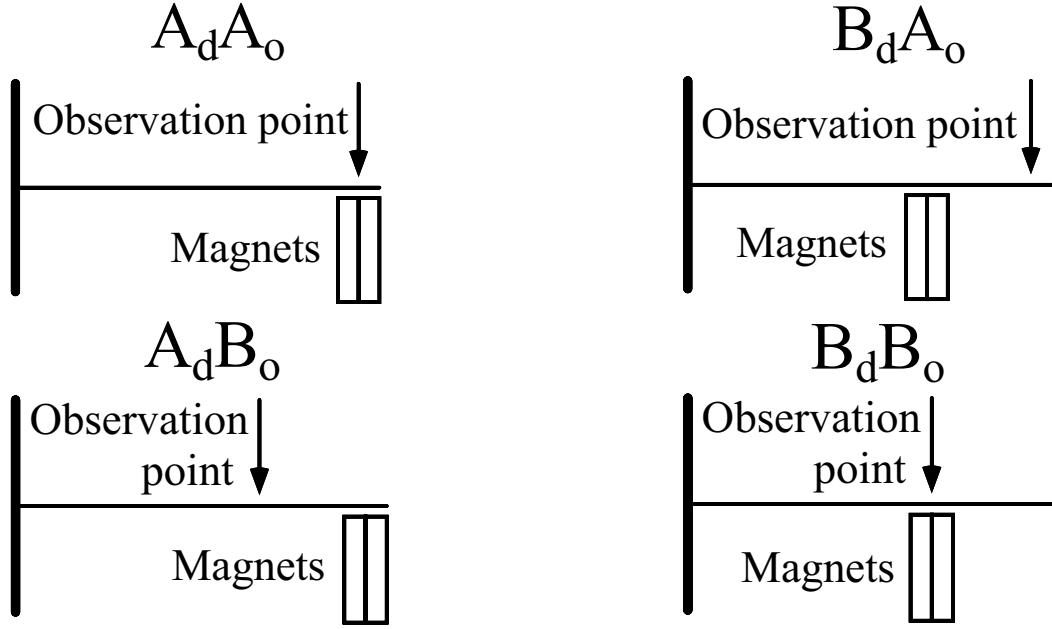


Figure 6.3: The configurations of the magnets. They are labelled $A_d A_o$, $A_d B_o$, $B_d A_o$, $B_d B_o$. For example, in $A_d B_o$, the center of the damped area is A and the observation point is B.

6.3.1 Estimation

The details of the evaluation of the traditional and advanced mode expansion and direct approach are introduced. Moreover, the results of these estimation methods are summarized. Since the dissipation caused by the eddy current is much larger than the original loss of the leaf spring, the original loss is neglected.

Estimation of traditional mode expansion

In the traditional mode expansion, the angular resonant frequency, ω_n , the effective mass, m_n , and the loss angle, ϕ_n , are evaluated. Inserting these values into Eq.(3.77), the spectrum of the thermal noise is obtained.

From Eq.(3.65), the observed coordinate, X , is defined as

$$X = \int u(x, t) P(x) dx, \quad (6.1)$$

where u is the transverse displacement of the leaf spring and P is its weighting function. Since the displacement of a point is observed, the weighting function, P , is expressed as

$$P(x) = \delta(x - x_o), \quad (6.2)$$

where $\delta(x)$ is the δ -function and x_o is the coordinate of the observation point.

The displacement, w_n , of the n -th mode of the leaf spring is the solution of the eigenvalue problem expressed as [48]

$$-\frac{h^2 E}{12(1 - \sigma^2)} \frac{\partial^4 w_n}{\partial x^4} = -\rho \omega_n^2 w_n, \quad (6.3)$$

where h, E, σ, ρ are thickness, Young's modulus, Poisson ratio, and density of the leaf spring respectively. Figure 6.1 shows that an end ($x = 0$) of the spring is fixed and that the other end ($x = l$) is free. The boundary conditions are written in the form [76]

$$w_n(0) = 0, \quad (6.4)$$

$$\left. \frac{dw_n}{dx} \right|_{x=0} = 0, \quad (6.5)$$

$$\left. \frac{d^2 w_n}{dx^2} \right|_{x=l} = 0, \quad (6.6)$$

$$\left. \frac{d^3 w_n}{dx^3} \right|_{x=l} = 0. \quad (6.7)$$

From Eqs.(6.3), (6.4), (6.5), (6.6), and (6.7), the n -th angular resonant frequency, ω_n , is given:

$$\omega_n = \frac{\alpha_n^2}{l^2} \sqrt{\frac{E h^2}{12 \rho (1 - \sigma^2)}}, \quad (6.8)$$

where α_n is the n -th solution of the equation expressed as

$$\cosh \alpha_n \cdot \cos \alpha_n + 1 = 0. \quad (6.9)$$

The values of α_n ($n = 1, 2, 3, 4$) are summarized in Table.6.1. The displacement of the n -th mode, w_n , is written in the form

$$w_n(x) = \cosh \left(\alpha_n \frac{x}{l} \right) - \cos \left(\alpha_n \frac{x}{l} \right) - \Phi_n \left[\sinh \left(\alpha_n \frac{x}{l} \right) - \sin \left(\alpha_n \frac{x}{l} \right) \right], \quad (6.10)$$

where Φ_n is a function of α_n defined by

$$\Phi_n = \frac{\sinh \alpha_n - \sin \alpha_n}{\cosh \alpha_n + \cos \alpha_n}. \quad (6.11)$$

The shapes of the modes are shown in Fig.6.4.

Table 6.1: α_n of leaf springs.

n	α_n
1	1.87510
2	4.69409
3	7.85476
4	10.99554

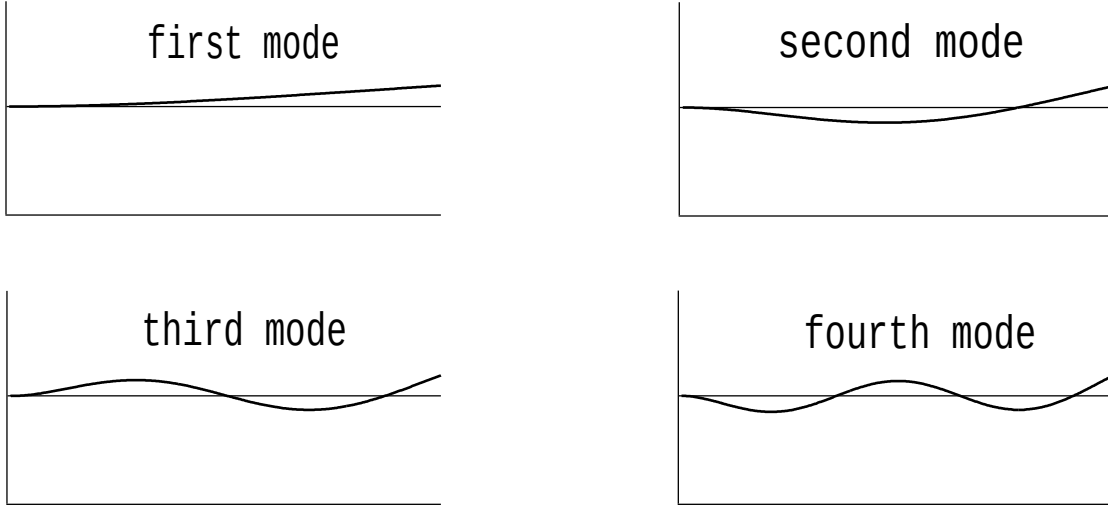


Figure 6.4: The shapes of modes of the leaf spring.

Inserting Eqs.(6.2) and (6.10) into Eq.(3.78), the effective masses are obtained. The formula, Eq.(3.78), is rewritten as

$$m_n = \frac{1}{[w_n(x_o)]^2} \int_0^l \rho A [w_n(x)]^2 dx, \quad (6.12)$$

where A is the cross section of the leaf spring.

In this case the loss of the leaf spring is dominated by the dissipation caused by the eddy current. The loss angle, ϕ_n , is described as

$$\phi_n(\omega) = \frac{\omega}{\omega_n Q_n}. \quad (6.13)$$

The Q-value, Q_n , of the n -th mode is evaluated from Eqs.(6.14) and (4.7).

Estimation of advanced mode expansion

In the advanced mode expansion, the coupling coefficient, α_{nk} , is evaluated. Inserting the coupling coefficients and the parameters in the traditional mode expansion, m_n, ω_n , and ϕ_n , into Eq.(4.55), the expression of the thermal noise is obtained.

The loss is dominated by the dissipation caused by the eddy current. Thus, the coupling coefficient, α_{nk} , is derived from Eq.(4.5). The basis function, w_n in Eq.(4.5), satisfies the normalization condition, Eq.(3.69). Since $w_n(x)$ in Eq.(6.10) is not normalized, Eq.(4.5) must be rewritten. From Eq.(6.2), the rewritten formula is expressed as

$$\alpha_{nk} = \frac{\omega}{w_n(x_0)w_k(x_0)} \int \rho \Gamma(x) w_n(x) w_k(x) dx. \quad (6.14)$$

In order to simplify the consideration, it is supposed that only the part of the leaf spring which faces the magnet is damped and that the strength of the dissipation is uniform in this damped region. Thus, $\Gamma(x)$ in $A_d A_o$ and $A_d B_o$ is written as

$$\Gamma(x) = \begin{cases} \Gamma & (31l/35 \leq x \leq l) \\ 0 & (0 \leq x \leq 31l/35) \end{cases}. \quad (6.15)$$

On the other hand, $\Gamma(x)$ in $B_d A_o$ and $B_d B_o$ is defined by

$$\Gamma(x) = \begin{cases} \Gamma & (10l/35 \leq x \leq 14l/35) \\ 0 & (0 \leq x \leq 10l/35, 14l/35 \leq x \leq l) \end{cases}. \quad (6.16)$$

Estimation of direct approach

The thermal noise of the leaf spring was calculated using Tsubono's approach. Since the observation point is not at the free end, the applied force at the observation point is treated as the external force. Thus, the transfer matrix includes the effect of this applied force, i.e. the transfer matrix is the same type as that of the matrix in Eq.(5.14).

The foundation of the calculation is the transfer matrix of the element shown in Fig.6.5. This element corresponds to a segment of the leaf spring with loss. The dissipation is the homogeneous eddy current damping. The transfer matrix of this system is $\mathbf{T}(l', \Gamma, l'', F)$. The length is l' . The force, F , is applied at the point which is l'' away from the right end (1). The value, Γ , is the strength of the eddy current damping in Eqs.(6.15) and (6.16).

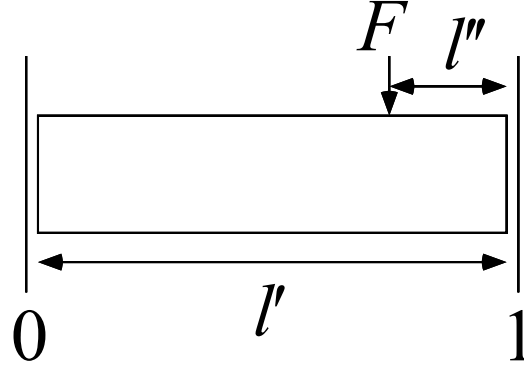


Figure 6.5: The part of the leaf spring with loss. The dissipation is the homogeneous eddy current damping. The transfer matrix of this system is $\mathbf{T}(l', \Gamma, l'', F)$. The length is l' . The force, F , is applied at the point which is l'' away from the right end (1). The value, Γ , is the strength of the eddy current damping in Eqs.(6.15) and (6.16).

The transfer matrix, $\mathbf{T}(l', \Gamma, l'', F)$, is described as [72, 73, 63]

$$\begin{pmatrix} X \\ V \\ \Psi \\ M \\ 1 \end{pmatrix}_1 = \frac{1}{2} \begin{pmatrix} C_+(\alpha l') & \frac{S_-(\alpha l')}{EI\alpha^3} & -\frac{S_+(\alpha l')}{\alpha} & \frac{C_-(\alpha l')}{EI\alpha^2} & -\frac{S_-(\alpha l'')F}{EI\alpha^3} \\ -\frac{S_+(\alpha l')EI}{\alpha^3} & C_+(\alpha l') & -\frac{C_-(\alpha l')EI}{\alpha^2} & -\frac{S_-(\alpha l')}{\alpha} & -C_+(\alpha l'')F \\ S_-(\alpha l')\alpha & -\frac{C_-(\alpha l')}{EI\alpha^2} & C_+(\alpha l') & \frac{S_+(\alpha l')}{EI\alpha} & \frac{C_-(\alpha l'')F}{EI\alpha^2} \\ \frac{C_-(\alpha l')EI}{\alpha^2} & \frac{S_+(\alpha l')}{\alpha} & -S_-(\alpha l')EI\alpha & C_+(\alpha l') & -\frac{S_+(\alpha l'')F}{\alpha} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ V \\ \Psi \\ M \\ 1 \end{pmatrix}_0. \quad (6.17)$$

The definition of the parameters, X , V , Ψ , and M , are the same as those in Fig.5.2. The functions, C_+ , C_- , S_+ , and S_- , are defined as

$$C_+(x) = \cos x + \cosh x, \quad (6.18)$$

$$C_-(x) = \cos x - \cosh x, \quad (6.19)$$

$$S_+(x) = \sin x + \sinh x, \quad (6.20)$$

$$S_-(x) = \sin x - \sinh x. \quad (6.21)$$

The parameter, α , is expressed as¹

$$\alpha = \left[\frac{12\rho(1 - \sigma^2)}{h^2 E} \omega^2 - i\rho\Gamma\omega \right]^{1/4}. \quad (6.22)$$

The value, I , is the moment of inertia of the cross section of the leaf spring. Since the cross section of the leaf spring is a rectangle, I is defined as [48]

$$I = \frac{wh^3}{12}, \quad (6.23)$$

where w and h are the width and thickness of the leaf spring, respectively.

The transfer matrix, $\mathbf{T}_{\text{total}}$, of the entire system is derived from Eqs.(6.15), (6.16), and (6.17). The matrix of the total spring is written in the form

$$\mathbf{T}_{\text{total}} = \begin{cases} \mathbf{T}(4l/35, \Gamma, 2l/35, F)\mathbf{T}(31l/35, 0, 0, 0) & (\text{in } A_d A_o) \\ \mathbf{T}(10l/35, 0, 2l/35, F)\mathbf{T}(4l/35, \Gamma, 0, 0)\mathbf{T}(21l/35, 0, 0, 0) & (\text{in } B_d A_o) \\ \mathbf{T}(4l/35, \Gamma, 0, 0)\mathbf{T}(31l/35, 0, 8l/35, F) & (\text{in } A_d B_o) \\ \mathbf{T}(10l/35, 0, 0, 0)\mathbf{T}(4l/35, \Gamma, 2l/35, F)\mathbf{T}(21l/35, 0, 0, 0) & (\text{in } B_d B_o) \end{cases}. \quad (6.24)$$

An end of the leaf spring is fixed, the other end is free. The state vectors of both ends are connected with each other by the overall transfer matrix, $\mathbf{T}_{\text{total}}$. This relation is expressed as

$$\begin{pmatrix} X \\ 0 \\ \Psi \\ 0 \\ 1 \end{pmatrix}_{\text{free}} = \mathbf{T}_{\text{total}} \begin{pmatrix} 0 \\ V \\ 0 \\ M \\ 1 \end{pmatrix}_{\text{fixed}}, \quad (6.25)$$

where free and fixed correspond to the free and fixed end, respectively. This matrix, $\mathbf{T}_{\text{total}}$, is expressed as

$$\mathbf{T}_{\text{total}} = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} & F_1 \\ t_{21} & t_{22} & t_{23} & t_{24} & F_2 \\ t_{31} & t_{32} & t_{33} & t_{34} & F_3 \\ t_{41} & t_{42} & t_{43} & t_{44} & F_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (6.26)$$

¹If the original loss of the leaf spring is considered, the complex Young's modulus, $E = E_0(1 + i\phi)$, is substituted for Young's modulus in Eqs.(6.17) and (6.22).

From Eq.(6.26), the formula described as

$$\begin{pmatrix} V \\ M \end{pmatrix}_{\text{fixed}} = \frac{1}{t_{22}t_{44} - t_{24}t_{42}} \begin{pmatrix} -t_{44}F_2 + t_{24}F_4 \\ t_{42}F_1 - t_{22}F_2 \end{pmatrix}. \quad (6.27)$$

is obtained.

The displacement of the observation point is calculated from the state vector at the fixed end, Eq.(6.27), and the transfer matrix, $\mathbf{T}'$, from the fixed end to the observation point;

$$\begin{pmatrix} X \\ V \\ \Psi \\ M \\ 1 \end{pmatrix}_{\text{obs}} = \mathbf{T}' \begin{pmatrix} 0 \\ V \\ 0 \\ M \\ 1 \end{pmatrix}_{\text{fixed}}, \quad (6.28)$$

where obs represents the observation point. This transfer matrix, $\mathbf{T}'$, is defined by

$$\mathbf{T}' = \begin{cases} \mathbf{T}(2l/35, \Gamma, 0, F)\mathbf{T}(31l/35, 0, 0, 0) & (\text{in } A_d A_o) \\ \mathbf{T}(8l/35, 0, 0, F)\mathbf{T}(4l/35, \Gamma, 0, 0)\mathbf{T}(21l/35, 0, 0, 0) & (\text{in } B_d A_o) \\ \mathbf{T}(23l/35, 0, 0, F) & (\text{in } A_d B_o) \\ \mathbf{T}(2l/35, \Gamma, 0, F)\mathbf{T}(21l/35, 0, 0, 0) & (\text{in } B_d B_o) \end{cases}. \quad (6.29)$$

The displacement at the observation point is derived from the above formulae as the function of the applied force, F . The transfer function is evaluated from this result. The thermal noise spectrum is calculated from the application of the fluctuation-dissipation theorem, Eq.(3.9), to this transfer function.

Results of the estimation

From above consideration, the thermal noise was evaluated using each method. The parameters of the leaf spring and the strength of the eddy current damping, Γ , are summarized in Table.6.2. The observation points are shown in Fig.6.2. The distribution of the eddy current damping is described as Eqs.(6.15) and (6.16). The thermal noise is estimated from these data.

The parameters in the mode expansion, resonant frequency (f_n), effective mass (m_n), Q-value (Q_n), and coupling coefficient (α_{nk}), were evaluated. In this experiment, the

Table 6.2: Parameters of leaf spring and damping.

parameter	
length (l)	35 mm
width (w)	5 mm
thickness (h)	0.1 mm
density (ρ)	2.67 g/cm ³
Young's modulus (E)	7.03×10^{10} Pa
Poisson ratio (σ)	0.345
Strength of damping (Γ)	4.98×10^2 /sec

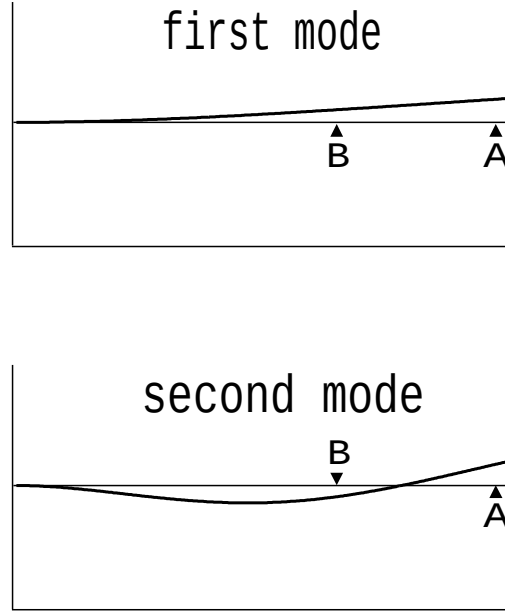


Figure 6.6: Displacement of the first mode (w_1) and the second mode (w_2). The positions of A and B are the same as those in Fig.6.2.

thermal motion was measured in the frequency range between the first and second modes. Since the thermal noise in this frequency range is dominated by that of the first and second modes, it was sufficient to know m_n, ω_n , and $Q_n (n = 1, 2)$ for estimation based

on the traditional mode expansion. In the advanced mode expansion, only α_{12} was calculated. The shape of the first and second modes and the points A and B are shown in Fig.6.6. The resonant frequencies, the effective masses, and Q-values of the leaf spring are shown in Tables.6.3, 6.4, and 6.5, respectively. The coupling coefficient is listed in Table.6.6. This value, α_{12} , was normalized by the maximum of the absolute value, $|\alpha_{12}|_{\max} = \sqrt{m_1\omega_1^2\phi_1m_2\omega_2^2\phi_2}$, in Eq.(4.40).

Table 6.3: Resonant frequencies of the leaf spring.

n	f_n
1	72.1 Hz
2	452 Hz

Table 6.4: Effective masses of the leaf spring.

Observation point	m_1	m_2
A	14 mg	22 mg
B	41 mg	58 mg

Table 6.5: Q-values of damped leaf spring.

Center of damped area	Q_1	Q_2
A	2.3	23
B	6.9	61

Table 6.6: Coupling coefficient of damped leaf spring.

	$\alpha_{12}/ \alpha_{12} _{\max}$
$A_d A_o$	0.99
$B_d A_o$	-0.96
$A_d B_o$	-0.99
$B_d B_o$	0.96

The results of the estimation of the thermal noise are shown in Fig.6.7. The positions of the magnets and the observation point are indicated above each graph. The thick and thin dotted lines are the computations of the advanced and traditional mode expansions, respectively. The solid line is the evaluation of the direct approach. Figure 6.7 proves that the spectra estimated using the advanced mode expansion were different from those evaluated from the traditional mode expansion in all cases. Moreover, the evaluation of the advanced mode expansion strongly depended on the observation point and the distribution of the loss although the estimation of the traditional mode expansion was not largely affected by the selection of the damped area and the observation point. These systems were appropriate for this experiment to test the estimation methods of the thermal noise because the difference between the results derived from the advanced and traditional mode expansion was large.

The estimation of the advanced mode expansion agreed with that of the direct approach. Thus, Fig.6.7 is an example of the consistency between the advanced mode expansion and the direct approach.

6.3.2 Physical interpretation

The difference between the estimation of the advanced and traditional mode expansion is discussed here [74, 75]. The evaluation of the advanced mode expansion is different from that of the traditional mode expansion in Fig.6.7. This discrepancy is due to the large coupling between the first and second modes. Table.6.6 shows that the absolute value of the coupling coefficient, α_{12} , is almost the maximum. This is because the dissipation is localized in a narrow region.

Figure 6.7 shows that the estimation of the traditional mode expansion in the four cases

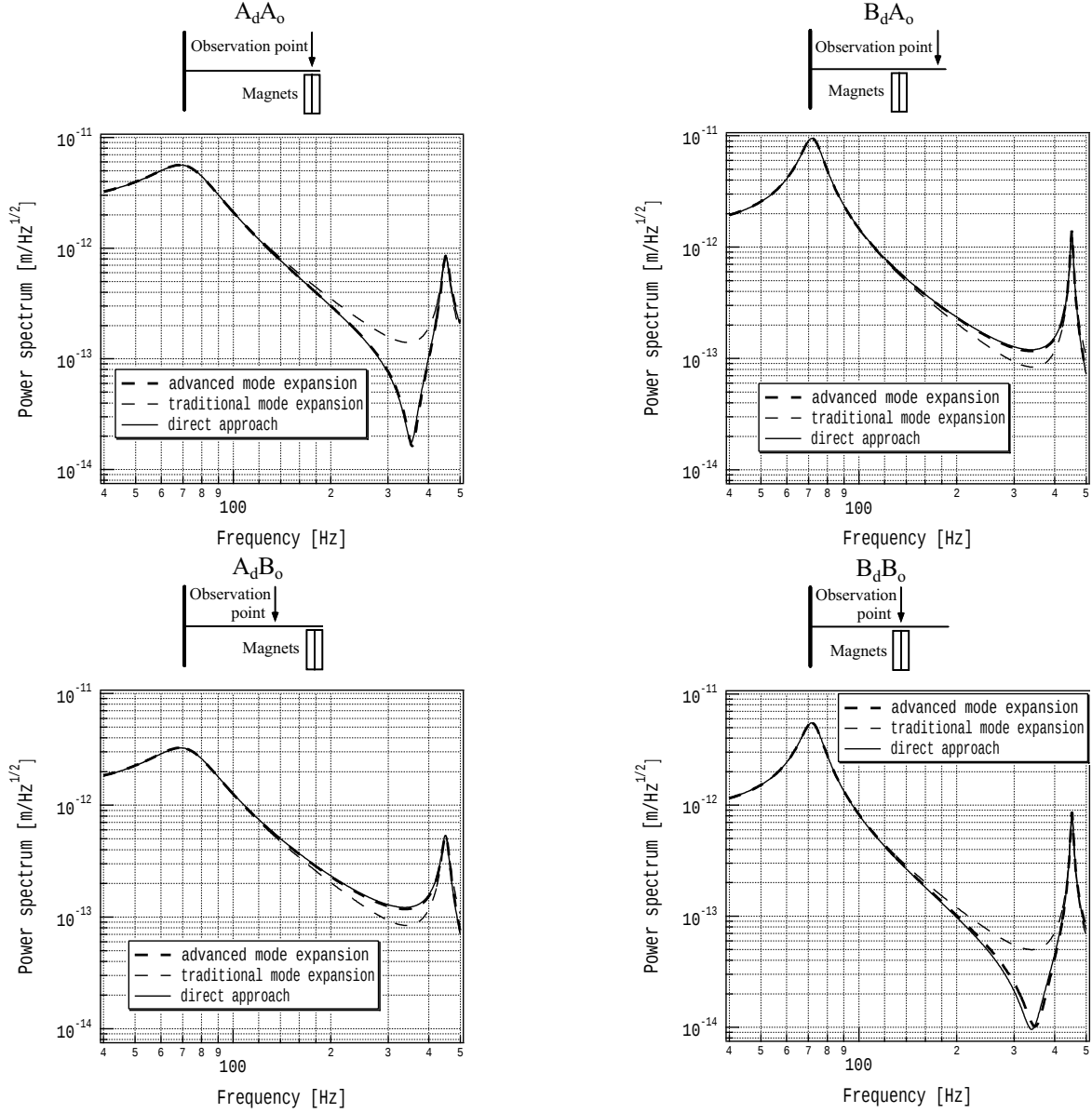


Figure 6.7: Results of estimation. The positions of the magnets and the observation point are indicated above each graph. The thick and thin dotted lines are the computations of the advanced and traditional mode expansions, respectively. The solid line is the evaluation of the direct approach.

has weak dependence on the observation point and the distribution of the dissipation. This is because there are not large differences of the effective masses and the Q -values among the four cases. Figure 6.6 shows that the amplitudes at A and B are almost the same in both the modes. This implies that the effective masses and the Q -values are not

sensitive to the selection of the positions of the observation and magnets, respectively.

The spectra derived from the advanced mode expansion are different in all four cases. This is because the effect of the correlations are large and the signs of them are different. Since the damping is localized to a small volume, ΔV , around x_d , the integration in Eq.(6.14) approximates to $\rho\Gamma(x_d)w_1(x_d)w_2(x_d)\Delta V$. From Eqs.(4.52) and (6.14), the cross spectrum, $G_{q_1q_2}$, between the first and second modes is expressed as

$$G_{q_1q_2} \approx \frac{4k_B T}{m_1 m_2 (-\omega^2 + \omega_1^2)(-\omega^2 + \omega_2^2)} \frac{w_1(x_d)w_2(x_d)}{w_1(x_o)w_2(x_o)} \rho\Gamma(x_d)\Delta V. \quad (6.30)$$

From this expression, the sign of the correlation depends on the distribution of the losses, the observation point, and the frequency. Figure 6.6 shows that the sign of $w_1(x_d)/w_1(x_o)$ is always positive, while that of $w_2(x_d)/w_2(x_o)$ depends on the distribution of the losses and on the observation point. In A_dA_o and B_dB_o , the sign of $w_2(x_d)/w_2(x_o)$ is positive because the center of the damped area and the observation point are the same. On the other hand, in B_dA_o and A_dB_o , the sign of this term is negative because the sign of w_2 at A is opposite to that at B. The sign of the remaining part of the correlation term, Eq.(6.30), depends on the frequency. This sign is negative in the frequency region between the first and second modes. From these discussions, the sign of the correlation in A_dA_o and B_dB_o is negative. Thus, the measured spectrum of the thermal motion is smaller than that calculated from the traditional mode expansion in A_dA_o and B_dB_o . On the contrary, the spectrum of the measured fluctuation is larger than that evaluated from the traditional mode expansion in B_dA_o and A_dB_o because the sign in B_dA_o and A_dB_o is positive.

The dependence of the thermal noise on the distribution of the loss is considered. The estimated spectra in A_dA_o is compared with those in B_dA_o (The same conclusion is derived from the comparison between A_dB_o and B_dB_o). The observation points in the both the cases are the same. The difference of the Q-values between the two cases is not large. However, there is large discrepancy between the spectra obtained from the advanced mode expansion because of the difference of the signs of the correlations. Therefore, the spectrum of the thermal noise is changed by moving the concentrated loss without relevant changes of Q-values.

The dependence of the thermal noise on the observation point is discussed. The calculated thermal fluctuation in A_dA_o is compared with those in A_dB_o (The same conclusion is derived from the comparison between B_dA_o and B_dB_o). The distribution of the eddy

current damping in both the cases are the same. Thus, these spectra correspond to the thermal motions at the different points in the same system. The evaluation of the advanced mode expansion is larger than that of the traditional mode expansion in $A_d B_o$. On the other hand, the estimation obtained from the advanced mode expansion is smaller than that of the traditional mode expansion in $B_d B_o$. This is because the sign of the coupling coefficient, α_{12} , depends on the observation point. When the sign of the difference between the measured thermal fluctuation and the evaluation from the traditional mode expansion depends on the observation point, this difference corresponds to the coupling due to the inhomogeneous loss. On the contrary, the sign of this difference is independent of the observation point, the chosen loss angle is not correct. Therefore, the information about the coupling, i.e. the distribution of the loss, can be derived from the measurements of the thermal motions at several points as discussed in the last section of Chapter 4.

6.4 Experimental method

In this experiment, the thermal motion and the transfer function, $H(\omega)$ in Eq.(3.7), of the leaf spring were measured to test the theory of the thermal noise caused by the inhomogeneous loss. The thermal fluctuation of the leaf spring was monitored with a Michelson interferometer. When the transfer function was measured, the force was applied using an electrostatic actuator. The motion of the leaf spring caused by this force was observed by the Michelson interferometer. Q-values were also measured for the estimation of the mode expansion. Experimental apparatus and the details of measurement of the thermal fluctuation and the transfer function and Q-values are introduced here.

6.4.1 Experimental apparatus

A schematic view of the experimental apparatus is shown in Fig.6.8. The leaf spring shown in Fig.6.1 was at one end of a differential Michelson interferometer to measure its motion. When the transfer function, $H(\omega)$, was measured, the leaf spring was excited by the electrostatic actuator. The interferometer was located on a seismic attenuation stack for the seismic isolation. All the apparatus, except for the laser source, were located in a

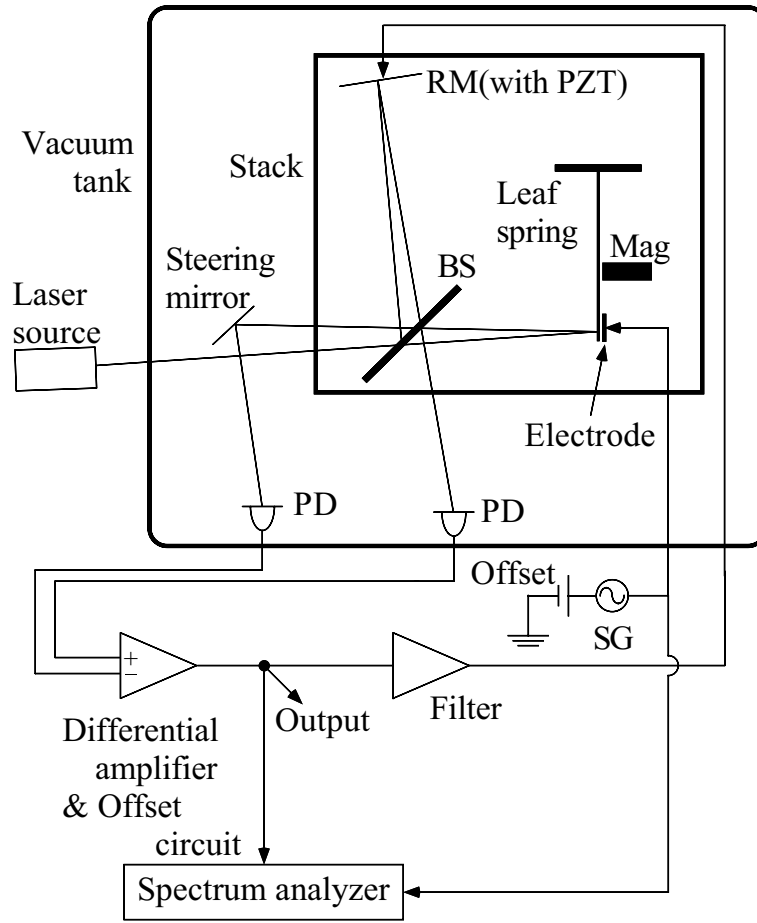


Figure 6.8: Schematic view of the experimental apparatus. BS, RM, Mag, PD, and SG stand for the beam splitter, the reference mirror, the magnets, the photo detector, and the signal generator, respectively. The sensor was a differential Michelson interferometer. To keep the interferometer at its operation point, the output signal of the interferometer was used to control the position of the reference mirror. The electrostatic actuator was used to measure the transfer function and Q -values. The interferometer was mounted on an attenuation stack for seismic isolation. All the apparatus, except for the laser source, were mounted in a vacuum chamber.

vacuum chamber. The details of each experimental apparatus components are described here.

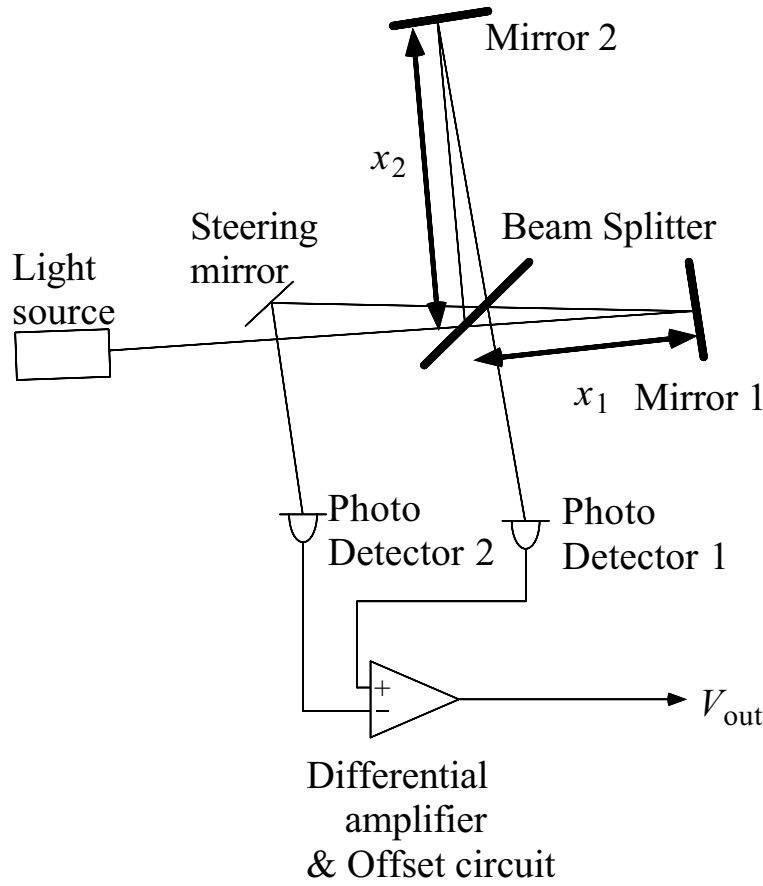


Figure 6.9: The fundamental configuration of a differential Michelson interferometer. The beam splitter divides the light from the laser source. These beams are reflected by mirrors. The reflected beams are recombined at the beam splitter. The recombined beams are detected by photo detectors. The output, V_{out} , of the interferometer is the difference between the outputs of the two photo detectors.

Michelson interferometer

Figure 6.9 shows the fundamental configuration of a differential Michelson interferometer. A beam splitter divides the light from the laser. These beams are reflected by mirrors. The complex amplitudes, E_1 and E_2 , of the reflected beams are expressed as

$$E_1 = A_1 \exp[i(2kx_1 - \omega t)], \quad (6.31)$$

$$E_2 = A_2 \exp[i(2kx_2 - \omega t)], \quad (6.32)$$

where A_i are arbitrary real numbers, ω is the angular frequency of the light, and x_i are lengths of arms of the interferometer in Fig.6.9. The parameter, k , is the wave number written as

$$k = \frac{2\pi}{\lambda}, \quad (6.33)$$

where λ is the wavelength of the light.

The reflected beams are recombined at the beam splitter. This combined beams go into photo detectors. The intensities of the lights caught by the photo detectors are considered here. To simplify the discussion, it is supposed that this beam splitter divides the beam into equal intensities. The intensity, I_1 , of the light which goes into the photo detector 1 is given by

$$I_1 = \frac{A_1^2 + A_2^2}{2} - A_1 A_2 \cos[2k(x_1 - x_2)]. \quad (6.34)$$

The intensity, I_2 , of the beam which goes into the photo detector 2 is written as

$$I_2 = r^2 \frac{A_1^2 + A_2^2}{2} + r^2 A_1 A_2 \cos[2k(x_1 - x_2)], \quad (6.35)$$

where r is the amplitude reflectivity of the steering mirror in Fig.6.9. The contrast, K , of the interferometer is defined by

$$K = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}, \quad (6.36)$$

where $I_{\max}$ and $I_{\min}$ are the maximum and minimum of the intensity measured by a photo detector, respectively. The contrast is the index of the symmetry of the interferometer. When the interferometer is perfectly symmetric, the contrast is unitary. A small contrast corresponds to a large asymmetry.

The output of the interferometer is the difference between the outputs of two photo detectors. The difference between the intensities of the beams caught by the photo detectors 1 and 2 is written in the form

$$I_1 - I_2 = (1 - r^2) \frac{A_1^2 + A_2^2}{2} - (1 + r^2) A_1 A_2 \cos[2k(x_1 - x_2)]. \quad (6.37)$$

The offset term, which is the first term in the right-hand side of Eq.(6.37), is canceled by the offset circuit in Fig.6.9. Consequently, the output voltage, V_{out} , of the interferometer is expressed as

$$V_{\text{out}} = A \cos(2kX), \quad (6.38)$$

where A is the constant, and X is the difference of the optical lengths of arms:

$$X = x_1 - x_2. \quad (6.39)$$

When V_{out} is nearly zero, V_{out} is the most sensitive to the change of X . Thus, the position of a mirror is controlled to make the output of the interferometer vanish at low frequency. When the interferometer is kept at this operation point, V_{out} is proportional to X . The proportional coefficient, H_{inter} , is described as

$$\begin{aligned} H_{\text{inter}} &= \left. \frac{dV_{\text{out}}}{dX} \right|_{V_{\text{out}}=0} \\ &= 2kA = \frac{4\pi A}{\lambda} \\ &= 2.0 \times 10^7 \text{ [V/m]} \left(\frac{\text{A}}{1 \text{ V}} \right) \left(\frac{633 \text{ nm}}{\lambda} \right). \end{aligned} \quad (6.40)$$

The schematic configuration of the interferometer used in this experiment is shown in Fig.6.8. The light source was a helium-neon laser ($\lambda=633 \text{ nm}$). The main part of the interferometer was comprised of a beam splitter, a leaf spring, and a reference mirror with a piezo-electric actuator (PZT). The PZT was an actuator to control the position of the reference mirror. The leaf spring was polished to increase its reflectivity. The fringe contrast of this interferometer was about 30%. The output signal of the interferometer was sent to the PZT of the reference mirror through servo filters to keep the interferometer at its best operation point. The output signal was recorded by a spectrum analyzer.

Electrostatic actuator

Figure 6.10 shows a basic configuration of an electrostatic actuator. The electrode faces the leaf spring. The voltage, V_{in} , is applied to this electrode. This voltage is an oscillatory voltage with an offset;

$$V_{\text{in}} = V_0 + v \cos(\omega t). \quad (6.41)$$

V_0 is chosen much larger than v , $V_0 \gg v$. The electric field applies a force, F , on the leaf spring. This force is expressed as

$$F = \frac{\varepsilon_0}{2} \frac{S}{d^2} V_{\text{in}}^2, \quad (6.42)$$

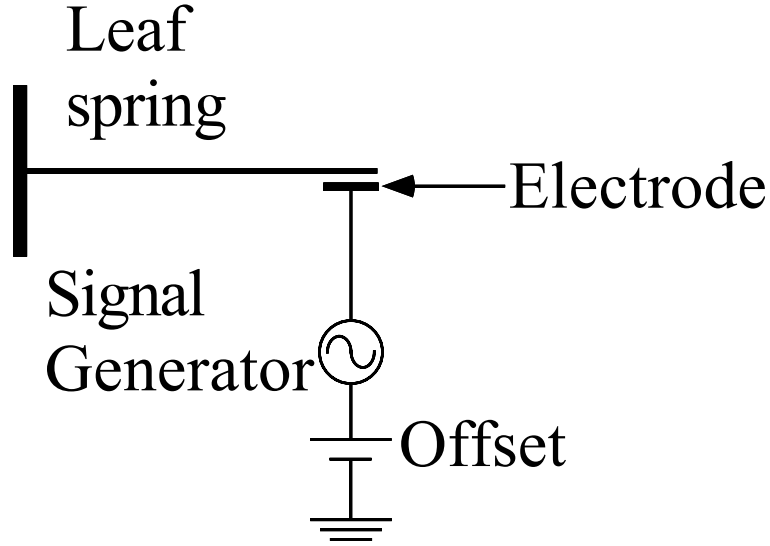


Figure 6.10: The basic configuration of an electrostatic actuator. The electrode faces the leaf spring. An oscillatory voltage with a large offset is applied on this electrode. The electric field applies a force on the leaf spring.

where ε_0 is the dielectric constant of the vacuum, S is the area of the electrode, d is the distance between the electrode and the leaf spring. Putting Eq.(6.41) into Eq.(6.42), the formula of the force is written as

$$\begin{aligned}
 F &= \frac{\varepsilon_0}{2} \frac{S}{d^2} [V_0 + v \cos(\omega t)]^2 \\
 &= \frac{\varepsilon_0}{2} \frac{S}{d^2} [V_0^2 + 2V_0 v \cos(\omega t) + v^2 \cos^2(\omega t)] \\
 &\approx \frac{\varepsilon_0}{2} \frac{S}{d^2} [V_0^2 + 2V_0 v \cos(\omega t)].
 \end{aligned} \tag{6.43}$$

This expression proves that an oscillatory force is applied on the leaf spring. The amplitude of this oscillatory force is proportional to the amplitude of the voltage, v . The proportional coefficient, H_{exciter} , is described as

$$\begin{aligned}
 H_{\text{exciter}} &= \varepsilon_0 \frac{V_0 S}{d^2} \\
 &= 2.1 \times 10^{-8} \text{ [N/m]} \left(\frac{V_0}{30 \text{ V}} \right) \left(\frac{S}{20 \text{ mm}^2} \right) \left(\frac{0.5 \text{ mm}}{d} \right)^2.
 \end{aligned} \tag{6.44}$$

The values obtained from the measurement using this exciter were the ratio of the displacement, $\tilde{X}$, of the leaf spring to the voltage, $v = \tilde{V}_{\text{in}}$, applied on the exciter. In

order to derive the transfer function, $H(\omega) = \tilde{X}/\tilde{F}$, from the observed value, $\tilde{X}/\tilde{V}_{\text{in}}$, the exciter efficiency, $H_{\text{exciter}} = \tilde{F}/\tilde{V}_{\text{in}}$, was calibrated. Since a precise measurement of the gap between the leaf spring and the electrode is difficult, Eq.(6.44) could not be used to estimate H_{exciter} in this experiment. The adopted method is introduced here. From the expression of the transfer function in the mode expansion, Eq.(3.76), the transfer function, H_{DC} , of the leaf spring in a frequency range lower than the first resonance is written in the form

$$H_{\text{DC}}(\omega) \approx \sum_n \frac{1}{m_n \omega_n^2}. \quad (6.45)$$

Since Eq.(6.8) and Table.6.1 show that the second resonant frequency is six times larger than the first resonant frequency, Eq.(6.45) is rewritten as

$$H_{\text{DC}}(\omega) \approx \frac{1}{m_1 \omega_1^2}. \quad (6.46)$$

Thus, H_{DC} for $\omega \ll \omega_1$ is derived from m_1 and ω_1 . The angular resonant frequency, ω_1 , was measured. The effective mass, m_1 , was calculated using Eq.(6.12). From H_{DC} and the measured data, $(\tilde{X}/\tilde{V}_{\text{in}})_{\text{DC}}$, in the low-frequency region, the exciter efficiency, H_{exciter} , was derived²:

$$H_{\text{exciter}} = \left(\frac{\tilde{X}}{\tilde{V}_{\text{in}}} \right)_{\text{DC}} \frac{1}{H_{\text{DC}}}. \quad (6.47)$$

The schematic configuration of the electrostatic actuator used in this experiment is shown in Fig.6.8. From the definition of the transfer function, $H(\omega)$, the point for the observation and the excitation were the same. When the observation point was in the damped area, the magnets, themselves, were used as the electrode. When the observation point was outside of the damped area, a stainless-steel electrode having the same width as that of the damped area was used. The applied voltage was recorded by the spectrum analyzer.

Seismic attenuation stack

The seismic motion is one of the main problems in precise measurements. The power spectrum, G_{seismic} , of the seismic motion in the suburb³ is expressed as [77, 78]

$$G_{\text{seismic}}(f) = \frac{10^{-7}}{f^2} [\text{m}/\sqrt{\text{Hz}}], \quad (6.48)$$

²The order of the value estimated from Eq.(6.47) was the same as that of the evaluation from Eq.(6.44).

³The seismic motion in a mine is one hundred times smaller than that in the suburb [78].

where f is the frequency. Figure 6.11 represents the seismic noise in the vacuum chamber used in this experiment. This seismic motion was measured with a seismograph. The spectrum in Fig.6.11 is comparable with Eq.(6.48). Figure 6.7 shows that the estimated thermal motion is about $10^{-13}\text{m}/\sqrt{\text{Hz}}$ at 300Hz. Therefore, the seismic motion is comparable to the estimated thermal motion.

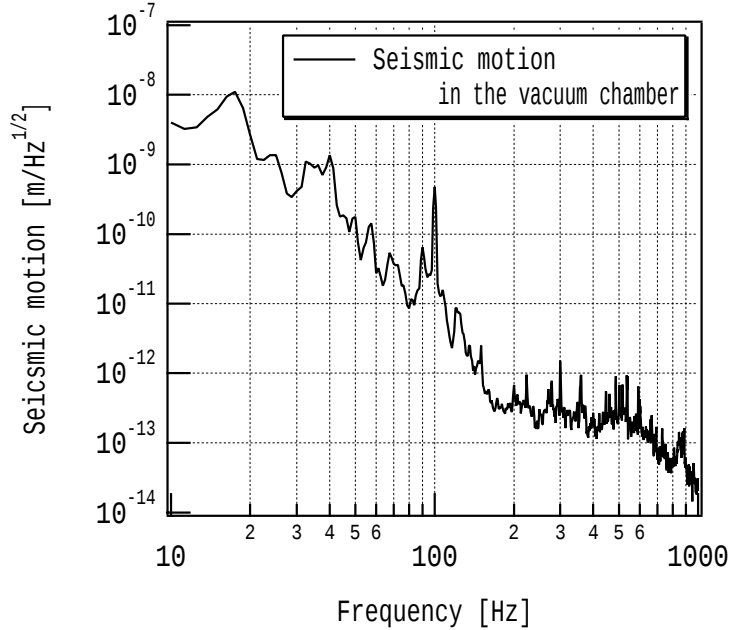


Figure 6.11: The seismic motion on the bottom plate of the vacuum chamber used in this experiment. This seismic motion was measured with a seismograph.

To avoid this problem, an isolation stack, which is a passive isolation system, was used to isolate the interferometer from the seismic vibration. This stack was a soft system which consisted of springs. The principle of the isolation stack is considered here using a spring. An end of the spring fixed to earth. Another end is connected to an object. The equation of motion of this spring in the frequency domain is written as

$$-m\omega^2\tilde{x}_{\text{obj}} = -k[1 + i\phi(\omega)](\tilde{x}_{\text{obj}} - \tilde{x}_s), \quad (6.49)$$

where m and x_{obj} are the mass and displacement of the object, respectively. The parameter, x_s , is the motion of the ground, k is the spring constant, and ϕ is the loss angle. The transfer function, $H_{\text{isolation}}$, from the seismic motion to the vibration of the object is

derived from Eq.(6.49);

$$H_{\text{isolation}}(\omega) = \frac{\tilde{x}_{\text{obj}}}{\tilde{x}_s} = \frac{\omega_0^2[1 + i\phi(\omega)]}{-\omega^2 + \omega_0^2[1 + i\phi(\omega)]}, \quad (6.50)$$

where $\omega_0 (= \sqrt{k/m})$ is the angular resonant frequency of the spring. Eq.(6.50) shows that the seismic vibration in the higher frequency range ($\omega \gg \omega_0$) is not transferred to the object. The Q-value of the stack should be decreased because the seismic motion at the resonant frequency is proportional to Q-value. Commonly, isolation stack has more than one stage to enhance the isolation ratio.

Figure 6.12 shows the configuration of the stack used in this experiment. The stack consisted of three layers which included an aluminium plate, springs, and rubbers. Springs were wrapped in lead ribbons to decrease the Q-values. Figure 6.13 shows that the output of the interferometer on the stack when the leaf spring was replaced by a fixed mirror. Comparing Fig.6.13 with Fig.6.7 proves that the interferometer is sufficiently isolated from the seismic vibration.

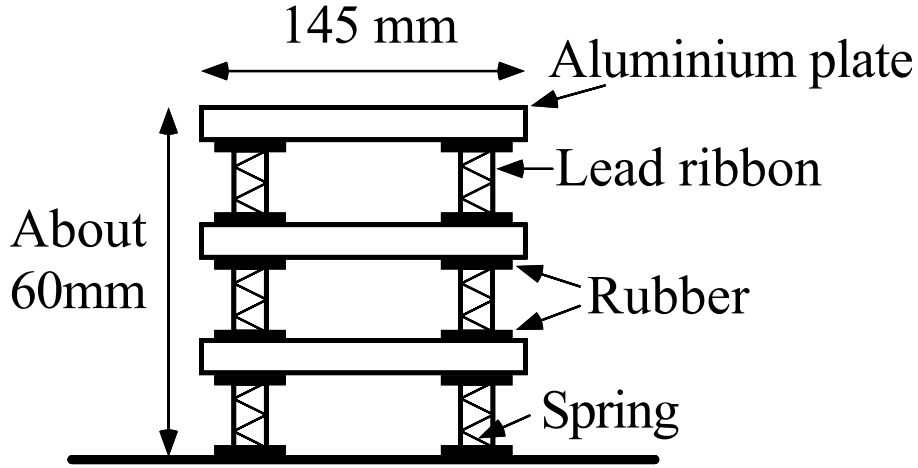


Figure 6.12: The schematic view of the stack used in this experiment. This stack consisted of three layers which included an aluminium plate, springs, and rubbers. Springs were wrapped in lead ribbons.

Vacuum chamber

All the apparatus, except for the helium-neon laser source, were put in a vacuum tank. The vacuum chamber was evacuated by a rotary pump; the pressure was about 10 Pa.

This pump was stopped during measurement. When the pressure was smaller than 40 Pa, the Q -values of the leaf spring without the eddy current damping was independent of the pressure. Thus, the loss caused by the residual gas was smaller than the original loss in the leaf spring.

6.4.2 Measurement of thermal noise

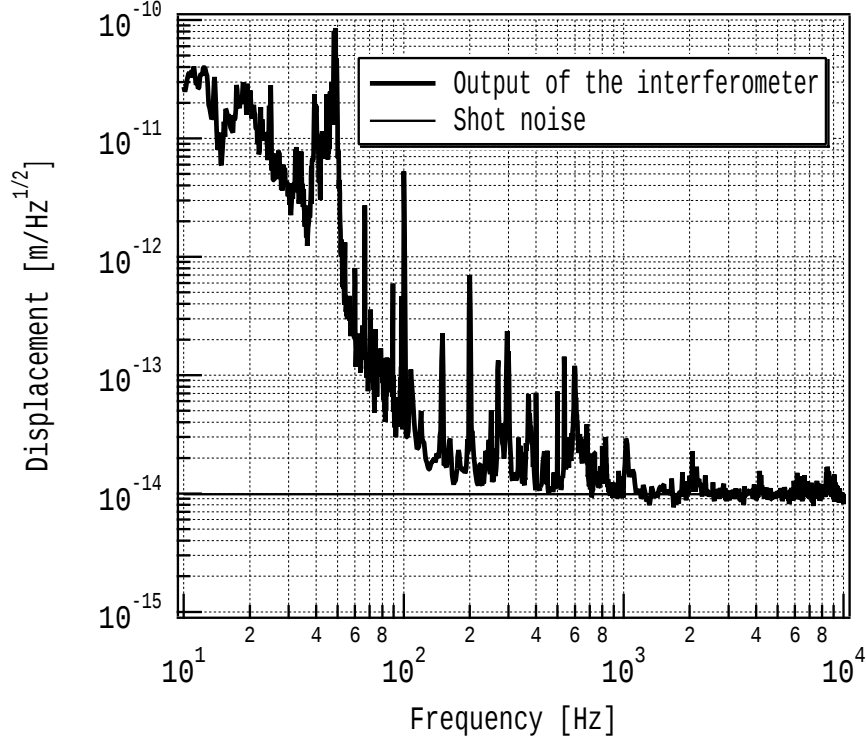


Figure 6.13: The spectrum of the output of the interferometer when the leaf spring is replaced with a fixed mirror. The thick line is the spectrum of the output of the interferometer. The thin line represents the shot noise. The spectrum in the frequency range lower than 150 Hz is probably dominated by the seismic motion. The sensitivity in the frequency region higher than 150 Hz is limited by the shot noise.

The thermal fluctuation of the leaf spring with inhomogeneous eddy current damping was observed with the Michelson interferometer. The output of the interferometer was recorded by the spectrum analyzer. In order to investigate the limit of the measurement of this interferometer, the leaf spring was replaced with a fixed mirror. The spectrum

of the interferometer in this case is shown in Fig.6.13. The thick line is the spectrum of the output of the interferometer. Comparing Fig.6.13 with Fig.6.7 proves that this interferometer sensitivity is sufficient to measure the thermal motion of the leaf spring with inhomogeneous eddy current damping near 300 Hz. The spectrum in the frequency range lower than 150 Hz is probably dominated by the seismic motion. The sensitivity in the frequency region higher than 150 Hz is limited by the shot noise. The shot noise is the length measurement error caused by the quantum fluctuation of the number of the photons. The power spectrum, G_i , of the shot noise of the photo-current in a photo detector is expressed as

$$G_i = 2eI_{DC}, \quad (6.51)$$

where e is the elementary electric charge⁴, I_{DC} is the DC photo-current. The output voltage of the photo detector is proportional to the photo-current. This proportional coefficient is R . The spectrum, G_{PD} , of the shot noise of the output voltage of the photo detector is written in the form

$$G_{PD} = 2eRV_{DC}, \quad (6.52)$$

where V_{DC} is the DC output voltage; $V_{DC} = RI_{DC}$. The interferometer in this experiment has two photo detectors. Thus, the shot noise, G_{shot} , of the interferometer is described as

$$G_{shot} = \frac{G_{PD1} + G_{PD2}}{H_{inter}} = \frac{2e(R_1V_{DC1} + R_2V_{DC2})}{H_{inter}}, \quad (6.53)$$

where R_n , V_{DCn} , and G_{PDn} are the R , V_{DC} , and G_{PD} of the n -th photo detector, respectively. The thin line in Fig.6.13 corresponds to the shot noise evaluated from Eq.(6.53).

In order to keep the interferometer at the operation point, the output signal of the interferometer was used to control the position of the reference mirror. Thus, the spectrum of the output of the interferometer was corrected in order to remove the effect of the feedback. The method of the correction of the output signal is discussed here. The outline of the control system of the interferometer is reviewed in Fig.6.8. The output signal of the interferometer was sent to the PZT of the reference mirror through frequency filters. The strain of the PZT is proportional to the applied voltage. Its voltage to strain coefficient⁵

⁴ $e = 1.6 \times 10^{-19}$ [C].

⁵In this experiment, $H_{PZT} = 1.6 \times 10^{-8}$ [m/V].

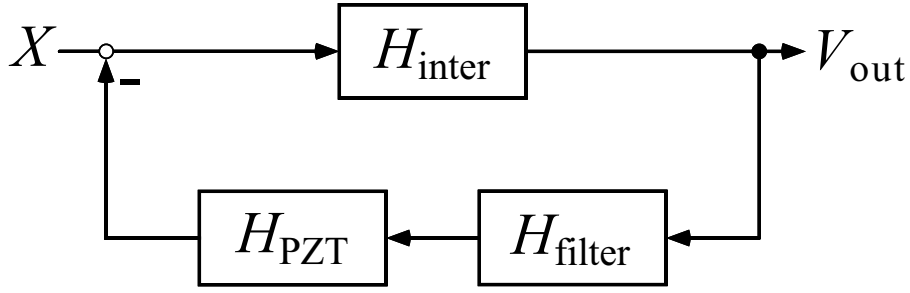


Figure 6.14: The block diagram of the control system of the interferometer. The parameter, X , is the displacement of the leaf spring. The value, V_{out} , is the output of the interferometer. The expression, H_{inter} , is the transfer function of the interferometer in Eq.(6.40). The values, H_{filter} and H_{PZT} , are the transfer function of the filter and the PZT, respectively.

is H_{PZT} . The position of the reference mirror was controlled to cancel the displacement of the leaf spring. The block diagram of this control system is shown in Fig.6.14. The parameter, X , is the displacement of the leaf spring. The value, H_{inter} , is the transfer function of the interferometer in Eq.(6.40). The expression, V_{out} , is the output of the interferometer. The values, H_{filter} and H_{PZT} , are the transfer functions of the filter and PZT, respectively. From Fig.6.14, the relation between X and V_{out} is obtained:

$$\tilde{V}_{\text{out}} = H_{\text{inter}}(\tilde{X} - H_{\text{filter}}H_{\text{PZT}}\tilde{V}_{\text{out}}). \quad (6.54)$$

From Eq.(6.54), the displacement of the leaf spring is expressed as

$$\tilde{X} = \frac{1 + G}{H_{\text{inter}}} \tilde{V}_{\text{out}}, \quad (6.55)$$

where G is the open loop gain defined by

$$G = H_{\text{inter}}H_{\text{filter}}H_{\text{PZT}}. \quad (6.56)$$

From Eq.(6.55), the power spectrum, G_X , of the fluctuation of the leaf spring is given

$$\sqrt{G_X} = \frac{|1 + G|}{H_{\text{inter}}} \sqrt{G_{V_{\text{out}}}}, \quad (6.57)$$

where $G_{V_{\text{out}}}$ is the power spectrum of V_{out} . The transfer function of the interferometer, H_{inter} , was derived from Eq.(6.40). The output, V_{out} , of the interferometer and its open loop gain, G , were measured. The open loop gain in this experiment⁶ is shown in Fig.6.15. From these values and Eq.(6.57), the spectrum, G_X , of the thermal motion is derived.

⁶The peak at 300 Hz is the resonance of the support of the PZT.

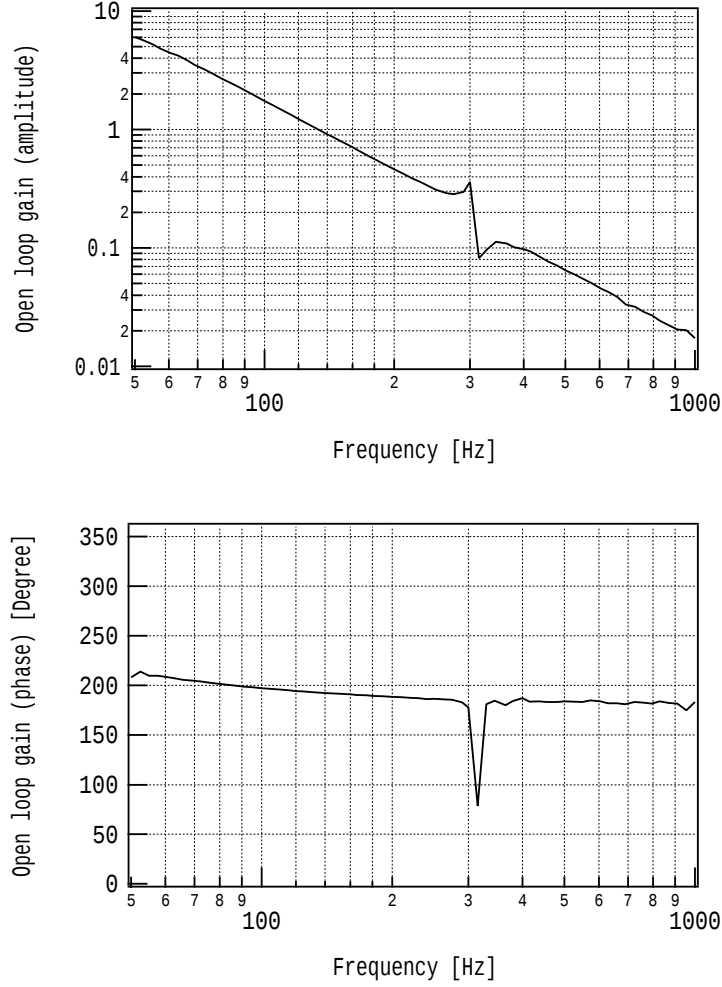


Figure 6.15: The open loop gain, G , in this experiment. The upper and lower graphs represent the amplitude and the phase of G , respectively.

6.4.3 Measurement of the transfer function

The details of the measurement of the transfer function, $H(\omega)$, which is the ratio of the displacement, $\tilde{X}$, of the leaf spring to the applied force, $\tilde{F}$, are described here. Using the electrostatic actuator, the oscillatory force was applied at the observation point of the leaf spring. The vibration of the leaf spring was observed with the interferometer. The voltage of the actuator and the output of the interferometer were recorded by the spectrum analyzer. From this measurement, we obtained the transfer function, $H(\omega)$.

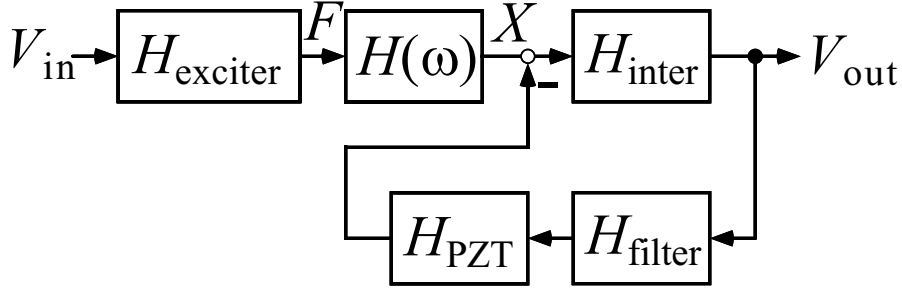


Figure 6.16: The block diagram of the control system of the measurement of the transfer function, $H(\omega)$, of the leaf spring. The voltage, V_{in} , is applied on the electrostatic actuator, H_{exciter} is the transfer function of the exciter in Eq.(6.44), and F is the force which the actuator applies on the leaf spring. The expression, X , is the displacement of the leaf spring. The value, H_{inter} , is the transfer function of the interferometer in Eq.(6.40), and V_{out} is the output of the interferometer. The expressions, H_{filter} and H_{PZT} , are the transfer function of the filter and the PZT, respectively.

In this experiment the correction of the feed back was necessary. The block diagram of the control system of this measurement is shown in Fig.6.16. The voltage, V_{in} , is applied on the electrostatic actuator; H_{exciter} is the transfer function of the exciter in Eq.(6.44). The value, F , is the force which the actuator applies on the leaf spring. The expression, $H(\omega)$, is the transfer function of the leaf spring, and X is the displacement of the leaf spring. The value, H_{inter} , is the transfer function of the interferometer in Eq.(6.40), and V_{out} is the output of the interferometer. The expressions, H_{filter} and H_{PZT} , are the transfer function of the filter and the PZT, respectively. From Fig.6.16, V_{out} is written in the form

$$\tilde{V}_{\text{out}} = H_{\text{inter}}[H(\omega)H_{\text{exciter}}\tilde{V}_{\text{in}} - H_{\text{PZT}}H_{\text{filter}}\tilde{V}_{\text{out}}]. \quad (6.58)$$

From Eq.(6.58), $H(\omega)$ is described as

$$H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} \frac{1 + G}{H_{\text{inter}}H_{\text{exciter}}}. \quad (6.59)$$

The ratio, $\tilde{V}_{\text{out}}/\tilde{V}_{\text{in}}$, and the open loop gain, G , were measured. The transfer functions, H_{inter} and H_{exciter} , were derived from Eqs.(6.40) and (6.47), respectively. The power spectrum of the thermal noise was obtained from the application of the fluctuation-dissipation theorem, Eq.(3.9), to the measured transfer function, $H(\omega)$ in Eq.(6.59).

6.4.4 Measurement of Q-values

In the estimation of the mode expansion, measured Q-values were used. The Q-value of the second mode of the leaf spring was derived from the decay time of its resonant motion and Eq.(3.49). The electrostatic actuator was used to excite the resonant vibration. The Q-value of the first mode was estimated from the width of its resonant peak in the transfer function and Eq.(3.51) because its Q-value was low. The results are summarized in Table.6.9.

6.5 Results

The measured spectra of the thermal motions were compared with the evaluation derived from the advanced and traditional mode expansion. The results proved that the advanced mode expansion is a correct method to estimate the thermal noise. On the contrary, the estimation of the traditional mode expansion did not agree with the measured values. This is the first experimental results which show the failure of the traditional mode expansion. The thermal fluctuations evaluated from the measured transfer functions were consistent with the motions of the leaf spring measured by the interferometer. Thus, it was confirmed that the measured motions were the thermal fluctuations. The parameters for the estimation of the mode expansion and these experimental results are summarized here.

6.5.1 Parameters for the estimation

In order to obtain the spectra from the advanced and traditional mode expansion, all the parameters, m_n, ω_n, Q_n ($n=1,2$), and α_{12} , were estimated⁷ from the measurements because these experimental values are slightly different from the calculated values in Tables.6.3, 6.4, 6.5, and 6.6. The angular resonant frequencies, ω_n , were measured directly. The Q-values, Q_1 and Q_2 , were derived from the measurements of the width of the resonant peak and the decay time of the resonant motion, respectively. The effective mass, m_n , is defined by Eq.(6.12). In this experiment, the point illuminated by the laser beam

⁷Since the dissipation was dominated by the eddy current damping, the loss angle was given by Eq.(6.13) and the measured Q-values.

was slightly different from the points A and B in Fig.6.2. The correction of the effective mass is considered here. Between the resonant frequencies, there is an anti-resonant frequency. At this frequency, the absolute value of the transfer function, $H(\omega)$, of the leaf spring is a local minimum. The angular anti-resonant frequency, ω_{anti} , between the first and second modes is expressed as [62, 63, 64]

$$\omega_{\text{anti}} = \sqrt{\frac{m_2}{m_1}} \omega_2. \quad (6.60)$$

The frequencies, ω_{anti} and ω_2 , were evaluated from the measurement of the transfer function. From Eq.(6.60) and these measured values, ω_2 and ω_{anti} , the ratio of m_2 to m_1 was calculated. The point, x_o , illuminated by the laser beam was obtained from m_2/m_1 because m_2/m_1 depends on x_o . Putting the estimated x_o into Eq.(6.12), m_1 and m_2 were given. The coupling coefficient, α_{12} , was obtained from multiplying the calculated value, $\alpha_{12}/|\alpha_{12}|_{\text{max}}$, in Table.6.6 times the maximum amplitude of α_{12} , $\sqrt{m_1 \omega_1^2 \phi_1 m_2 \omega_2^2 \phi_2}$, derived from the measured values, m_n , ω_n , and Q_n .

The resonant frequencies⁸, effective masses⁹, and Q-values derived from the experiments are summarized in Tables.6.7, 6.8, and 6.9. The measured Q-values in Table.6.9 were about the same as the calculated values in Table.6.5. Thus, the thermal motions of the leaf spring in this experiment were roughly comparable to the evaluated spectra in Fig.6.7. The leaf spring was sufficiently damped for this experiment. The Q-values of the leaf spring without the eddy current damping were about 1000; Table.6.9 then shows that the original loss of the leaf spring was negligible.

6.5.2 Test of new methods of estimation

The measured and evaluated thermal motion spectra for all of four configurations are shown in Fig.6.17. The positions of the magnets and the observation point are indicated above each graph. Each graph contains the spectrum of the measured thermal motion (thick solid line), the estimated spectra obtained by the advanced (thin solid line) and

⁸The measured resonant frequencies in Table.6.7 are smaller than the calculated resonant frequencies in Table.6.3. Since the dimensions of the leaf spring was measured precisely, we conclude that the effective Young's modulus is smaller than that in Table.6.2.

⁹From the correlation of the effective masses, the distance between the observed point and points A and B in Fig.6.2 are about 1 mm. In all cases, the observation points were shifted toward the free end.

Table 6.7: Resonant frequencies of the leaf spring (measurement).

	f_1	f_2
A _d A _o	60.5 Hz	360.6 Hz
B _d A _o	61.2 Hz	361.2 Hz
A _d B _o	61.3 Hz	361.9 Hz
B _d B _o	61.3 Hz	361.2 Hz

Table 6.8: Effective masses of the leaf spring (measurement).

	m_1	m_2
A _d A _o	13.2 mg	17.7 mg
B _d A _o	13.5 mg	20.7 mg
A _d B _o	35.9 mg	94.7 mg
B _d B _o	38.2 mg	67.8 mg

Table 6.9: Q-values of damped leaf spring (measurement).

	Q_1	Q_2
A _d A _o	3.5	31
B _d A _o	6.0	67
A _d B _o	3.2	27
B _d B _o	6.2	77

traditional (thin dashed line) mode expansions, and the shot noise level (thin long dashed line) of the interferometer. Figure 6.17 shows that the measured spectra in the frequency range between the first and second modes agreed with the spectra estimated from the advanced mode expansion in all cases. Therefore, the validity of the advanced mode expansion was proved. It was also proved that the traditional mode expansion is not a correct estimation method when the loss is distributed inhomogeneously. These were the first experimental results which show the failure of the traditional mode expansion.

The fluctuation spectra obtained from the direct measurements and evaluation based on the measured transfer function are shown in Fig.6.18. Each graph contains the spec-

trum of the directly measured motion (thick solid line), the estimation from the measured transfer function using the fluctuation-dissipation theorem (open circles), the spectrum obtained from the advanced mode expansion (thin solid line), and the shot noise level (thin long dashed line) of the interferometer. Figure 6.18 shows that the measured motions agreed well with the FDT-based estimation in the frequency region between the first and second modes. Thus, it was confirmed that the measured spectra corresponded to the thermal fluctuations. Consequently, Figs. 6.17 and 6.18 prove that the advanced mode expansion is valid even when the dissipation is not homogeneous. In addition, the failure of the traditional mode expansion was shown experimentally.

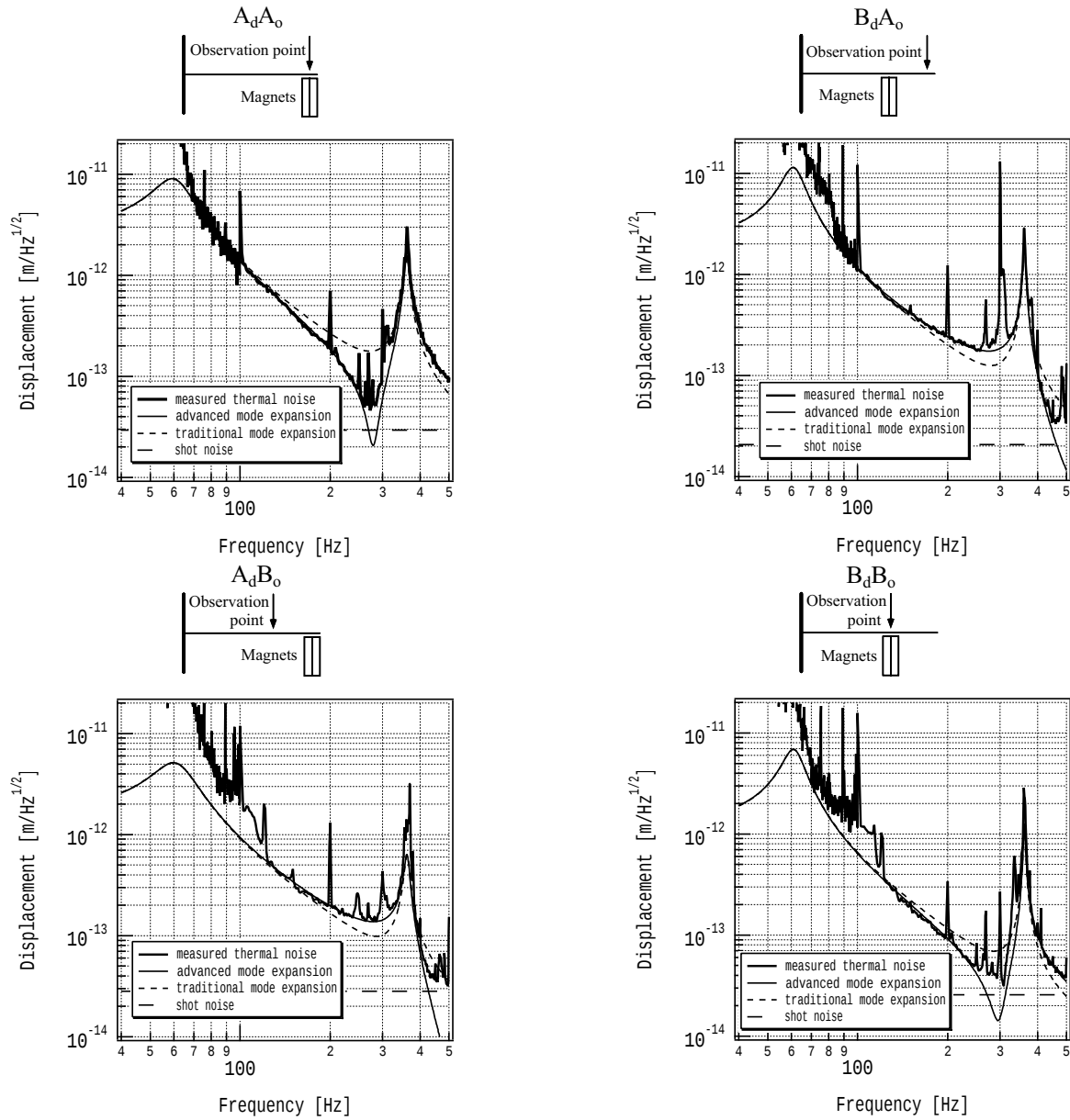


Figure 6.17: Results of measurements. The positions of the magnets and the observation point are indicated above each graph. The thick solid line is the measured spectrum of the thermal motion. The thin solid and dashed lines are estimation using advanced and traditional mode expansions, respectively. The thin long dashed line is the shot noise level of the interferometer.

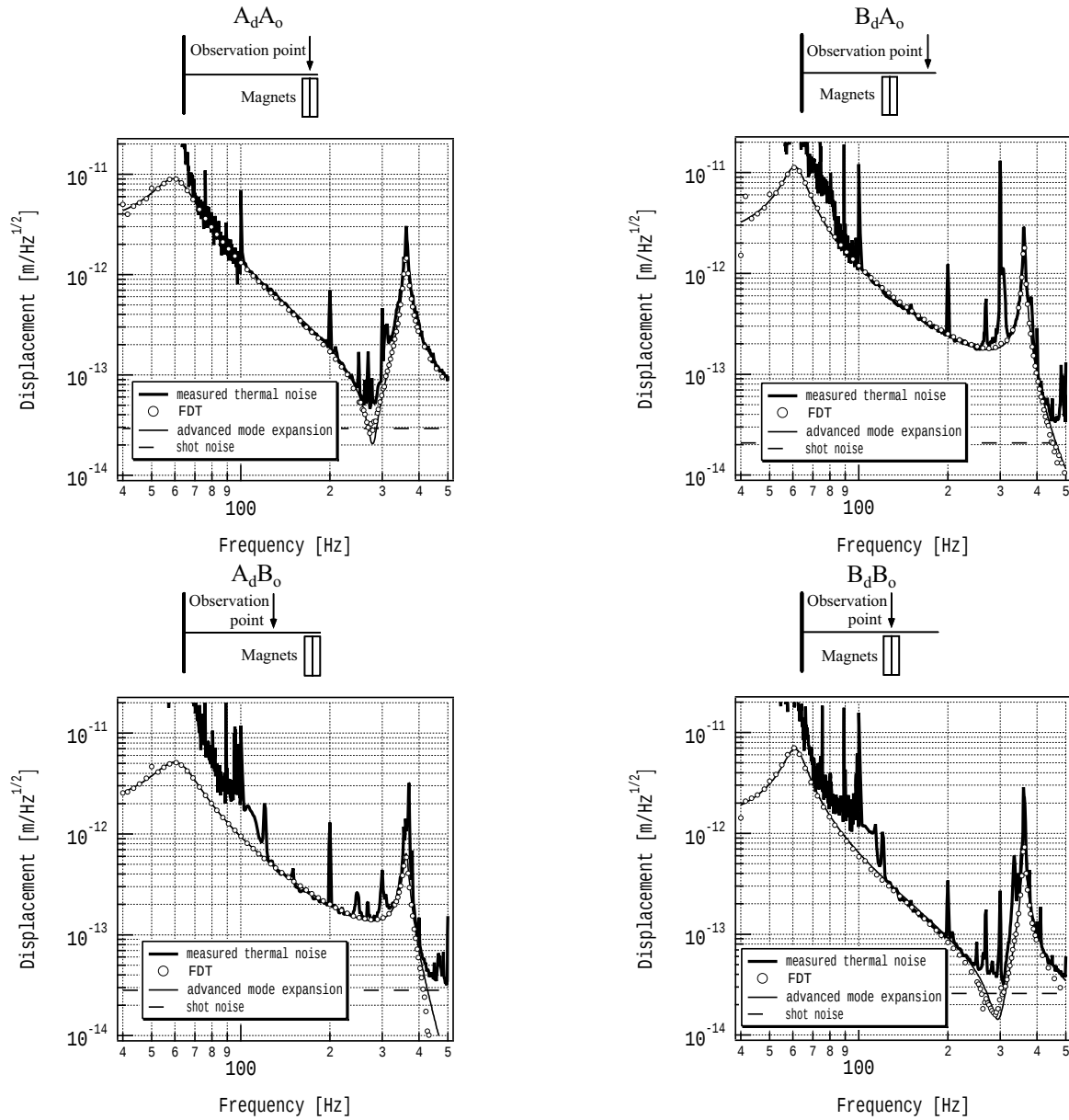


Figure 6.18: Results of measurements. The positions of the magnets and the observation point are indicated above each graph. The thick solid line is the measured spectrum of the thermal motion. The open circles are the spectrum evaluated from the measured transfer function using the fluctuation dissipation theorem. The thin solid line is estimation using the advanced mode expansion. The thin long dashed line is the shot noise level of the interferometer.

Figure 6.18 shows that the evaluation from the measured transfer function was well consistent with the calculation of the advanced mode expansion¹⁰. However, there were discrepancies between the directly measured spectra and those obtained from the measured transfer function near 300 Hz in A_dA_o and B_dB_o . These differences were attributed to the contribution of the shot noise of the interferometer. Fig.6.19 shows that the measured thermal fluctuation (thick solid line) was consistent with the summation (black dots) of the estimation obtained from the transfer function and the shot noise.

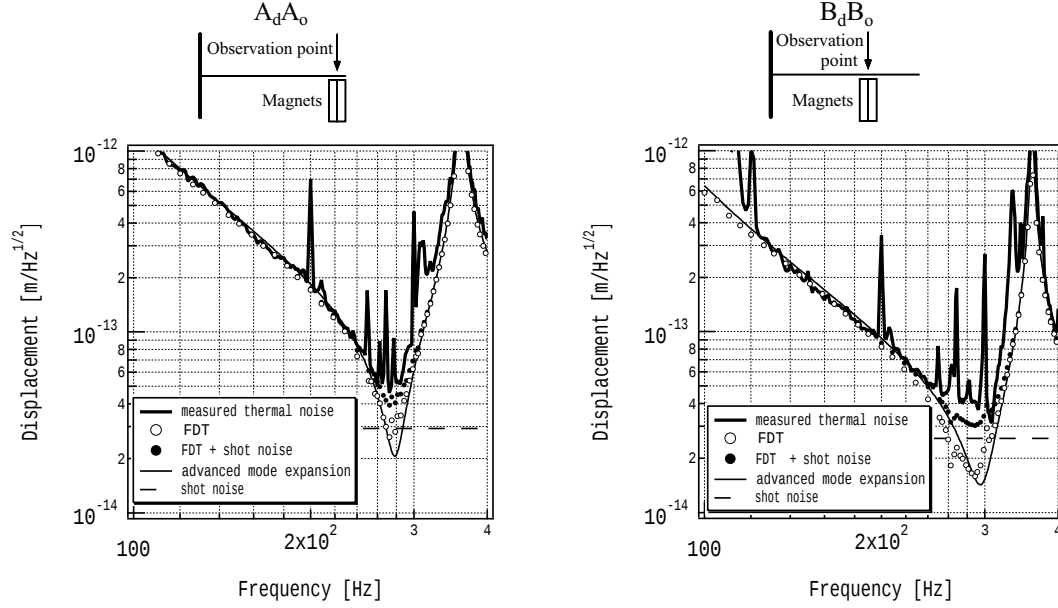


Figure 6.19: The spectra including the contribution of the shot noise of the interferometer. The positions of the magnets and the observation point are indicated above each graph. The thick solid line is the measured spectrum of the thermal fluctuation. The open circles are the thermal motion given from the measured transfer function. The black dots represent the summation of the thermal fluctuation evaluated from the transfer function and the shot noise. The thin solid line is the estimation derived from the advanced mode expansion. The thin broken line is the shot noise of the interferometer.

In no case the measured spectra of the thermal motion agreed with the evaluation from the measured transfer function below 100 Hz although the spectra derived from the measured transfer function were consistent with the estimation of the advanced mode

¹⁰There were slight differences between the spectrum obtained from the measured transfer function and the estimation from the advanced mode expansion near 300 Hz in A_dA_o and B_dB_o . These discrepancies corresponded to the thermal fluctuations caused by the original loss in the leaf spring.

expansion. In this frequency range, the motion of the leaf spring was not dominated by the thermal motion. Figure 6.20 show the fluctuation of the leaf spring with (thick line) and without (thin line) the eddy current damping. Even though the magnets for the eddy current damping were removed, the fluctuation in the motion below 100 Hz did not diminish. Thus, the motion in this frequency region was dominated by other than the thermal motion. Probably, this fluctuation was caused by seismic motion.

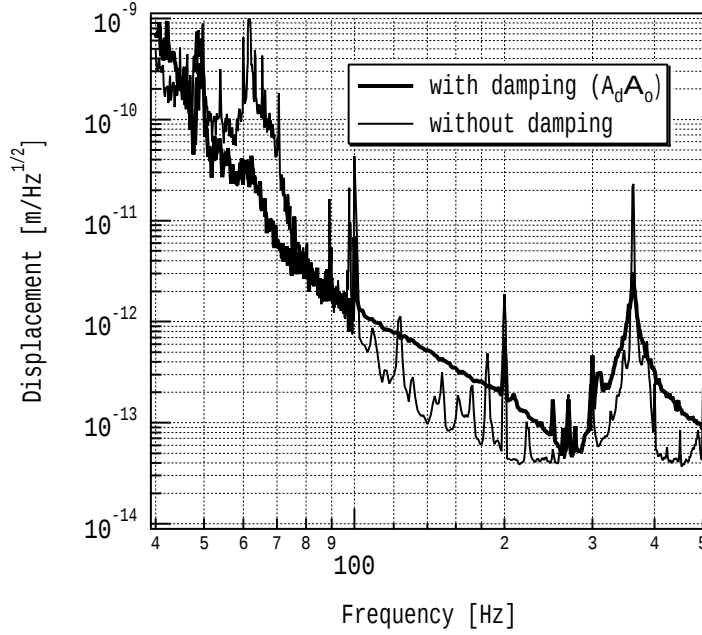


Figure 6.20: The fluctuation of the leaf spring with and without the eddy current damping. The solid and thin lines correspond to the motion with and without eddy current damping, respectively.

Chapter 7

Mirror with inhomogeneous loss: Estimation

From the discussions in Chapter 4, it has been shown that the traditional mode expansion used frequently to estimate the thermal noise is not correct when the dissipation is inhomogeneous. On the other hand, the new methods, the advanced mode expansion in Chapter 4 and the direct approaches described in Chapter 5, appear to be valid even when the loss is not homogeneous as confirmed by the experiments in the previous chapter. In this chapter, the new methods are applied to the evaluation of the thermal noise of mirrors with inhomogeneous losses in the interferometric gravitational wave detectors. In the next chapter, this estimation will be checked experimentally in an aluminum drum.

The thermal noise of the mirrors has been estimated using the traditional mode expansion [50, 51, 52]. However, in general, the dissipation in mirrors are inhomogeneous. For example, the measurements of Q-values suggest that the loss is localized near the magnets [80] and stand-off's [53] glued on the mirrors and on the surfaces of the mirror [54, 82, 83]. If the loss is concentrated in a small volume, the thermal noise of the mirrors is largely different from the estimation of the traditional mode expansion as discussed in Chapter 4; since the thermal fluctuation of the mirror consists of contributions of many modes, the correlations caused by highly inhomogeneous losses yield large contributions. Moreover, the thermal noise derived from the traditional mode expansion is already the limiting factor of sensitivity around few hundreds Hz in all current projects. Thus, the

large discrepancy from the estimation of the traditional mode expansion will have large effects on the strategy to handle the thermal noise in the interferometric detectors. Nevertheless, the thermal noise of mirrors caused by the inhomogeneous loss was seldom studied¹. In this chapter, the difference between the thermal noise estimated from direct approach and the calculation of the traditional mode expansion are investigated quantitatively.

7.1 Qualitative consideration

In order to understand the outline of the thermal noise of the mirror with inhomogeneous loss, the discrepancy between the actual thermal noise and the estimation obtained from the traditional mode expansion are first considered qualitatively [68]. In the traditional mode expansion, the thermal noise is derived from Q -values of resonant modes. On the other hand, the observation band of the gravitational wave detectors is lower than the first resonant frequency of the mirror. Therefore, it is discussed how an inhomogeneous loss contributes to the Q -values and to the thermal noise in the observation band.

Figure 7.1 shows the deformation of a mirror when the oscillatory pressure is applied on it. The arrows correspond to the pressure. The profile of the pressure is the same as the beam profile. In the left and right sides of Fig.7.1, the frequency of the pressure is at the mirror's first mechanical resonance and in the observation band, respectively. The Q -values represent dissipated energy in the left-hand side of Fig.7.1. The thermal noise in the observation band is related to the loss in the right-hand side of Fig.7.1 governed by the fluctuation-dissipation theorem. The left-hand side of Fig.7.1 suggests that every part of the mirror has strain on resonance. This implies that the energy is distributed over the mirror. Thus, the inhomogeneous loss decrease Q -values almost independently from its location. The right-hand side of Fig.7.1 shows that below resonance only the parts near the beam spot have strain. The strain energy is concentrated near the beam spot. Thus, if the localized loss is near the beam spot, the loss contributes to the thermal noise in the observation band more efficiently than to the Q -values. On the other hand, when

¹Logan et al. has researched the loss of the mirror with stand-off's (lugs) [84]. Gillespie has studied the effects of the loss in spacers between a mirror and magnets [85]. Levin has estimated roughly the thermal noise of the mirror in which the loss is concentrated on the reflective coating [68].

- (1) At resonant frequency (2) In observation band
 ($\ll$ resonant frequency)

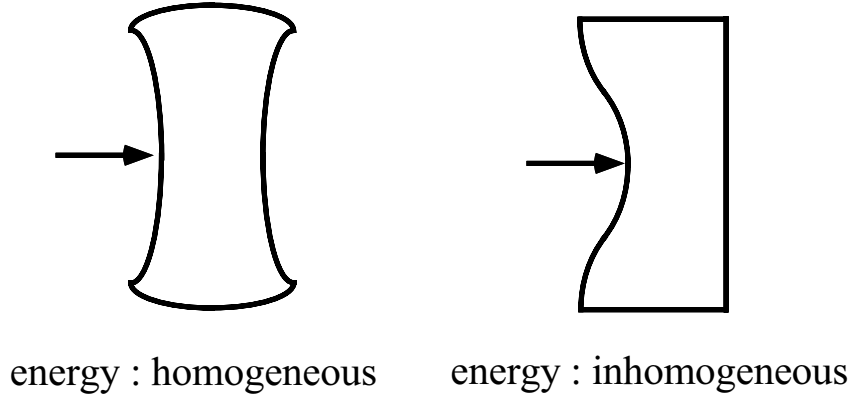


Figure 7.1: The deformation of a mirror when the oscillatory pressure is applied on the mirror. The arrows correspond to the pressure. The profile of the pressure is the same as the beam profile. The frequency of the oscillatory pressure is the resonant frequencies (left-hand side) and in the observation band which is lower than the resonant frequencies (right-hand side).

the concentrated loss is far from the beam spot, the loss has a far smaller effect on the thermal noise in the observation band than on Q -values. Consequently, the loss near the beam spot generates the thermal noise larger than what estimated with the traditional mode expansion. On the contrary, if the loss is far from the beam, the thermal noise will be smaller.

The thermal noise is much different from the estimation obtained from the traditional mode expansion when the loss is localizes in a small volume in the mirrors. From the discussion in Chapter 4, the large inhomogeneity of the dissipation causes a lot of large correlations between the fluctuations in the motions of the modes. Since the thermal motion of the mirror includes contributions of many modes, the large correlations cause large discrepancy between the actual thermal noise and the estimation of the traditional mode expansion. Therefore, it is important to investigate this discrepancy quantitatively.

7.2 Distribution and property of loss

The distributions and properties of loss in the mirror are introduced here.

7.2.1 Distribution

There are three categories of the models for the distribution of the dissipation: Homogeneous, Surface, and Point-like².

Homogeneous loss corresponds to the intrinsic dissipation of the material of the mirror. The thermal noise caused by the homogeneous loss is consistent with the evaluation obtained from the traditional mode expansion. Thus, this model is used to check the validity of the estimation of new methods: the direct approaches and the advanced mode expansion.

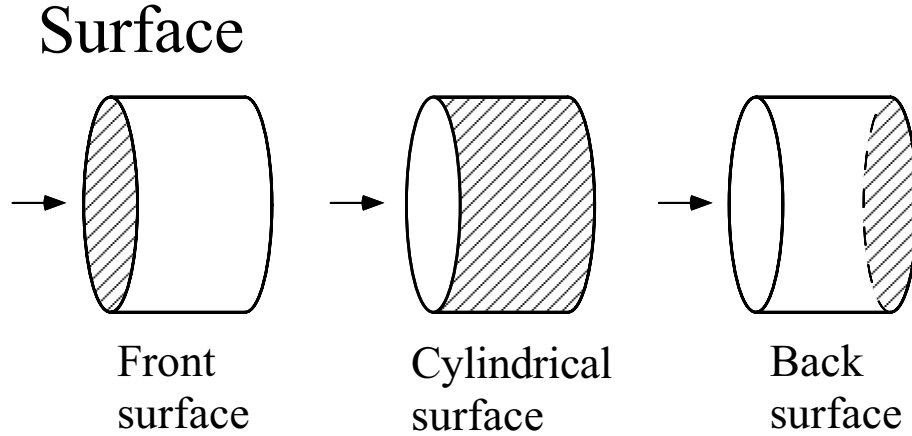


Figure 7.2: The three models in the category of Surface loss. The shadows show surfaces on which the dissipation is localized. The arrows represent the laser beams.

Surface loss models corresponds to the surface damage caused by the inadequate polish or the loss of the dielectric reflective coating or simply surface accumulation of substrate stress. Figure 7.2 shows the three main models of Surface loss distributions considered here. The shadows show the surfaces on which the dissipation can be localized. The arrows represent the laser beams. The left side of Fig.7.2 corresponds to the mirror

²In this thesis, the dissipation induced by wires is not considered.

with the dissipation localized on the beam reflecting surface. This model is called Front surface loss. The right side of Fig.7.2 represents the mirror with loss concentrated on the opposite surface. This distribution is called Back surface loss. The middle of Fig.7.2 shows that the cylindrical surface is damped. This model is called Cylindrical surface loss.

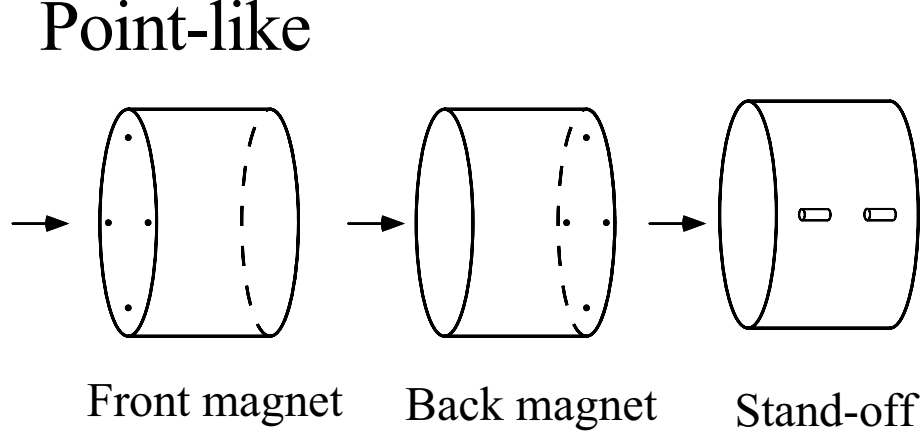


Figure 7.3: The three models in the category of Point-like loss. The black point are the position of the drive magnets. The small cylinders are the suspension stand-off's. The arrows represent the laser beams.

The models in the Point-like loss category correspond to mirror with dissipation localizing at points. The losses in this category are typically caused by attachments to mirrors. For example, the thin magnets are glued on the mirrors back to control their positioning. The stand-off's, thin short bars, are glued on the cylindrical surface to fix the suspension wires to the mirror. The experiments show that the adoption of the stand-off's increases the Q-value of a violin mode [55]. Figure 7.3 shows the three models of Point-like loss distributions taken into consideration here. The arrows represent the laser beams. The left and center parts of Fig.7.3 show the mirrors with drive magnets. The black points indicate the positions at which the magnets are glued. On the left side, magnets are glued on the flat surface illuminated by the beam. This model is called Front magnet. The center part represents magnets glued on the opposite flat surface. This model is called Back magnet. The configuration of the Back magnet model is the same as that of the mirror in TAMA300 and most other interferometers. The right part of Fig.7.3 corresponds to a mirror with suspension stand-off's. The cylinders in Fig.7.3 represent

the stand-off's. This model is called "Stand-off". Since the mirrors in TAMA300 are suspended by two-loop wires, four stand-off's are glued on a mirror. All stand-off's are on a horizontal plane which includes the center of the mirror.

7.2.2 Properties

It is supposed that the mirrors are subject to inhomogeneous structure damping. This dissipation is described using the complex Young's modulus as Eq.(4.18):

$$E = E_0[1 + i\phi(\mathbf{r})], \quad (7.1)$$

where ϕ is the loss angle. The loss angle is independent of the frequency. However, ϕ depends on the position, $\mathbf{r}$. In order to simplify the consideration, ϕ is not zero only in the damped region. In addition, the loss angle is assumed to be a constant in the damped volume. Thus, the loss angle is defined by

$$\phi(\mathbf{r}) = \begin{cases} \phi & (\mathbf{r} \text{ in the damped region}) \\ 0 & (\text{otherwise}) \end{cases}. \quad (7.2)$$

7.3 Method of estimation

In order to study quantitatively the discrepancy between the actual thermal noise of the mirror with inhomogeneous losses and the evaluation of the traditional mode expansion, one of the direct approaches was used. The advanced mode expansion was not adopted because it would be necessary to take many modes into account. The detail of the calculation of the direct approach and the traditional mode expansion are considered here.

7.3.1 Direct approach

In order to calculate the actual thermal noise of mirrors with inhomogeneous losses, a direct approach was used. In this study, Levin's approach was adopted. In this method, the fluctuation-dissipation theorem is expressed as Eq.(5.1):

$$G_X(f) = \frac{2k_B T}{\pi^2 f^2} \frac{W_{\text{loss}}}{F_0^2}. \quad (7.3)$$

The function, G_X , is the power spectrum of the fluctuation of the observed coordinate, X , and f is the frequency. Since the deformation of the surface of the mirror is monitored with the interferometer, X is defined by

$$X(t) = \int_{\text{surface}} u_z(\mathbf{r}, t) P(\mathbf{r}) dS, \quad (7.4)$$

where u_z is the z -component of the displacement, $\mathbf{u}$, of the mirror. The z -axis is parallel to the optical axis. The weighting function, P , represents the beam profile. Since the beam profile is Gaussian and the center of the mirror is on the optical axis in interferometers, the beam profile can be written in the form,

$$P(\mathbf{r}) = \frac{2}{\pi r_0^2} \exp\left(-\frac{2r^2}{r_0^2}\right), \quad (7.5)$$

where r_0 is the beam radius, r is the distance from the optical axis.

The value, W_{loss} , in Eq.(7.3) is the average dissipated power when the oscillatory pressure, $F_0 \cos(2\pi ft) P(\mathbf{r})$, is applied on the mirror. The dissipated power, W_{loss} , is written in the form

$$W_{\text{loss}} = 2\pi f \int \mathcal{E}(\mathbf{r}) \phi(\mathbf{r}) dV, \quad (7.6)$$

where $\mathcal{E}$ is the elastic energy density when the strain is maximum, and ϕ is the loss angle in Eq.(7.1). The elastic energy density, $\mathcal{E}$, is described as

$$\mathcal{E}(\mathbf{r}) = \frac{E_0}{2(1+\sigma)} \left[\sum_{i,j=1}^3 u_{ij}^2 + \frac{\sigma}{1-2\sigma} \left(\sum_{i=1}^3 u_{ii} \right)^2 \right], \quad (7.7)$$

where E_0 is the Young's modulus and σ is the Poisson ratio. The strain tensor, u_{ij} , is defined by

$$u_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (7.8)$$

where u_i is the i -th component of the displacement, $\mathbf{u}$. Since $\phi(\mathbf{r})$ is given as Eq.(7.2), the problem on the estimation of W_{loss} in Eq.(7.6) is the calculation of $\mathcal{E}(\mathbf{r})$. The observation band of the gravitational wave detectors is lower than the first resonant frequency of mirrors. Thus, it is an appropriate approximation that the static pressure, $F_0 P(\mathbf{r})$, is applied on the surface to estimate $\mathcal{E}(\mathbf{r})$.

Since a static pressure is applied on a mirror, the center of the mirror is accelerated uniformly. This accelerated motion is a difficulty in the calculation of the strain of the mirror. Inertial relief is a method to avoid this difficulty. In this method, an inertial force, which cancels the static pressure, is applied on the mirror. Thus, the accelerated motion vanishes. This situation corresponds to that the observer moves at the same acceleration as that of the mirror. Therefore, the strain of the mirror is the same as that of the mirror supported by the pressure, $P(\mathbf{r})$, in an uniform gravitational field.

The analytical evaluation of $\mathcal{E}(\mathbf{r})$ of a finite elastic cylinder is difficult. ANSYS, a common program for the finite element numerical analysis method (FEM), was used. Fig.7.4 shows a solution derived from ANSYS. In the FEM, a mirror is divided into small meshes. Test functions are used to describe motions of these meshes. Parameters of the test functions are derived from the equation of motion and boundary conditions. The dissipated power, W_{loss} , is derived from the elastic energy density, $\mathcal{E}$, of each mesh. In this estimation, the tetrahedron meshes were adopted. The size of the mesh was ten times smaller than the radius of the mirror. The number of the meshes was about twenty thousands. In order to confirm that the estimation of the thermal noise did not depend on how to divide a mirror into meshes, the thermal noise was evaluated from three kinds of meshing. The results proved that the estimated value was independent of the meshing.

7.3.2 Traditional mode expansion

The dissipation is described using the structure damping model. In addition, the observation band is lower than the resonant frequencies of the mirror. Thus, from Eq.(3.77), the formula of the thermal noise derived from the traditional mode expansion is written as

$$G_{\text{mirror}}(f) = \sum_n \frac{4k_B T}{m_n \omega_n^2 Q_n} \frac{1}{\omega}, \quad (7.9)$$

where m_n , ω_n , and Q_n are the effective mass, angular resonant frequency, and Q-value of the n -th mode, respectively. Thus, the problem is reduced to the calculation of m_n , ω_n , and Q_n .

The method proposed by Hutchinson [49] is the most convenient to calculate the parameters in Eq.(7.9). This method is a semi-analytical algorithm to simulate the

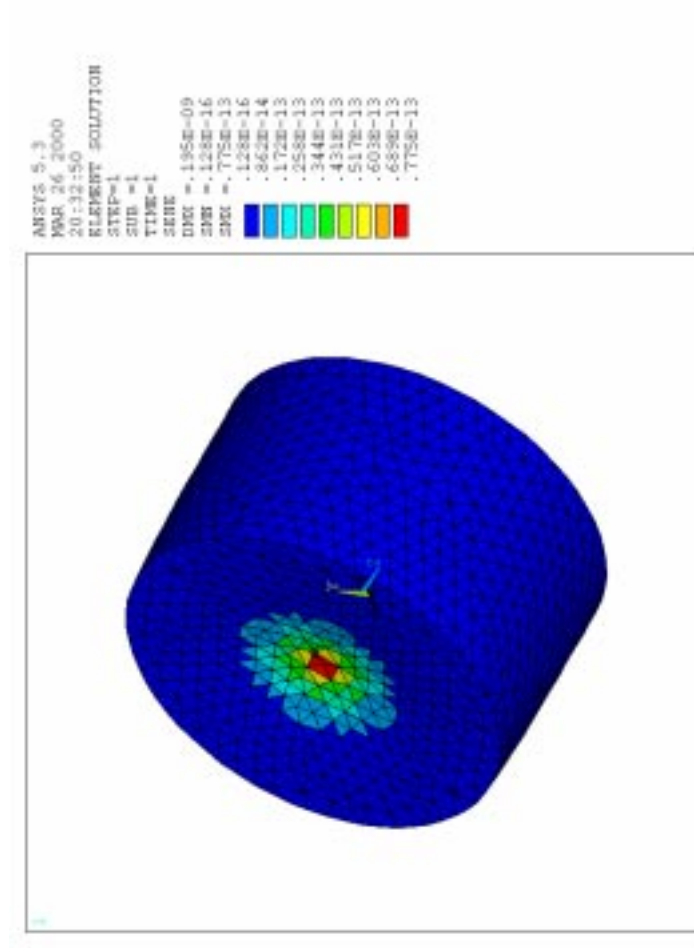


Figure 7.4: A solution derived from ANSYS.

resonant modes of an isotropic elastic cylinder. The resonant angular frequency, ω_n , and the displacement, $\mathbf{w}_n$, of the n -th mode are derived from this method. From Eq.(3.78), the effective mass, m_n , is expressed as

$$m_n = \frac{\int_{\text{volume}} \rho |\mathbf{w}_n(\mathbf{r})|^2 dV}{\left| \int_{\text{surface}} w_{n,z}(\mathbf{r}) P(\mathbf{r}) dS \right|^2}, \quad (7.10)$$

where ρ is the density, $w_{n,z}$ is the z -component of $\mathbf{w}_n$. The Q -value, Q_n , is written as

$$Q_n = \frac{\int \mathcal{E}_n(\mathbf{r}) dV}{\int \mathcal{E}_n(\mathbf{r}) \phi(\mathbf{r}) dV}, \quad (7.11)$$

where $\mathcal{E}_n$ is the elastic energy density of the n -th mode. This elastic energy density, $\mathcal{E}_n(\mathbf{r})$, is derived from substituting the i -th component of the displacement of the n -th mode, $w_{n,i}$, for u_i in Eq.(7.7). Thus, all the parameters in Eq.(7.9) are derived from the Hutchinson's method.

7.4 Results

The results of the quantitative research of the discrepancy between the evaluated thermal fluctuations from Levin's approach and from the traditional mode expansion are summarized here. The mirror in this discussion was the same of the TAMA project. The material was fused silica. The mirror was 50mm in radius and 60mm in height. The center of the beam spot was on the center of the flat surface of the mirror. Since the selection of the Fabry-Perot cavity geometry affects the beam radii at the mirrors, the dependence of the amplitude of the thermal noise on the beam radius were investigated³.

7.4.1 Homogeneous

The thermal noise caused by the homogeneous loss is consistent with the traditional mode expansion. Thus, the mirror with the homogeneous dissipation was used to confirm the validity of Levin's approach and ANSYS. Figure 7.5 shows the estimation of the amplitude of the thermal noise at 100 Hz on the beam radius. Black dots and open squares represent the results of Levin's approach and the traditional mode expansion, respectively. In this calculation, the loss angle, ϕ , is 10^{-6} . The results of Levin's approach agree with those of the traditional mode expansion. Therefore, Levin's approach and ANSYS do not have serious problems.

7.4.2 Surface

The thermal noise caused by the loss concentrated on surfaces are considered here.

³The beam radius of TAMA300 is 8mm at the front mirror and 15mm at the end mirror.

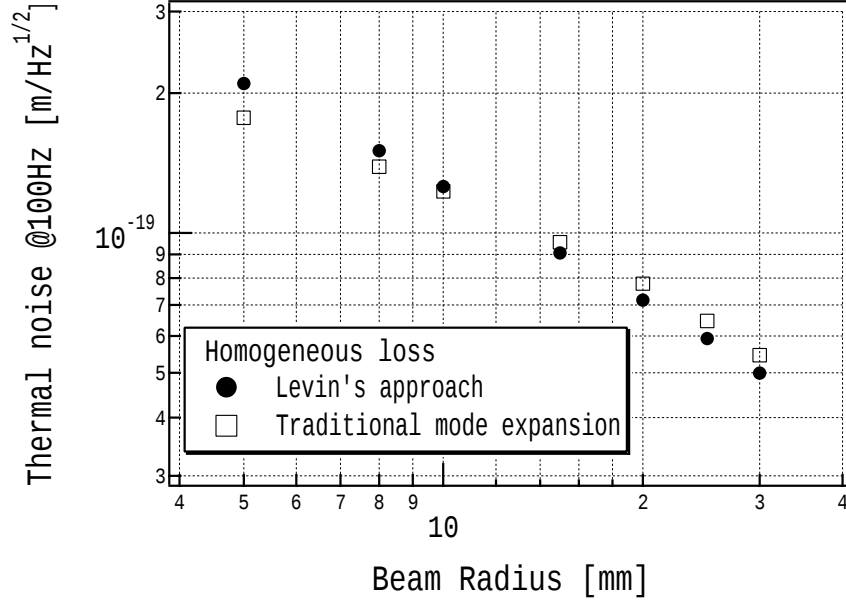


Figure 7.5: Thermal noise caused by homogeneous loss. This graph shows the dependence of the amplitude of the thermal noise at 100 Hz on the beam radius. Black dots and open squares represent the results of Levin's approach and the traditional mode expansion, respectively.

Front and Back surface

The thermal fluctuations of the Front and Back surface models in Fig.7.2 are discussed. In this estimation, the loss layer was $15\mu\text{m}$ in thickness, the same as the dielectric coating in TAMA. In the loss layer, ϕ was assumed to be 1.2×10^{-2} . From this distribution of losses and Eq.(7.11), the resulting Q-values were of the order of 10^5 . These numbers were the same as those observed in the suspended mirror of TAMA [80, 81].

The results are shown in Fig.7.6. Closed and open circles and open squares represent the thermal motions of Front and Back surface models calculated from Levin's approach and the estimation of the traditional mode expansion, respectively. The results of the traditional mode expansion for Front and Back surface models are the same because the Q-values are the same in both cases. These results show that there is the large discrepancy between the values derived from Levin's approach and the traditional mode expansion. The thermal noise of the Front surface model is two or three times larger than the evaluation from the traditional mode expansion. On the contrary, the thermal

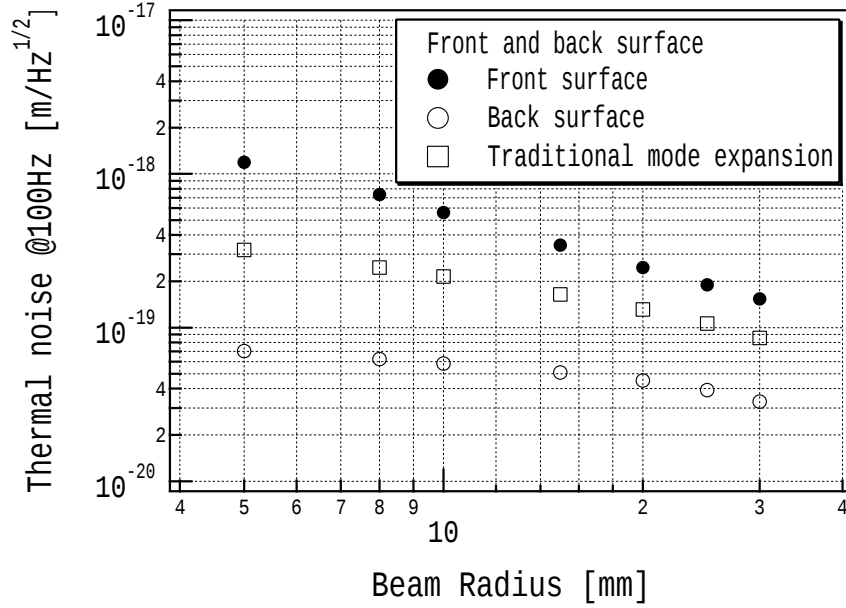


Figure 7.6: Thermal fluctuations caused by losses localized on the mirror's flat surfaces. This graph shows the beam radius dependence of the amplitude of the thermal noise at 100 Hz. Closed and open circles and open squares represent the thermal motions of the Front and Back surface models calculated from Levin's approach and the estimation of the traditional mode expansion, respectively. The results of the traditional mode expansion for the Front and Back surface models are the same because the Q-values are the same in both cases.

motion of Back surface model is three or four times smaller. Levin predicted that the amplitude of the thermal noise of Front surface model is at least $\sqrt{R/r_0}$ times larger than what calculated with the traditional mode expansion [68], where R is the radius of the mirror. Levin's conclusion is consistent with Fig.7.6.

The beam radius dependence of the values estimated from Levin's approach is different from what derived from the traditional mode expansion. The traditional mode expansion predicts that the amplitude is inversely proportional to the square root of the beam radius. However, the amplitude of noise in the Front surface model is inversely proportional to the beam radius. This result is also consistent with Levin's discussion [68]. On the contrary, the amplitude of noise in the Back surface model is independent of the beam radius. This is because the loss is far from the the beam spot and the elastic energy density at the lossy areas is independent of the beam radius.

The thermal fluctuations of the Front and Back surface models calculated from the traditional mode expansion are the same. Instead, Levin's approach predicts that the thermal noise of the Front surface is larger than that of Back surface. This is an evidence which shows that the traditional mode expansion is not a appropriate method when the loss is inhomogeneous. Levin predicted that the thermal noise caused by the loss which is near the beam spot is larger than that caused by the loss which is far from the beam spot [68]. This prediction agrees with Fig.7.6.

Cylindrical surface

The thermal noise of the mirror of the Cylindrical surface loss model are described. In this estimation, the loss layer was 0.1 mm in thickness. In the loss layer, ϕ was assumed to be 1×10^{-3} . From this distribution of the loss and Eq.(7.11), the estimated Q-values were of the order of 10^5 as observed in the suspended mirror of the TAMA [80, 81].

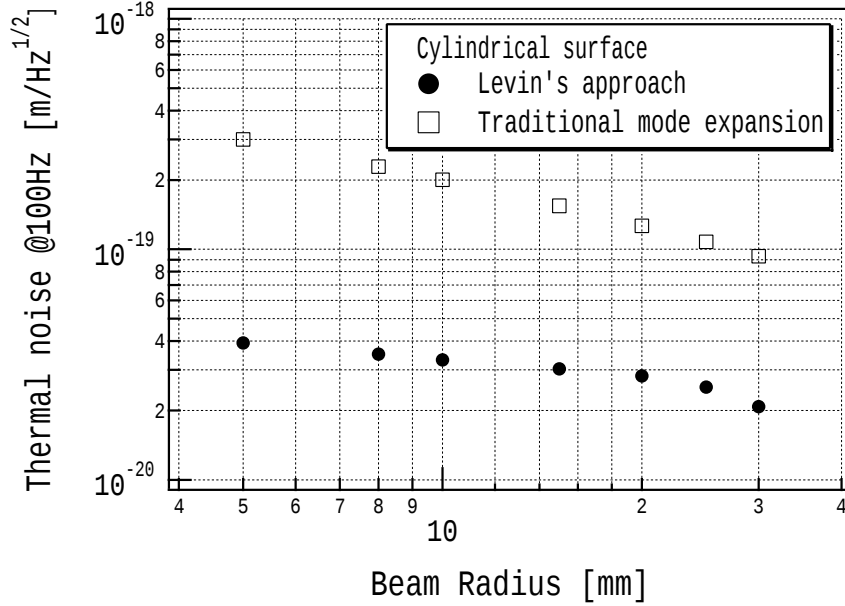


Figure 7.7: Thermal noise caused by the loss localized on cylindrical surface. This graph shows the dependence of the amplitude of the thermal noise at 100 Hz on the beam radius. Dots and open squares represent the results of Levin's approach and the traditional mode expansion, respectively.

The results are shown in Fig.7.7. Dots and open squares represent the estimated values of Levin's approach and the traditional mode expansion, respectively. These results show

that there is a large difference between the values obtained from Levin's approach and the traditional mode expansion. The thermal noise calculated with the Levin's approach is about five times smaller than the evaluation of the traditional mode expansion.

The beam radius dependence of the thermal fluctuations estimated from Levin's approach is different from that obtained from the traditional mode expansion. The traditional mode expansion predicted that the amplitude is inversely proportional to the square root of the beam radius. However, the noise amplitude calculated with Levin's approach is independent of the beam radius. Since the loss is far from the beam spot, the elastic energy density near the loss is independent of the beam radius.

7.4.3 Point-like

The thermal noise caused by point-like losses are considered here.

Front and Back magnet

The thermal motions of the Front and Back magnet models are discussed. In this estimation, the magnets were 45 mm away from the center of the flat surface. The radius of a magnet was 0.5 mm. The distances from the center and the dimensions of the magnets in this model were the same as in TAMA. The thickness of the equivalent loss layer was 0.1 mm. In these damped regions, ϕ was assumed to be 1. From this distribution of losses and Eq.(7.11), the calculated Q-values were of the order of 10^5 , the same as in the mirror with magnets of TAMA [80, 81].

The results are shown in Fig.7.8. Closed and open circles and open squares represent the thermal fluctuations of the Front and Back magnet models evaluated from Levin's approach and the estimation of the traditional mode expansion, respectively. The estimated values derived from the traditional mode expansion in both cases are the same because the Q-values are the same. In the Front and Back magnet models, the amplitude estimated from Levin's approach is about ten and twenty times smaller than the evaluation from the traditional mode expansion, respectively. Thus, the discrepancy between the calculations of Levin's approach and the traditional mode expansion is larger than that of the Surface models in Fig.7.2. The thermal noise induced by the loss localizing in small regions is greatly different from the estimation derived from the traditional mode expansion.

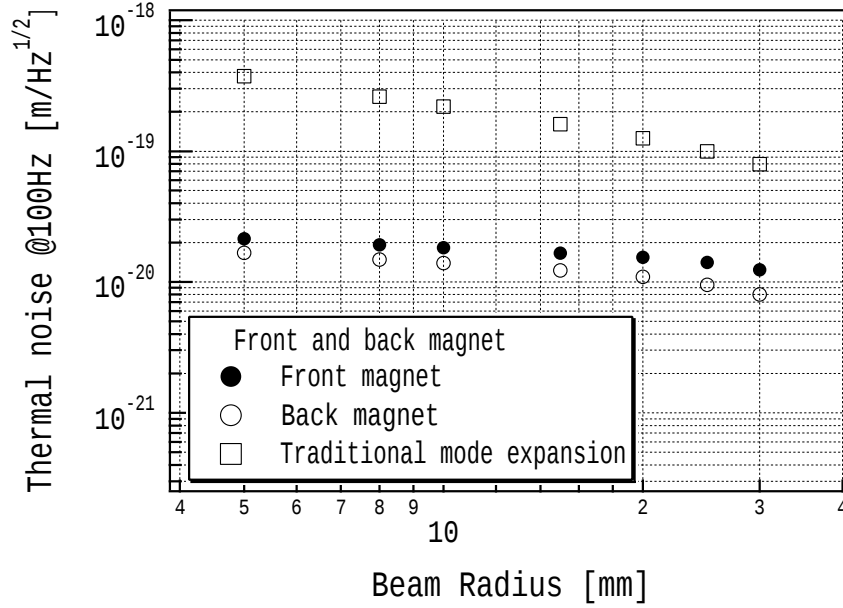


Figure 7.8: Thermal motions of the Front and Back magnet models. This graph shows the dependence of the amplitude of the thermal noise at 100 Hz on the beam radius. Closed and open circles and open squares represent the thermal noise of the Front and Back magnet models evaluated from Levin's approach and the estimation of the traditional mode expansion, respectively. The computations of the traditional mode expansion for the Front and Back magnet models are the same because the Q-values are the same in both cases.

The beam radius dependence of the thermal noise derived from Levin's approach is different from that obtained from the traditional mode expansion. The traditional mode expansion predicts that the amplitudes in both cases are inversely proportional to the square root of the beam radius. However, noise amplitudes calculated with Levin's approach are independent of the beam radius. Since the loss is far from the beam spot, the elastic energy near the loss is independent of the beam radius.

The thermal motions of the Front and Back magnet models calculated from the traditional mode expansion are the same. However, Levin's approach predicts that the thermal noise of the Back magnet model is smaller than that of the Front magnet model. This is an evidence which shows that the traditional mode expansion is not an appropriate method when the loss is inhomogeneous. Levin predicted that the thermal noise caused by the loss which is near the beam spot is larger than that caused by the loss which is

far from the beam spot [68]. Levin's conclusion agrees with Fig.7.8.

Stand-off

The thermal noise caused by the stand-off's are described here. In this calculation, all four stand-off's were on a horizontal plane which included the center of the mirror. The distance between a stand-off and a flat surface of the mirror was 20 mm. The radius and length of a stand-off were 1 mm and 4 mm, respectively. This specification was the same as of TAMA. The thickness of the equivalent loss layer was 0.1 mm. In these damped regions, ϕ was assumed to be 0.08. From this distribution of the loss and Eq.(7.11), the calculated Q-values were of the order of 10^5 , the same as observed with stand-off's of TAMA [80, 81].

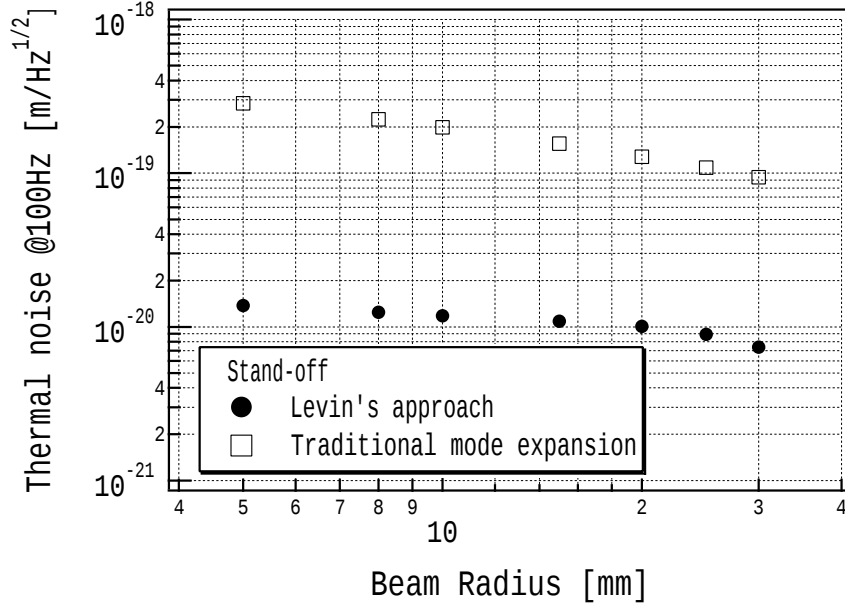


Figure 7.9: Thermal noise caused by the stand-off's. This graph shows the dependence of the amplitude of the thermal noise at 100 Hz on the beam radius. Dots and open squares represent the computations of Levin's approach and the traditional mode expansion, respectively.

The results are shown in Fig.7.9. Dots and open squares represent the results of Levin's approach and the traditional mode expansion, respectively. These results show that there is the large difference between the values obtained from Levin's approach and

the traditional mode expansion. The thermal noise estimated with Levin's approach is about twenty times smaller than the evaluation of the traditional mode expansion. The discrepancy between the calculations of Levin's approach and the traditional mode expansion is larger than that of the Surface models in Fig.7.2. The thermal noise induced by the loss localizing in small regions is greatly different from the estimation derived from the traditional mode expansion.

The beam radius dependence of the thermal motions obtained from Levin's approach is different from what derived from the traditional mode expansion. The traditional mode expansion predicts that the amplitude is inversely proportional to the square root of the beam radius. However, the noise amplitude calculated with Levin's approach is independent of the beam radius. Since the loss is far from the the beam spot, the elastic energy density near the loss is independent of the beam radius.

7.5 Inverse problem

The inverse problem shown in Fig.1.2 is to derive the information of the distribution and properties of the losses separately from the thermal noise. The discussion in Chapter 4 suggests that the measurements of the spectrum of the thermal noise at various points should give clues of this inverse problem. The inverse problem of the mirror with inhomogeneous losses are discussed here.

One of the ways to address the inverse problem is the observation of the motions at the centers of the two flat surfaces. The computations of the traditional mode expansion at both points are the same because the effective mass of all modes at both points are the same. However, if the distribution of the loss is not symmetric, the fluctuation at the center near the localized loss is larger. For example, a mirror with dissipation concentrated on one flat surface is considered. The thermal motions at the centers of the two flat surfaces correspond to those of the Front and Back surface models in Fig.7.6. The thermal motion at the center of the surface on which the loss is concentrated is expected to be ten times larger than that of the opposite surface. Consequently, the comparison between the thermal motions at centers of the two flat surfaces suggests which flat surface is plagued by the losses.

The observation of the fluctuations at points other than the centers of the flat surfaces can indicate the distribution of the losses. As an example, the dependence of the thermal

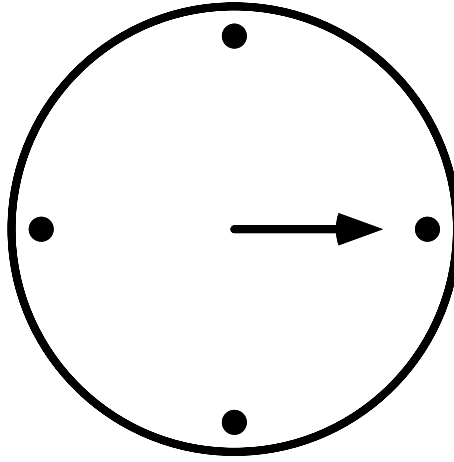


Figure 7.10: The direction of the displacement of the beam spot. The large open circle and small black closed circles represent the mirror and the magnets, respectively. The arrow shows the direction of the displacement of the beam spot.

noise on the distance between the beam spot and the center of the mirror is discussed in the Front and Back magnet models. The direction of the displacement of the beam spot is shown in Fig.7.10. The dependence of the thermal noise on the distance between the beam spot and the center is shown in Fig.7.11. Closed and open circles represent the thermal noise of the Front and Back magnet models, respectively. The parameters are the same as those in Fig.7.8. The beam radius is 10 mm. The results derived from the traditional mode expansion in both cases are the same because the effective masses and Q-values are the same. Nevertheless, the thermal fluctuations in the Front and Back magnet models are different from each other. Since the beam spot and the magnets are on the same surface in the Front magnet model, the thermal noise is substantially larger near the edge. On the contrary, the thermal noise has weak dependence on the distance from the center in the Back magnet model because the distance between the magnets and the beam spot is larger than the thickness of the mirror. Therefore, the observations at various points on the flat surfaces may indicate the distance between the localized loss and the beam spot.

Even without moving the beam spot, it is possible to obtain clues of the inverse problem. The dependence of the thermal noise on the beam radius is investigated. From the consideration in this chapter, the evaluation of the traditional mode expansion is

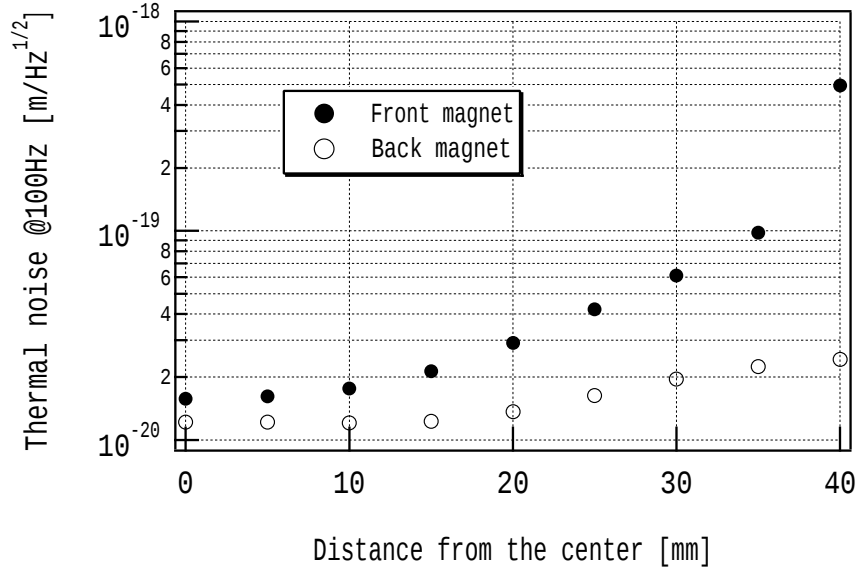


Figure 7.11: Dependence of the thermal noise on the distance between the beam spot and the center of the flat surface in the Front and Back magnet models. Closed and open circles represent the thermal noise of the Front and Back magnet models, respectively. The parameters are the same as those in Fig.7.8. The beam radius is 10 mm. The calculations of the traditional mode expansion in both the models are the same because the Q-values and the effective masses are the same.

always roughly inversely proportional to the square root of the beam radius when the beam spot is on the center of the flat surface. However, from Fig.7.6, the amplitude of the thermal noise caused by the loss concentrated on the surface illuminated by the laser beam is inversely proportional to the beam radius. On the other hand, the thermal motions induced by losses far from the beam spot have weak dependence on the beam radius. Thus, the dependence of the thermal noise on the beam radius indicates the distance between the loss and the beam spot.

From the these consideration, the dependence of the thermal noise on the mirror parameters shows the distribution of the dissipation. Levin's approach and ANSYS are useful methods to connect the dependence of the thermal noise to the distribution of the loss.

Chapter 8

Mirror with inhomogeneous loss: Experiment

In the previous chapter, it was proved theoretically that there is large differences between actual thermal fluctuations of mirrors with inhomogeneous loss and computations of the traditional mode expansion. These results have large impacts on the research of the thermal noise of gravitational wave detectors. Therefore, these theoretical predictions were checked experimentally. The thermal noise of a mechanical model of mirrors with inhomogeneous losses was derived from the measured mechanical response using the fluctuation-dissipation theorem. The results obtained from this experiment supported the calculations in the previous chapter. The details of this experiment are described in this chapter.

8.1 Outline of experiment

In order to test the theoretical predictions in the previous chapter, a mechanical model of the mirror with exaggerated loss characteristics was used because the measurement of the real mirror is too difficult for a small experiment. This model was called drum. The drum was made of aluminum. Permanent magnets were positioned near one of the drum membranes to introduce inhomogeneous dissipation through localized eddy

current. Although the loss caused by the eddy current was large, direct measurement of the thermal noise was still too difficult. Therefore, the thermal motion was evaluated from the application of the fluctuation-dissipation theorem to the measured transfer function. This experimental results were compared with the evaluation of both the advanced and the traditional mode expansions.

8.2 Drum

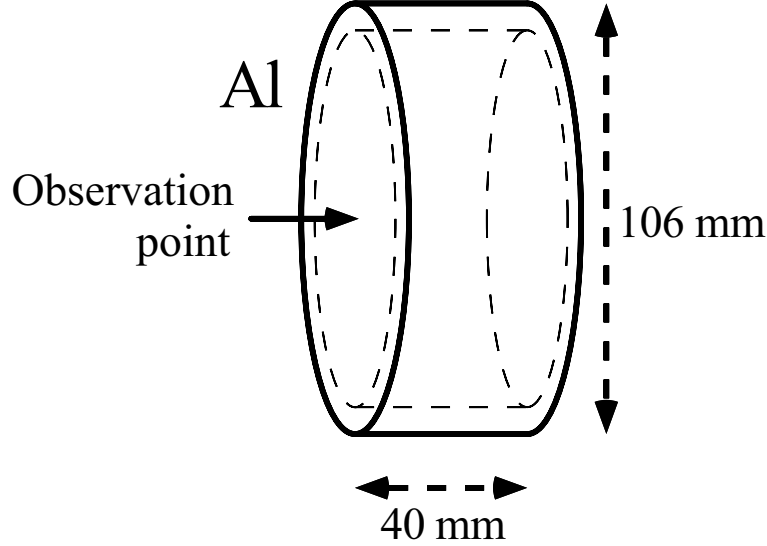


Figure 8.1: Shape and dimensions of the drum. The drum was made of aluminum alloy (Al5056). The shape of the drum was a hollow cylinder: 106 mm in diameter and 40 mm in height.

The drum is shown in Fig.8.1. The drum was made of aluminum alloy (Al5056). The shape of the drum was a hollow cylinder: 106 mm in diameter and 40 mm in height. Figure 8.2 shows the cross section of the drum. The shell of the cylindrical surface was 3 mm thick. The membranes of the drum were 0.5 mm thick. Thus, in practice, in the resonant modes of the drum, only the flat surfaces vibrate. Since the drum was a mechanical model of interferometer mirrors, the thermal fluctuation of the center of the membrane was investigated. The frequency range of this measurement was lower than the first resonant frequency (about 500 Hz) of the drum, just as observation band of the gravitational wave detectors is lower than the first resonant frequency of mirrors.

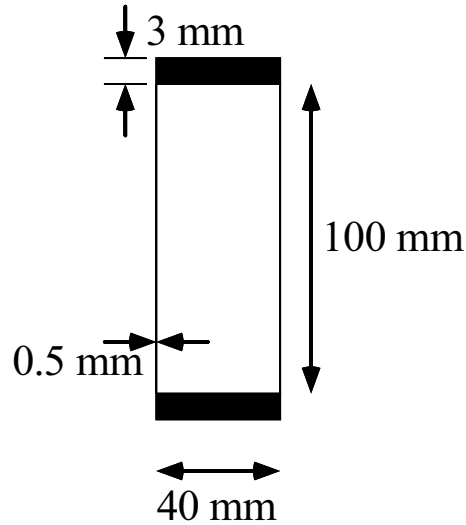


Figure 8.2: Cross section of the drum. The shell of the cylindrical surface was 3 mm thick. The membranes were 0.5 mm thick.

Figure 8.3 shows the configurations of magnets used in order to introduce the desired inhomogeneous dissipation. The arrows represent the probe laser beams. The small squares correspond to the magnets. In both cases, only one side of the drum has large dissipation induced by eddy current. In the left-hand side of Fig.8.3, the magnets are positioned near the flat surface measured by the laser. This configuration is called Front disk. In the right-hand side of Fig.8.3, the other flat surface has large losses. This configuration is called Back disk. The distributions of the losses in Fig.8.3 correspond to the models of Front and Back surface in Fig.7.2. The magnets in this experiment were strong neodymium permanent magnets. The shape of each magnet was a disk: 5 mm in diameter and 2 mm in thickness. The number of magnets was 100 with checker board alternating field direction. A region within a radius of about 30 mm from the center faced these magnets. The gap between the magnets and the drum was about 1mm.

In order to show that the drum was an appropriate mirror model, the thermal fluctuations of the drum and a mirror are compared qualitatively. Figure 8.4 shows the shape of some of the displacements of the mirror and the drum. An oscillatory pressure which has the same profile as the laser beam is applied on the beam spot. The arrows correspond to this pressure. In the upper and lower parts of Fig.8.4, the frequencies of the pressure are at resonance and in the observation band, respectively. The observation band

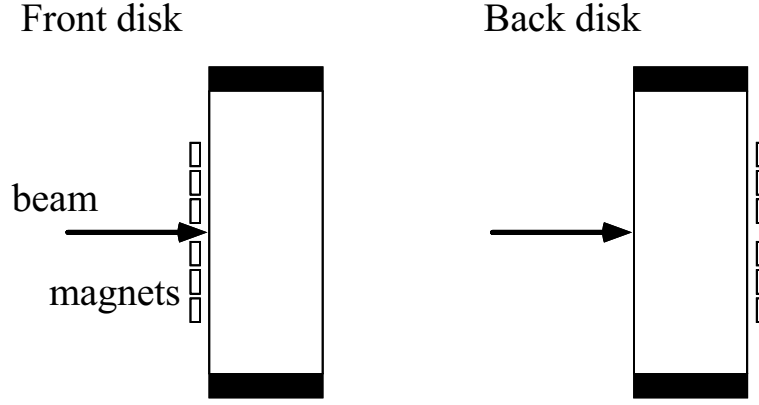


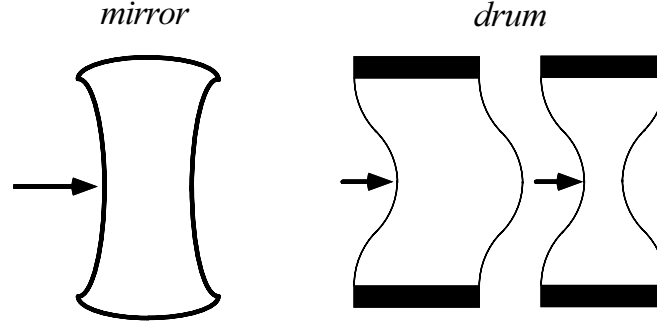
Figure 8.3: Configurations of magnets in order to introduce inhomogeneous dissipation caused by eddy current. The arrows represent beams of laser. Small squares correspond to the magnets. In the left-hand side, the magnets are put near the flat surface which reflects the beam. This configuration is called Front disk. In the right-hand side, another flat surface has large loss. This configuration is called Back disk.

is lower than all resonant frequencies. In the left and right sides of Fig.8.4, vibrations of the mirror and the drum are shown, respectively. It shows that the displacement of the drum is similar to that of the drum. On the resonance, all parts of the mirror and both the disks of the drum vibrate. On the other hand, in the observation band, only the parts of the mirror and the drum near the beam spot vibrate. From this similarity of displacement and the fluctuation-dissipation theorem, the dependence of the thermal noise of the drum on the distribution of the loss is transferred to that of the mirror. In order to further check this consideration, the thermal noise of the drum is evaluated in the next subsection.

8.3 Estimation of thermal noise of the drum

Figure 8.4 suggests that the dependence of the thermal noise of the drum on the distribution of loss is similar to that of mirrors. In order to confirm the validity of this assertion, the thermal motions of the drum were calculated. In practice, in the resonant modes of the drum, only the two thin flat disks oscillate. Thus, the resonant modes of the disk were researched. Based on this research of resonant modes of disks, a simple model was adopted to estimate the thermal noise of the drum. The results of the calculations

(1) At resonant frequency



(2) In observation band ($\ll$ resonant frequency)

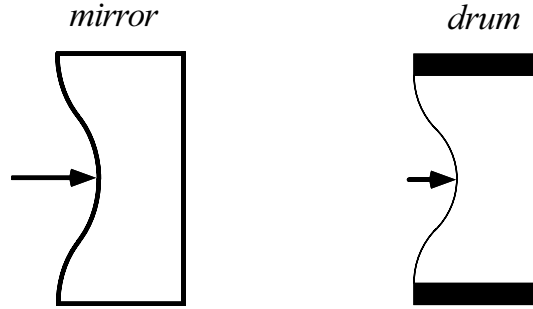


Figure 8.4: The comparison between displacement of a mirror and that of the drum. An oscillatory pressure which has the same profile as the laser beam is applied on the beam spot. The arrows correspond to this pressure. In the upper and lower parts, the frequencies of the pressure are at resonance and in the observation band, respectively. The observation band is lower than all resonant frequencies. In the left and right sides, vibrations of the mirror and the drum are shown, respectively.

proved that the dependence of the thermal noise of the drum on the distribution of loss is similar to that of mirrors. Thus, the drum was an appropriate mechanical model for the mirror behavior. The details of the consideration are described here.

8.3.1 Modes of disk

The resonant modes of the disk are investigated here. The cylindrical coordinates are used. Since the motion of the center was observed, only axially symmetric modes are considered. It is supposed that the edge of the disk is fixed because the shell of the cylindrical surface of the drum is rigid and heavy. From Eq.(3.65), the observed coordinate, X , is defined as

$$X = \int u(r, t) P(r) dS, \quad (8.1)$$

where u is the transverse displacement of the disk and P is the weighting function. Since the beam radius is much smaller than the radius of the disk in this experiment and the motion of the center of the disk is observed, the weighting function, P , can be approximated as

$$P(r) = \delta(r), \quad (8.2)$$

where $\delta(r)$ is the δ -function.

The displacement, w_n , of the n -th mode of the disk is the solution of the eigenvalue problem expressed as [48]

$$-\frac{h^2 E}{12(1 - \sigma^2)} \Delta^2 w_n(r) = -\rho \omega_n^2 w_n(r), \quad (8.3)$$

where h, E, σ, ρ are thickness, Young's modulus, Poisson ratio, and density of the disk, respectively. Since only axial symmetric modes are calculated, The Laplacian is defined by

$$\Delta = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}. \quad (8.4)$$

Since the edge of the disk is fixed, the boundary conditions are written in the form [48]

$$w_n(a) = 0, \quad (8.5)$$

$$\left. \frac{dw_n}{dr} \right|_{r=a} = 0, \quad (8.6)$$

where a is the radius of the disk.

From Eqs.(8.3), (8.5), and (8.6), the n -th angular resonant frequency, ω_n , is given by

$$\omega_n = \frac{\alpha_n^2}{a^2} \sqrt{\frac{E h^2}{12 \rho (1 - \sigma^2)}}, \quad (8.7)$$

where α_n is the n -th solution of the equation expressed as

$$J_1(\alpha_n)I_0(\alpha_n) + J_0(\alpha_n)I_1(\alpha_n) = 0. \quad (8.8)$$

The functions, J_n and I_n , are the n -th Bessel function of the first kind and modified Bessel function of the first kind, respectively. The values of α_n ($n = 1, 2, 3, 4$) are summarized in Table.8.1. The displacement of the n -th mode, w_n , is written in the form

$$w_n(x) = J_0\left(\alpha_n \frac{r}{a}\right) + \Phi_n I_0\left(\alpha_n \frac{r}{a}\right) \quad (8.9)$$

where Φ_n is the function of α_n defined by

$$\Phi_n = -\frac{J_0(\alpha_n)}{I_0(\alpha_n)}. \quad (8.10)$$

The dependence of the displacement, w_n , of the n -th mode on the distance, r , from the center are shown in Fig.8.5.

Table 8.1: α_n of disk.

n	α_n
1	3.19622
2	6.30644
3	9.4395
4	12.5771

Introducing Eqs.(8.2) and (8.9) into Eq.(3.78), the effective masses are obtained. The formula, Eq.(3.78), is rewritten as

$$m_n = \frac{2\pi}{[w_n(0)]^2} \int_0^a \rho h [w_n(r)]^2 r dr. \quad (8.11)$$

The effective masses, m_n ($n = 1, 2, 3, 4$), are summarized in Table.8.2, normalized to the real mass of the disk, m_{disk} .

The observation band in this experiment was lower than the first resonant frequency. Thus, the transfer function and the thermal noise of the disk in the low frequency region are considered. From Eq.(3.76), the transfer function, H_{diskDC} , in the low frequency range can be expressed as

$$H_{\text{diskDC}}(\omega) \approx \sum_n \frac{1}{m_n \omega_n^2}. \quad (8.12)$$

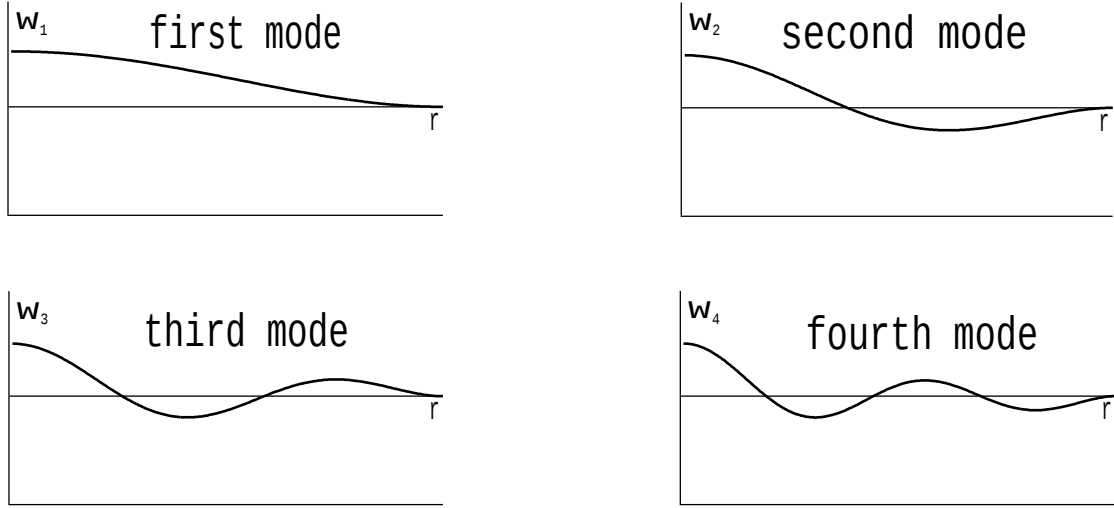


Figure 8.5: Shapes of modes of the disk. The dependence of the displacement, w_n , on the distance, r , from the center are shown.

Table 8.2: m_n/m_{disk} of disk.

n	m_n/m_{disk}
1	0.182834
2	0.101896
3	0.0675435
4	0.0506647

From Eqs.(3.77) and (3.35), the thermal noise, G_{diskDC} , caused by the viscous damping in the low frequency range is written as

$$G_{\text{diskDC}}(f) \approx \sum_n \frac{4k_B T}{m_n \omega_n^3 Q_n}. \quad (8.13)$$

From Eq.(3.36), Q_n in Eq.(8.13) is proportional to ω_n . From Eqs.(8.12), (8.13), (8.7), and Tables.8.1 and 8.2, it can be seen that the transfer function, H_{diskDC} , and the thermal noise, G_{diskDC} , of the disk in the low frequency range are dominated by the contribution of the first mode. Therefore, in this experiment, it was a good approximation to take into account only the contribution of the first mode of the disk.

8.3.2 Thermal noise of drum

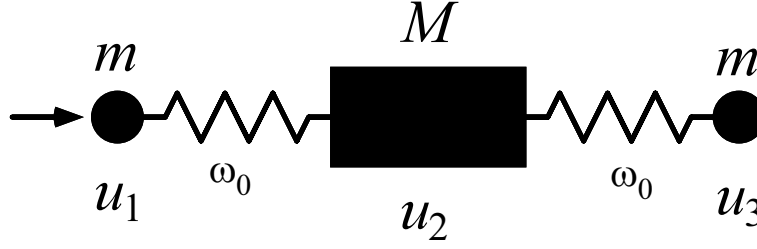


Figure 8.6: A simple model for the drum. The middle part of this model is a heavy rigid body. Equivalent harmonic oscillators are attached to both ends of this body. The heavy rigid body and the harmonic oscillators correspond to the thick cylindrical shell and the thin flat disks of the drum, respectively. The mass of the heavy rigid body, M , is the mass of the cylindrical shell. The angular resonant frequency, ω_0 , of the harmonic oscillators is the angular resonant frequency of the first mode of the disks. In this discussion, it was supposed that the mass of the harmonic oscillator, m , is a half of the mass of the disk. This was because the difference between the first and second resonant frequencies of this model agreed with the measured value of the drum when m was a half of the mass of the disk. The arrow represents the probe laser beam. The friction force caused by eddy current is applied on only a mass point of an oscillator. The values, u_i ($i = 1, 2, 3$), are the displacement of the left, middle, and right mass, respectively.

In resonant motions of the drum, only the flat disks oscillate. Moreover, from the previous discussion, the transfer function and the thermal noise of the drum in the low frequency region are dominated by the contribution of the first mode. Thus, a simple model shown in Fig.8.6 was used to calculate the thermal noise of the drum. To simplify the consideration, no other sources of the dissipation were taken into account except the applied eddy current friction.

The thermal noise of the drum was calculated using this model. The estimation of the advanced mode expansion was compared with that of the traditional mode expansion. In addition, it was confirmed that the results of the advanced mode expansion were consistent with those of the direct approach. The details of these calculations and the results are summarized here.

Estimation of traditional mode expansion

From Eq.(3.65), the observable coordinate, X , is defined as

$$X = \sum_{i=1}^3 P_i u_i, \quad (8.14)$$

where u_i ($i = 1, 2, 3$) are the displacement of the left, middle, and right mass in Fig.8.6, respectively. The parameter, P_i , is the i -th weighting factor. Since the displacement of the left mass in Fig.8.6 are observed, the weighting factor, P_i , is expressed as

$$P_i = \delta_{i1}, \quad (8.15)$$

where δ_{ij} is the Kronecker's δ symbol.

The displacement, $w_{n,i}$, of the i -th mass in the n -th resonant mode is the solution of the eigenvalue problem expressed as

$$\begin{pmatrix} -m\omega_0^2 & m\omega_0^2 & 0 \\ m\omega_0^2 & -2m\omega_0^2 & m\omega_0^2 \\ 0 & m\omega_0^2 & -m\omega_0^2 \end{pmatrix} \begin{pmatrix} w_{n,1} \\ w_{n,2} \\ w_{n,3} \end{pmatrix} = -\omega_n^2 \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} w_{n,1} \\ w_{n,2} \\ w_{n,3} \end{pmatrix}. \quad (8.16)$$

The n -th angular resonant frequency, ω_n , is given by¹

$$\omega_1 = \omega_0, \quad (8.17)$$

$$\omega_2 = \omega_0 \sqrt{1 + \frac{2m}{M}}. \quad (8.18)$$

The displacement of the n -th mode, $w_{n,i}$, is written in the form

$$\begin{pmatrix} w_{1,1} \\ w_{1,2} \\ w_{1,3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad (8.19)$$

$$\begin{pmatrix} w_{2,1} \\ w_{2,2} \\ w_{2,3} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}. \quad (8.20)$$

The shapes of the modes of the drum are shown in Fig.8.7.

¹There is another solution, $\omega_n = 0$. However, this solution is neglected in this discussion because this mode corresponds to the translation motion of the drum.

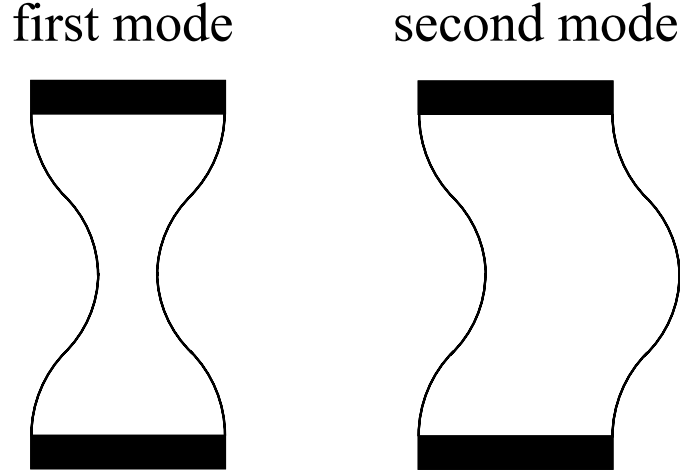


Figure 8.7: Shapes of the first and second resonant modes of the drum.

From Eqs.(8.15), (8.19), (8.20), and (3.78), the effective masses are obtained. The expression, Eq.(3.78), is rewritten as

$$m_n = \frac{1}{[w_{n,1}]^2} [m(w_{n,1})^2 + M(w_{n,2})^2 + m(w_{n,3})^2]. \quad (8.21)$$

The effective masses, m_n ($n = 1, 2$), are reduced as

$$m_1 = 2m, \quad (8.22)$$

$$m_2 = 2m \left(1 + \frac{2m}{M} \right). \quad (8.23)$$

In this experiment, the drum is damped strongly by eddy current. The loss angle, ϕ_n , is described as

$$\phi_n(\omega) = \frac{\omega}{\omega_n Q_n}. \quad (8.24)$$

The Q-value, Q_n , of the n -th mode is evaluated from Eqs.(8.27), (8.28), (8.19), (8.20), and (4.7). The results are described as

$$Q_1 = \frac{2\omega_1}{\Gamma}, \quad (8.25)$$

$$Q_2 = \frac{2\omega_2}{\Gamma} \left(1 + \frac{2m}{M} \right). \quad (8.26)$$

The definition of Γ is in Eq.(8.28).

Estimation of advanced mode expansion

The formula of the coupling coefficient, α_{12} , is derived from Eq.(6.14) because the source of the loss in the drum is only eddy current. From Eq.(8.15), Eq.(6.14) is rewritten as

$$\alpha_{nk} = \frac{\omega}{w_{n,1}w_{k,1}}[m\Gamma_1w_{n,1}w_{k,1} + M\Gamma_2w_{n,2}w_{k,2} + m\Gamma_3w_{n,3}w_{k,3}]. \quad (8.27)$$

The parameter, Γ_i , in Eq.(8.27) represents the friction force applied on the i -th mass in Fig.8.6. The friction force caused by the eddy current is the product of the mass, Γ_i , and the velocity. The coefficient, Γ_i , in this experiment is written as

$$\Gamma_i = \begin{cases} \Gamma\delta_{i1} & (\text{in Front disk}) \\ \Gamma\delta_{i3} & (\text{in Back disk}) \end{cases}. \quad (8.28)$$

Inserting Eqs.(8.28), (8.19), and (8.20) into Eq.(8.27), α_{12} is obtained:

$$\alpha_{12} = \begin{cases} |\alpha_{12}|_{\max} = \sqrt{m_1\omega_1^2\phi_1m_2\omega_2^2\phi_2} & (\text{in Front disk}) \\ -|\alpha_{12}|_{\max} = -\sqrt{m_1\omega_1^2\phi_1m_2\omega_2^2\phi_2} & (\text{in Back disk}) \end{cases}. \quad (8.29)$$

The value, $|\alpha_{12}|_{\max}$, is the maximum of the absolute value of α_{12} in Eq.(4.40).

Estimation of direct approach

The thermal noise of the drum is calculated using Tsubono's approach. Since the observation point is at the end of the model in Fig.8.6, the boundary condition includes the generalized force applied on the observation point. Thus, the generalized force is not treated as the external force, i.e. the transfer matrix is the same type as that of the matrix in Eq.(5.13).

The calculation is based on the transfer matrices of a mass point and a spring. The transfer matrix, $\mathbf{T}_{\text{mass}}(m, \Gamma)$, of the mass point with eddy current damping (The parameter, m , is the mass and Γ is in Eq.(8.28).) is described as [63, 72, 73]

$$\begin{pmatrix} X \\ F \end{pmatrix}_1 = \begin{pmatrix} 1 & 0 \\ -m\omega^2 + im\Gamma\omega & 1 \end{pmatrix} \begin{pmatrix} X \\ F \end{pmatrix}_0. \quad (8.30)$$

The transfer matrix, $\mathbf{T}_{\text{spring}}(m, \omega_0)$, of the spring (The value, m , is the mass and ω_0 is the resonant angular frequency.) is similarly written as²

$$\begin{pmatrix} X \\ F \end{pmatrix}_1 = \begin{pmatrix} 1 & \frac{1}{m\omega_0^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ F \end{pmatrix}_0. \quad (8.31)$$

The transfer matrix, $\mathbf{T}_{\text{total}}$, of the total system is derived from Eqs.(8.30) and (8.31). The matrix of the total system is written in the form

$$\mathbf{T}_{\text{total}} = \begin{cases} \mathbf{T}_{\text{mass}}(m, \Gamma) \mathbf{T}_{\text{spring}}(m, \omega_0) \mathbf{T}_{\text{mass}}(M, 0) \mathbf{T}_{\text{spring}}(m, \omega_0) \mathbf{T}_{\text{mass}}(m, 0) & \text{(in Front disk)} \\ \mathbf{T}_{\text{mass}}(m, 0) \mathbf{T}_{\text{spring}}(m, \omega_0) \mathbf{T}_{\text{mass}}(M, 0) \mathbf{T}_{\text{spring}}(m, \omega_0) \mathbf{T}_{\text{mass}}(m, \Gamma) & \text{(in Back disk)} \end{cases}. \quad (8.32)$$

An end of this model is the observation point. The other end is free. The state vectors of both ends are connected by the total transfer matrix, $\mathbf{T}_{\text{total}}$. This relation is expressed as

$$\begin{pmatrix} X \\ F \end{pmatrix}_{\text{obs}} = \mathbf{T}_{\text{total}} \begin{pmatrix} X \\ 0 \end{pmatrix}_{\text{free}}, \quad (8.33)$$

where obs and free represent the observation point and the free end, respectively. The matrix, $\mathbf{T}_{\text{total}}$, is rewritten as

$$\mathbf{T}_{\text{total}} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}. \quad (8.34)$$

From Eq.(8.34), the transfer function, $H(\omega)$, is derived as

$$H(\omega) = \left. \frac{X}{F} \right|_{\text{obs}} = \frac{t_{11}}{t_{21}}. \quad (8.35)$$

The thermal noise spectrum is calculated from the application of the fluctuation-dissipation theorem, Eq.(3.9), to this transfer function, Eq.(8.35).

Results of the estimation

From above consideration, the thermal noise of the drum is evaluated using each method. The parameters of the disk of the drum are summarized in Table.8.3. Also, the parameters of the drum are summarized³ in Table.8.4. The distribution of the loss

Table 8.3: Parameters of disk in drum.

parameter	
radius (a)	50 mm
thickness (h)	0.5 mm
density (ρ)	2.67 g/cm ³
Young's modulus (E)	7.03×10^{10} Pa
Poisson ratio (σ)	0.345

Table 8.4: Parameters of drum.

parameter	
mass of the rigid shell (M)	0.183 kg
mass of the harmonic oscillator (m)	5.24 g
resonant frequency of the harmonic oscillator (f_0)	513 Hz
strength of damping (Γ)	38.1 /sec

Table 8.5: f_n, m_n, Q_n of the drum.

n	f_n	m_n	Q_n
1	513 Hz	10.5 g	169
2	528 Hz	11.1 g	184

induced by eddy current is described as Eq.(8.28). The thermal noise is estimated from these data.

The parameters in the mode expansion, resonant frequency (f_n), effective mass (m_n), Q-value (Q_n), and coupling coefficient (α_{12}), were calculated. The resonant frequencies, the effective masses, and Q-values of the drum are summarized in Table.8.5. The coupling coefficient, α_{12} , was obtained from Eq.(8.29) and Table.8.5.

The thermal noise obtained from each method are shown in Fig.8.8. The upper and

²If the original loss of the spring is considered, the complex spring constant, $m\omega_0^2(1 + i\phi)$, is substituted for the spring constant, $m\omega_0^2$, in Eq.(8.31).

³The mass of the harmonic oscillator, m , is a half of the mass of the disk.

lower graphs in Fig.8.8 show the evaluation of the Front and Back disk models in Fig.8.3, respectively. The thick and thin dashed lines are the computations based on the advanced and traditional mode expansions, respectively. The solid line represents the calculation of the direct approach. Figures 8.8 and 7.6 prove that the dependence of thermal noise of the drum on the distribution of the loss is similar to that of the mirror. In the Front disk model, the thermal noise estimated from the advanced mode expansion is larger than that from the traditional mode expansion. On the contrary, in the Back disk model, the thermal noise evaluated from the advanced mode expansion is smaller than that from the traditional mode expansion. Consequently, the drum is an appropriate mechanical model of mirrors.

The estimation of the direct approach agrees with that of the advanced mode expansion. These results prove that the direct approach is consistent with the advanced mode expansion.

8.4 Experimental method

Apparatus and methods in this experiment are almost the same as those in the measurement⁴ of the leaf spring in Chapter 6. The transfer function, $H(\omega)$, of the drum was measured in order to evaluate the thermal noise using the fluctuation-dissipation theorem. In this measurement, a force was applied on the drum using an electrostatic actuator. The motion of the drum caused by the actuator was monitored by a Michelson interferometer. Q-values were also measured for the estimation of the mode expansion. The details of the Experimental apparatus and the methods of measurement of the transfer function and Q-values are introduced here.

8.4.1 Experimental apparatus

A schematic view of the experimental apparatus is shown in Fig.8.9. The drum membrane was one of the end mirrors of a differential Michelson interferometer. In order to measure the transfer function, $H(\omega)$, the drum was excited by an electrostatic actuator

⁴Since the principle of the apparatus and methods are described in Chapter 6, they are omitted from the explanations.

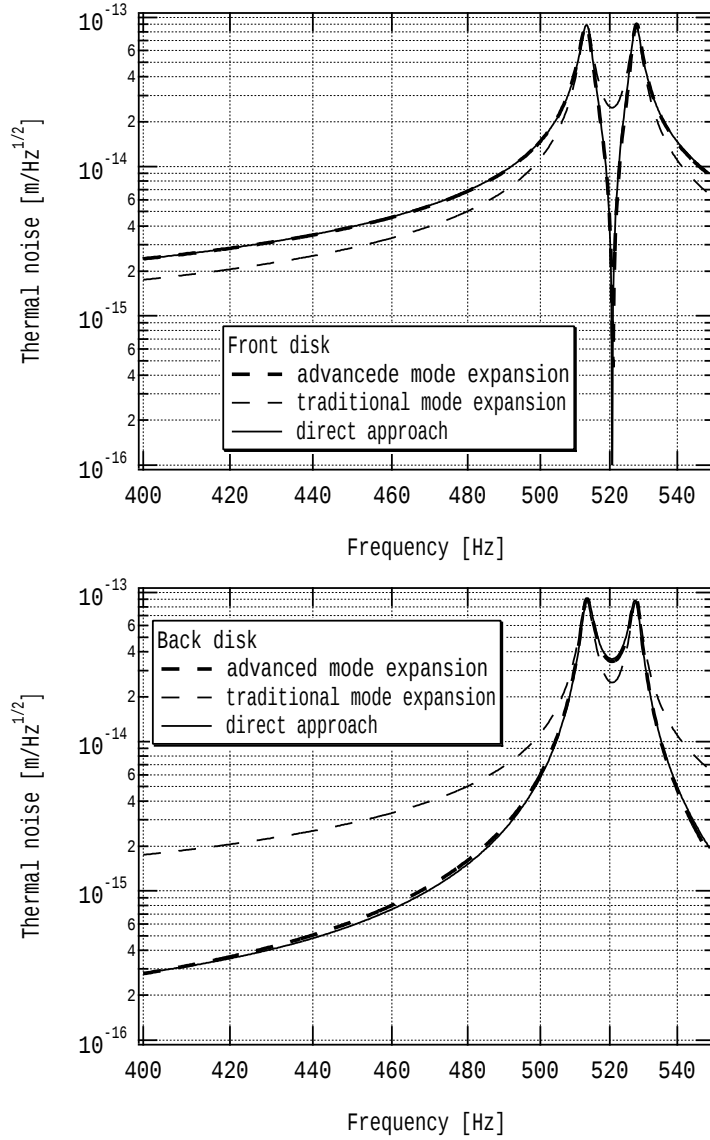


Figure 8.8: The evaluation of the thermal noise of the drum. The upper and lower graphs show the evaluation of the Front and Back disk models in Fig.8.3, respectively. The thick and thin dashed lines are the computations of the advanced and traditional mode expansions. The solid line represents the calculation of the direct approach.

placed inside it. All apparatus, except for a laser source, were put in a vacuum chamber. The details of each experimental apparatus are introduced here.

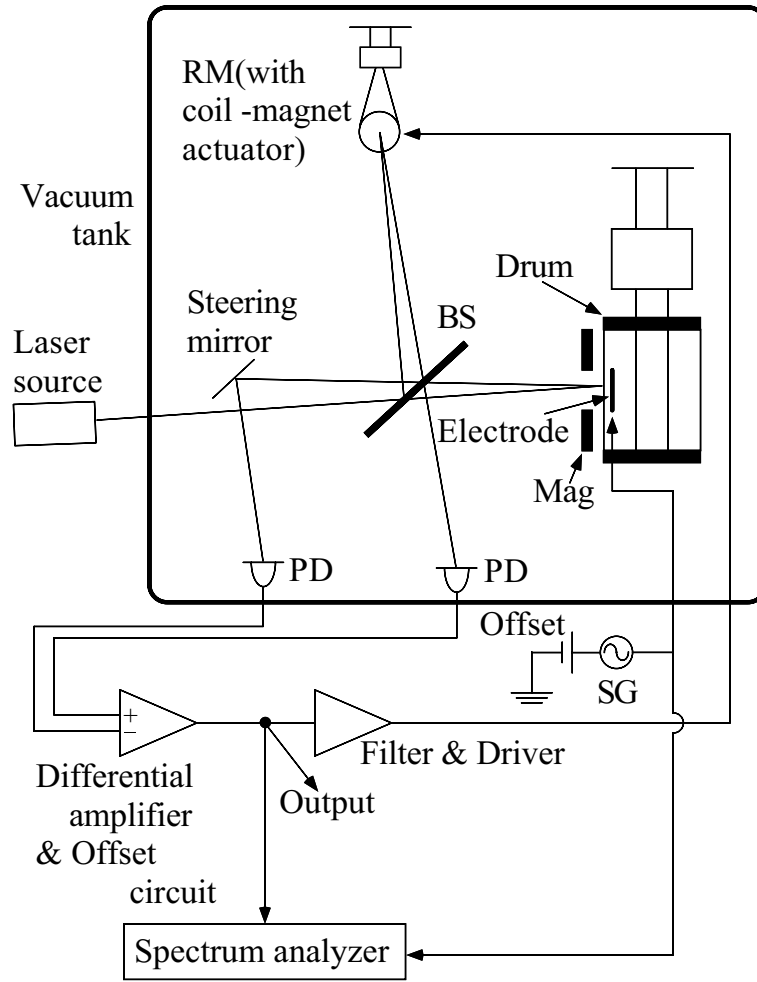


Figure 8.9: Schematic view of the experimental apparatus. BS, RM, Mag, PD, and SG stand for the beam splitter, the reference mirror, the magnets, the photo detector, and the signal generator, respectively. The sensor was a differential Michelson interferometer. To keep the interferometer at an operation point, the output signal of the interferometer was used to control the position of the reference mirror. The drum and the reference mirror were suspended for seismic isolation. An electrostatic actuator was used to excite the drum. All apparatus, except for the laser source, were put in a vacuum chamber.

Michelson interferometer

The schematic configuration of the interferometer used in this experiments is shown in Fig.8.9. This Michelson interferometer was almost the same as that used in the measurement of the leaf spring in Chapter 6. The light source was helium-neon laser ($\lambda=633$

nm). The main part of the interferometer was comprised of a beam splitter, a drum, and a reference mirror with coil-magnet actuators. The drum and the reference mirror were suspended for seismic isolation. Moreover, since mirrors in interferometric gravitational wave detectors are also suspended, the treatment of the drum in this measurement was the same as that of mirrors in detectors. The suspension systems for the drum and the reference mirror were double pendulums. The principle of the suspension system is the same as that of the stack used in the measurement of the leaf spring. From Eq.(6.50), the seismic vibration in the frequency range higher than the resonant frequencies of the suspension system is not transferred to the drum and to the reference mirror. In order to suppress the motion at the resonant frequencies of the suspension, the intermediate masses of the double pendulums were damped by eddy current induced by strong permanent magnets [29, 86, 87]. The disks of the drum was polished or coated with aluminum by vacuum evaporation to increase its reflectivity. The fringe contrast of the interferometer was about 30%. The output signal of the interferometer was sent to the coil-magnet actuators of the reference mirror through filters and drivers to keep the interferometer at an operation point. The principle of the coil-magnet actuator is that the magnetic field induced by the coils applies a force on magnets glued on the reference mirror. Thus, the force applied on the reference mirror is proportional to current in the coils. The output signal of the interferometer and the voltage of the signal generator for the electrostatic exciter were recorded by a spectrum analyzer to evaluate the transfer function, $H(\omega)$.

Electrostatic actuator

The schematic configuration of the electrostatic actuator used in this experiment is shown in Fig.8.9. This electrode was mounted on the inside of the drum and faced the center of the disk. Thus, the actuator applied force on the observation point without intercepting the laser beam. The size of the electrode was 8 mm $\times$ 8mm. The gap between the electrode and the disk of the drum was about 1 mm.

The actuator was calibrated based on Eq.(6.47). In this experiment, H_{DC} was calculated as the static response of the disk. It was assumed that the edge of the disk was fixed because the disk was fixed on the rigid heavy cylindrical shell. Thus, H_{DC} was described as [48]

$$H_{DC} = \frac{3(1 - \sigma^2)a^2}{4h^3\pi E}, \quad (8.36)$$

where a, h, E , and σ are radius, thickness, Young's modulus, and Poisson ratio of the disk⁵.

Vacuum chamber

All apparatus, except for the helium-neon laser source, were put in a vacuum tank. The vacuum chamber was evacuated by a rotary pump; the pressure was about 5 Pa. This pump was stopped during measurement. When the pressure was smaller than 20 Pa, the Q-values of the drum without eddy current damping was independent of the pressure. Thus, the loss caused by the residual gas was smaller than the original loss of the drum.

8.4.2 Measurement of the transfer function

Using the electrostatic actuator, the oscillatory force was applied at the center of the disk of the drum. The vibration of this center was observed with the interferometer. The voltage of the actuator and the output of the interferometer were recorded by the spectrum analyzer. From this measurement, we obtained the transfer function of the drum, $H(\omega)$, which is the ratio of the displacement, $\tilde{X}$, to the applied force, $\tilde{F}$. Equation (6.59) was used in order to remove the effects of the feedback of the interferometer. The measured open loop gain, G , of this experiment is shown in Fig.8.10.

8.4.3 Measurement of Q-values

In the estimation of the mode expansion, measured Q-values were used. These Q-values of the drum were derived from the decay time of the resonant motions excited by the electrostatic actuator and Eq.(3.49). The results are summarized in Table.8.8.

⁵The order of magnitude of H_{exciter} evaluated from Eqs.(6.47) and (8.36) was the same as that of the estimation obtained from Eq.(6.44).

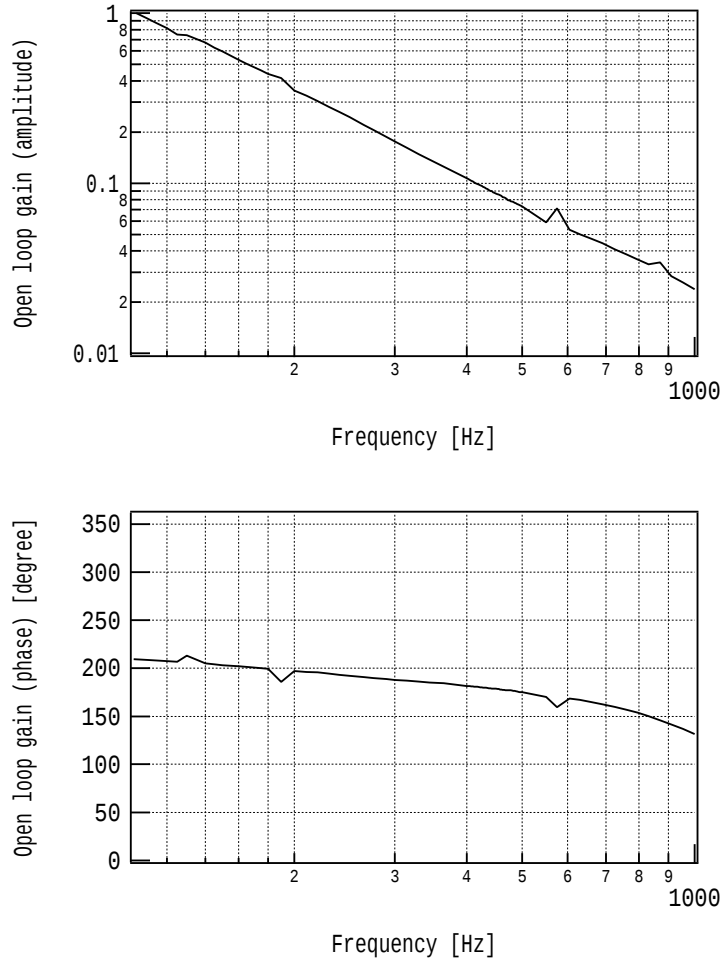


Figure 8.10: The open loop gain, G , in this experiment. The upper and lower graphs represent the amplitude and the phase of G , respectively.

8.5 Results

The thermal noise of the drum was derived from the measured transfer function and compared with the calculations of the advanced and traditional mode expansions. The experimental results agree with the advanced mode expansion predictions. Therefore, this experiment suggested that the estimation of the thermal noise of the mirror with inhomogeneous loss in the previous chapter is correct. Parameters for the mode expansion and the results of this experiment are summarized here.

8.5.1 Parameters for the estimation

In order to obtain the spectrum from the advanced and traditional mode expansions, angular resonant frequencies (ω_n), effective masses (m_n), Q-values (Q_n), of the first and second modes, and the coupling coefficient, α_{12} , were estimated from the experiment. In the simple model of the drum in Fig.8.6, it was supposed that the resonant frequencies of two harmonic oscillators (disks) were the same. However, in reality, there was a slight difference between the two resonant frequencies. Since the parameters, ω_n, m_n, Q_n , and α_{12} , depend strongly on this difference between the resonant frequencies of the disks, these parameters were obtained from the experiment.

The angular resonant frequencies, ω_n , were measured directly. The effective masses, m_n , were evaluated from the measured anti-resonant frequency between the first and second modes and the calculated absolute value of the transfer function in the low frequency range. At the anti-resonant frequency, the absolute value of the transfer function is the local minimum. The anti-resonant angular frequency, ω_{anti} , is expressed as Eq.(6.60):

$$\omega_{\text{anti}} = \sqrt{\frac{m_2}{m_1}} \omega_2. \quad (8.37)$$

Since the electrostatic actuator was calibrated using Eq.(8.36), the transfer function in the low frequency region, Eq.(6.45), is rewritten as

$$H_{\text{DC}} = \frac{3(1 - \sigma^2)a^2}{4h^3\pi E} = \frac{1}{m_1\omega_1^2} + \frac{1}{m_2\omega_2^2}. \quad (8.38)$$

Inserting the measured resonant and anti-resonant frequencies and the calculated transfer function in the low frequency range into Eqs.(8.37) and (8.38), m_n were derived. The Q-values were calculated from the measured decay time of the resonant motion and Eq.(3.49). In order to evaluate the contribution of the eddy current damping to the Q-values, the Q-values of the n -th mode of the drum with ($Q_{\text{total},n}$) and without ($Q_{\text{ori},n}$) the eddy current damping were measured. The contribution of the eddy current damping, $Q_{\text{eddy},n}$, is written in the form

$$\frac{1}{Q_{\text{eddy},n}} = \frac{1}{Q_{\text{total},n}} - \frac{1}{Q_{\text{ori},n}}. \quad (8.39)$$

The resonant frequencies⁶, effective masses⁷, and Q-values derived from the experiment

⁶The measured resonant frequencies in Table.8.6 are smaller than the calculated resonant frequencies in Table.8.5. Since the dimensions of the drum was measured precisely, the Young's modulus is assumed to be smaller than that in Table.8.3.

⁷The effective masses derived from the measurements in Table.8.7 are about the same as the calculated

are summarized in Tables.8.6, 8.7, and 8.8. Table.8.8 shows that the original loss of the drum was comparable to the eddy current damping. Thus, it is expected that the discrepancy between the actual thermal noise and the evaluation of the traditional mode expansion will be smaller than that in Fig.8.8 where the original loss of the drum is not taken into account.

Table 8.6: Resonant frequencies of the drum (measurement).

	f_1	f_2
Front disk	455.0 Hz	472.5 Hz
Back disk	455.6 Hz	471.2 Hz

Table 8.7: Effective masses of the drum (measurement).

	m_1	m_2
Front disk	3.11 g	5.53 g
Back disk	3.03 g	5.83 g

Table 8.8: Q-values of the drum (measurement).

	$Q_{\text{total},1}$	$Q_{\text{total},2}$	$Q_{\text{ori},1}$	$Q_{\text{ori},2}$	$Q_{\text{eddy},1}$	$Q_{\text{eddy},2}$
Front disk	56	82	140	165	93	163
Back disk	114	85	180	190	310	154

The loss angle, $\phi_n(\omega)$, of the n -th mode of this drum is the sum of the loss angles of the eddy current damping and the original loss of the drum,

$$\phi_n(\omega) = \phi_{\text{ori},n}(\omega) + \phi_{\text{eddy},n}(\omega). \quad (8.40)$$

The term, $\phi_{\text{eddy},n}$, is the loss angle of the n -th mode of the eddy current damping:

$$\phi_{\text{eddy},n} = \frac{\omega}{\omega_n Q_{\text{eddy},n}}. \quad (8.41)$$

values in Table.8.5.

The Q -values, $Q_{\text{eddy},n}$, of the eddy current damping are summarized in Table.8.8. The loss angles of the original loss, $\phi_{\text{ori},n}$, are explained in the next subsection. This original loss was homogeneous. Thus, the original dissipation has no effects on the coupling coefficient, α_{12} . The formula of α_{12} , Eq.(8.29), are rewritten as

$$\alpha_{12} = \begin{cases} \sqrt{m_1\omega_1^2\phi_{\text{eddy},1}m_2\omega_2^2\phi_{\text{eddy},2}} & (\text{in Front disk}) \\ -\sqrt{m_1\omega_1^2\phi_{\text{eddy},1}m_2\omega_2^2\phi_{\text{eddy},2}} & (\text{in Back disk}) \end{cases}. \quad (8.42)$$

Inserting the values in Tables.8.6, 8.7, 8.8 into Eq.(8.42), the coupling coefficient, α_{12} , was derived.

8.5.2 Original loss of drum

From Table.8.8, the original loss of the drum was as large as the dissipation induced by eddy current. In order to investigate the original dissipation of the drum, the transfer function of the drum without the eddy current damping was measured. From this measured transfer function, the thermal noise without the eddy current damping was derived using the fluctuation-dissipation theorem. This result is shown in Fig.8.11. The open circles are the spectrum evaluated from the measured transfer function. The solid line in Fig.8.11 is the thermal noise induced by the thermoelastic damping in the disks of the drum. Figure 8.11 proves that the original loss of the drum was dominated by the thermoelastic damping.

The details of the thermoelastic damping is discussed here. The estimation of the thermal noise caused by the thermoelastic damping in Fig.8.11 was derived from the traditional mode expansion. This was because the strength of the thermoelastic damping in both the disks were the same. The loss angles of the original loss of the drum, $\phi_{\text{ori},n}$, are the same as that of the thermoelastic damping in disks [88]. Thus, the loss angles, $\phi_{\text{ori},n}$, is described as Eq.(3.58):

$$\phi_{\text{ori},n}(\omega) = \Delta \frac{\omega\tau}{1 + (\omega\tau)^2}. \quad (8.43)$$

When the displacement of the disk is axially symmetric, the strength of the damping, Δ , is expressed as [88]

$$\Delta = \frac{E\alpha^2T}{C\rho} \frac{1 + \sigma}{1 - \sigma}, \quad (8.44)$$

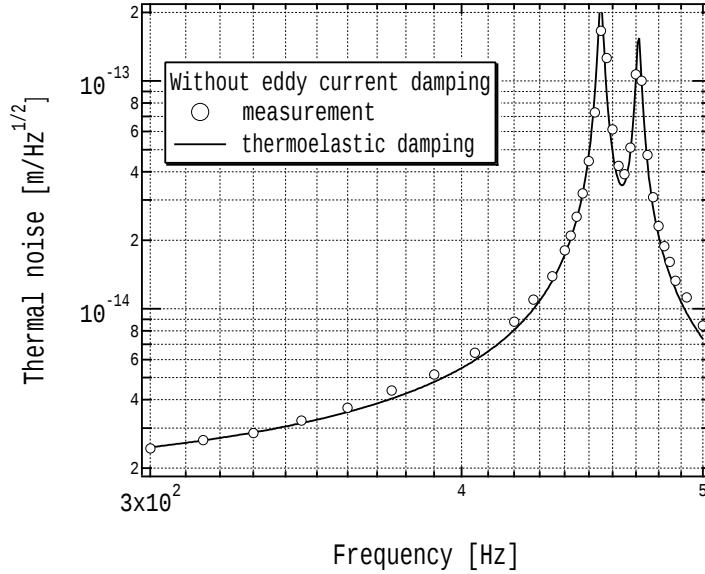


Figure 8.11: The thermal noise of the drum without the eddy current damping. The open circles are the spectrum evaluated from the measured transfer function using the fluctuation-dissipation theorem. The solid line represents the thermal noise caused by the thermoelastic damping in the disks of the drum. The original loss of the drum was dominated by the thermoelastic damping.

where E is Young's modulus, α is the linear coefficient of the thermal expansion, T is the temperature, C is the specific heat, ρ is the density, and σ is Poisson ratio. The frequency, f_0 , which corresponds to the relaxation time, τ in Eq.(8.43), is written as [88]

$$f_0 = \frac{1}{2\pi\tau} = \frac{\pi}{2} \frac{\kappa}{C\rho h^2}, \quad (8.45)$$

where κ is the thermal conductivity, h is the thickness of the disk. The parameters for the estimation of the thermoelastic damping are shown in Table.8.9.

8.5.3 Comparison between measurement and estimation

The evaluated thermal motion spectra from the measured transfer functions are shown in Fig.8.12. The upper and lower graphs show the results of the Front and Back disk models in Fig.8.3, respectively. Each graph contains the estimation of thermal noise from the measured transfer function using the fluctuation-dissipation theorem (open circles),

Table 8.9: Parameters for the estimation of the thermoelastic damping.

parameter	
Young's modulus (E)	7.03×10^{10} Pa
linear coefficient of the thermal expansion (α)	23.1×10^{-6} /K
temperature (T)	300 K
specific heat (C)	0.903 J/g/K
density (ρ)	2.67 g/cm ³
Poisson ratio (σ)	0.345
thermal conductivity (κ)	2.37 W/cm/K
thickness (h)	0.5 mm

the estimated spectra obtained by the advanced (solid line) and traditional (dashed line) mode expansions. Figure 8.12 shows the spectra evaluated from the measured transfer functions agreed with the calculations of the advanced mode expansion in both the cases. The actual thermal noise in the Front disk model was larger than the estimation of the traditional mode expansion. On the other hand, the comparison between Figs.8.11 and 8.12 proves that the thermal noise of the Back disk model in the lower frequency range was dominated by the contribution of the original loss of the drum in spite of the existence of the eddy current damping. Therefore, it was proved that the thermal motions of the drum, which was a representative mechanical model of mirrors, were calculated correctly using the advanced mode expansion and the direct approaches. Consequently, the results of this experiment suggested that the estimation of the thermal noise of the mirror with inhomogeneous dissipation in the previous chapter is correct.

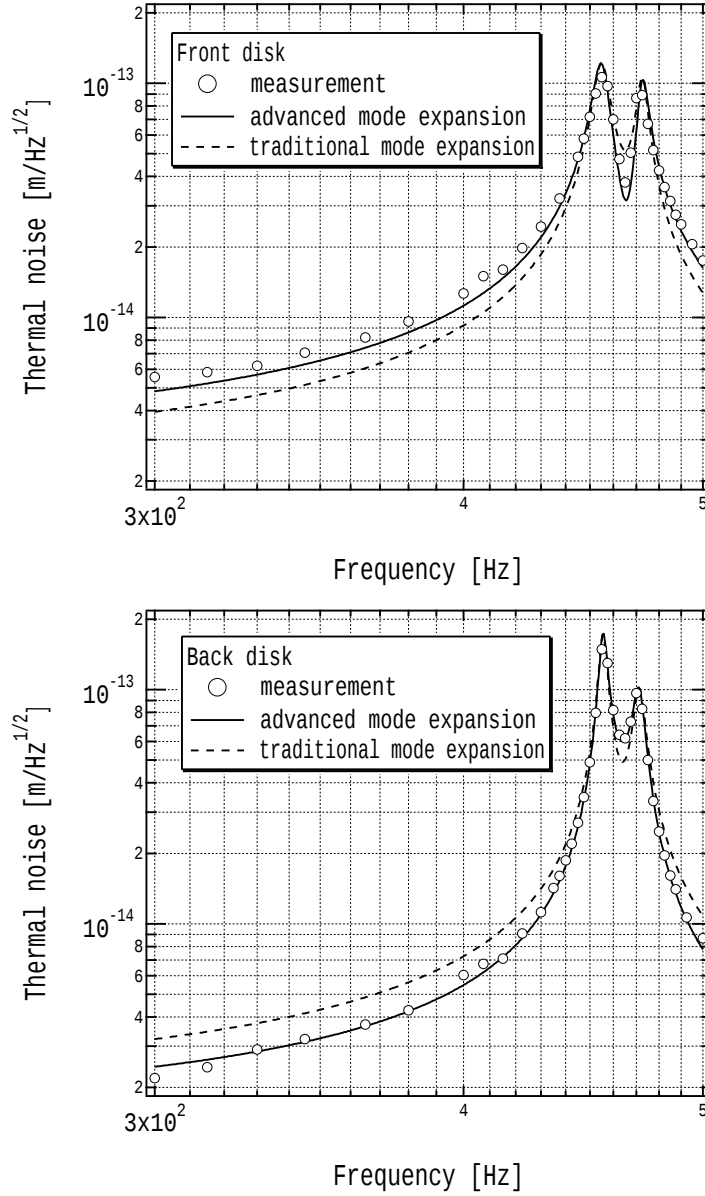


Figure 8.12: Results of measurements of the drum. The upper and lower graphs show the results of the Front and Back disk models in Fig.8.3, respectively. The open circles are the spectrum evaluated from the measured transfer function using the fluctuation-dissipation theorem. The solid and dashed lines represent the estimation of the advanced and traditional mode expansions, respectively.

Chapter 9

Discussion

The experiments in Chapters 6 and 8 proved that the traditional mode expansion used frequently to evaluate the thermal noise is wrong when the dissipation is distributed inhomogeneously. Moreover, these experiments showed that new methods (advanced mode expansion and direct approaches) are valid even when the loss is not homogeneous. These new methods showed that the actual thermal noise of the mirrors¹ with inhomogeneous losses in the gravitational wave detectors is largely different from the computation of the traditional mode expansion (Chapter 7). This is a serious problem because the research of reducing thermal noise is based on the estimation of the the traditional mode expansion. Therefore, the required upper limits of the losses of the mirrors are evaluated using the results of the calculation in Chapter 7. In addition, the future works about the estimation of the thermal noise caused by inhomogeneous losses are considered.

9.1 Required upper limit of loss in mirrors

The results in Chapter 7 show that the actual thermal noise of the mirror with inhomogeneous losses is at least two or three times larger or smaller than the estimation with the traditional mode expansion. This is a large discrepancy because it is expected that

¹The discrepancy between the actual thermal noise of the suspension systems in the gravitational wave detectors and the estimation derived from the traditional mode expansion is negligible (Chapter 4).

the number of the detectable sources are inversely proportional to the cube of the mirror displacement sensitivity for gravitational wave detectors² [13]. The actual required limits are evaluated from the results of the direct approach in Chapter 7.

9.1.1 Coating

A flat surface of each mirror in the gravitational wave detectors is coated with dielectric layers. The loss in the coating is represented by the front surface model in Fig.7.2. Fig.7.6 shows that the actual thermal noise of Front surface model is about three times³ larger than the computation of the traditional mode expansion. Since the amplitude of the thermal noise is proportional to the square root of the dissipation, the required upper limit of the loss in the coating becomes ten times severer. Nevertheless, the loss of the coating has been seldom investigated. It is an important issue to measure the dissipation in the coating.

The upper limit of the loss in the coating is evaluated here. The upper limit is defined as following; the contribution, G_{coating} [m^2/Hz], of the loss in the coating to the thermal noise must be at least ten times smaller than the contribution, $G_{\text{intrinsic}}$, of the intrinsic loss of the mirror,

$$G_{\text{coating}} < \frac{1}{10} G_{\text{intrinsic}}. \quad (9.1)$$

The spectrum, $G_{\text{intrinsic}}$, is proportional to the loss angle of the intrinsic loss, $\phi_{\text{intrinsic}}$. The spectrum, G_{coating} , is about ten times larger than the result derived from the traditional mode expansion. In the traditional mode expansion, the thermal fluctuation caused by the coating loss is inversely proportional to the contribution of the coating loss to Q-values, Q_{coating} . Equation (9.1) is rewritten as

$$10 \times \frac{1}{Q_{\text{coating}}} < \frac{1}{10} \phi_{\text{intrinsic}}. \quad (9.2)$$

Since the elastic energy is distributed almost homogeneously in resonance, Q_{coating} is expressed as

$$Q_{\text{coating}} \approx \frac{l}{\delta l} \frac{1}{\phi_{\text{coating}}}, \quad (9.3)$$

²This is because the maximum distance of the detectable sources is inversely proportional to the sensitivity.

³The beam radius of TAMA300 is 8 mm at the near mirror and 15 mm at the end mirror.

where l and δl are the thicknesses of the mirror and the coating layer, respectively. The angle, ϕ_{coating} , is the loss angle in the coating. Introducing Eq.(9.3) into Eq.(9.2), the requirement of the coating loss is obtained as

$$\phi_{\text{coating}} < \frac{1}{100} \frac{l}{\delta l} \phi_{\text{intrinsic}}. \quad (9.4)$$

In most cases, the thickness, l , of the mirror is about 10 cm. The thickness, δl , of the coating layer is about 10 μm . Thus, the upper limit of the loss in the coating is expressed as

$$\phi_{\text{coating}} < 100 \phi_{\text{intrinsic}}. \quad (9.5)$$

The required upper limits of $\phi_{\text{intrinsic}}$ in current and future projects are 10^{-6} and 10^{-8} , respectively. The required ϕ_{coating} is described as

$$\phi_{\text{coating}} < \begin{cases} 10^{-4} & (\text{in current projects}) \\ 10^{-6} & (\text{in future projects}) \end{cases}. \quad (9.6)$$

Figure 7.1 shows that the actual thermal motion is larger than the results derived from the traditional mode expansion when the dissipation is localized near the beam spot. Since the reflective coating is nearest part in the mirror, Fig.7.6 suggests that the maximum ratio of the actual thermal noise to the estimation of the traditional mode expansion is about three⁴.

9.1.2 Polished surface

The loss is concentrated on the surfaces of the mirrors [54, 82, 83] because of the polish and so on. From Figs.7.6 and 7.7, the requirement on the loss angle, ϕ_{front} , on the surface illuminated by the laser beam is the severest. This requirement is the same as that of the coating loss, Eq.(9.4). Since the value derived from the experiment is the product of the loss angle on the surface and the depth, d , of the loss layer [54, 82, 83], Eq.(9.4) is rewritten as

$$\phi_{\text{front}} d < \frac{1}{100} l \phi_{\text{intrinsic}}. \quad (9.7)$$

⁴If loss is concentrated unfortunately near the center of the flat surface illuminated by the laser beam, this ratio becomes about ten.

The thickness of the mirror, l , is about 10 cm. Equation (9.7) is reduced as

$$\phi_{\text{front}} d < \phi_{\text{intrinsic}} \times 10^{-3} [\text{m}]. \quad (9.8)$$

The required upper limits of $\phi_{\text{intrinsic}}$ in current and future projects are 10^{-6} and 10^{-8} , respectively. The required ϕ_{front} is described as

$$\phi_{\text{front}} d < \begin{cases} 10^{-9} [\text{m}] & (\text{in current project}) \\ 10^{-11} [\text{m}] & (\text{in future project}) \end{cases}. \quad (9.9)$$

The measurement of the Q-values of a polished fused silica mirror [54] shows that $\phi_{\text{front}} d$ is 10^{-9} m at most. Thus, in current projects, the loss caused by the polish on the surface illuminated by the laser beam is not a serious problem. In the future project, it is necessary to investigate this loss.

The requirements of the loss angles of the cylindrical surface, $\phi_{\text{cylindrical}}$, and of the opposite flat surface, ϕ_{back} , are less severe than that of the coated surface. From Figs. 7.6 and 7.7, the actual amplitudes of the thermal noise caused by the loss in the cylindrical surface and in the opposite surface are about six and four times smaller than the estimation of the traditional mode expansion, respectively. Equation (9.8) is rewritten as

$$\phi_{\text{cylindrical}} d < \phi_{\text{intrinsic}} \times 3 \times 10^{-1} [\text{m}], \quad (9.10)$$

$$\phi_{\text{back}} d < \phi_{\text{intrinsic}} \times 10^{-1} [\text{m}]. \quad (9.11)$$

Since the required upper limit of $\phi_{\text{intrinsic}}$ in future projects are 10^{-8} , $\phi_{\text{cylindrical}} d$ and $\phi_{\text{back}} d$ must be smaller than 10^{-9} m. The loss in the cylindrical and opposite surfaces are not a serious problem even in future projects because the measurement of the polished fused silica mirror [54] shows that these values are smaller than 10^{-9} m.

9.1.3 Magnet and stand-off

In order to control the positions of the mirrors, coil-magnet actuators are used in TAMA300. The magnets of these actuators are glued on the mirrors back. The stand-off's, thin short bars, are glued on the cylindrical surface to fix wires to the mirror. The experiments show that the adoption of the stand-off's increases the Q-value of a violin mode [55].

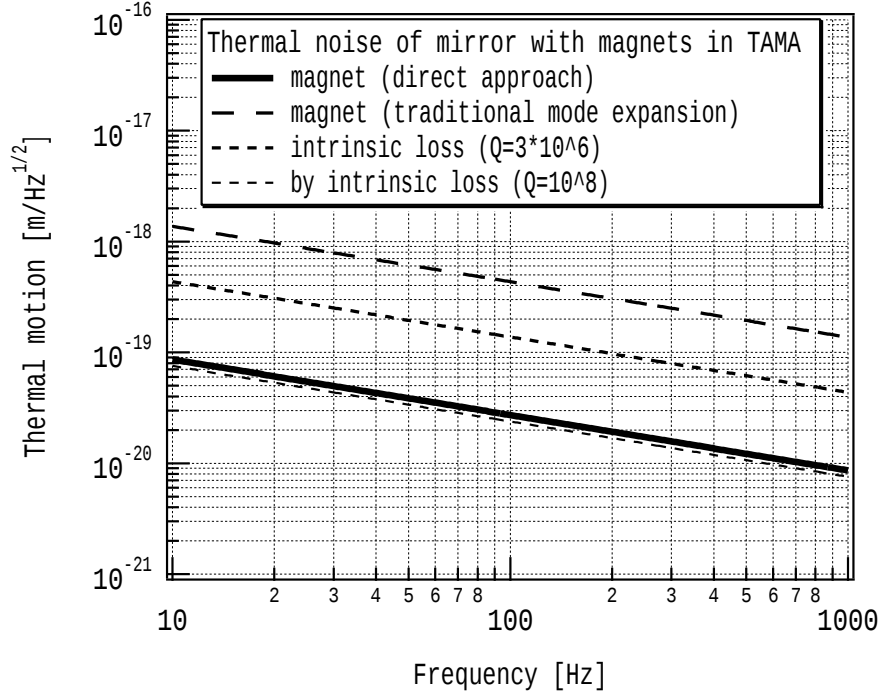


Figure 9.1: The estimations of the thermal fluctuation of the mirrors with magnets in TAMA300. The solid and long dashed lines are the contributions of the loss at the magnets to the thermal noise derived from the direct approach and from the traditional mode expansion, respectively. The dashed lines are the contributions of the intrinsic loss of the mirrors. The Q -values of the thick and thin dashed line are 3×10^6 (TAMA) and 10^8 (future projects), respectively.

Figs.7.8 and 7.9 show that the thermal fluctuations of magnets and stand-off's are similar. Only the magnets are considered. The estimations of the thermal fluctuation of the mirrors with the magnets in TAMA300 are shown in Fig.9.1. The solid line is the actual contribution of the magnet loss to the thermal noise derived from the direct approach⁵ in Fig.7.8. The long dashed line represents the contribution of the magnet loss obtained from the traditional mode expansion. The actual amplitude is 15 times smaller than the traditional estimation. Since the intrinsic loss of the mirror was not taken into account in the calculations in Fig.7.8, it is considered. The dashed lines in Fig.9.1 are the contributions of the intrinsic loss in the mirrors to the thermal fluctuations. The Q -

⁵The mirrors with magnets are the same as the Back magnet model in Fig.7.3. The beam radius of TAMA300 is 8 mm at the near mirror and 15 mm at the end mirror.

values of the thick and thin dashed lines are 3×10^6 (TAMA) and 10^8 (future projects), respectively.

It was thought that the loss at the magnets increase the thermal noise. The initial intrinsic Q-values, $Q_{\text{intrinsic}}$, of the mirrors of TAMA is 3×10^6 [54]. The Q-values of the mirrors decrease to about 10^5 when the magnets are glued on them [79, 80, 81]. Since the estimation of the traditional mode expansion is inversely proportional to the square root of Q-values, this increase of the thermal noise due to the glued magnets was expected to be a serious problem in the improvement of the sensitivity of the interferometer. Fig.9.1 shows that the traditional estimation of the thermal motion caused by the loss at the magnets is three times larger than the thermal noise caused by the intrinsic loss ($Q_{\text{intrinsic}} = 3 \times 10^6$). However, the actual thermal noise caused by the magnet loss is much smaller than the estimation of the traditional mode expansion. Fig.9.1 shows that the actual contribution of the magnet loss should be five times smaller than the contribution of the intrinsic loss. The thermal noise of the mirror with the magnets should be dominated by the intrinsic loss in spite of the large decrease of the Q-values caused by the glued magnets. Consequently, the thermal noise induced by the magnets is expected to be negligible in TAMA project.

The required intrinsic Q-values in future projects are about 10^8 . The thin dashed line in Fig.9.1 represents the contribution of the intrinsic loss in future projects. Figure 9.1 shows that the thermal noise caused by the glued magnets is comparable to the fluctuations induced by the intrinsic loss of the mirrors for future projects. Therefore, it is necessary to decrease the loss induced by the magnets or to develop other type actuators with low loss. The upper limit of the loss caused by the magnet is evaluated. The upper limit is defined as that of the coating loss; the contribution of the loss at the magnets, G_{magnet} [m^2/Hz], must be at least ten times smaller than the contribution of the intrinsic loss, $G_{\text{intrinsic}}$, of the mirror,

$$G_{\text{magnet}} < \frac{1}{10} G_{\text{intrinsic}}. \quad (9.12)$$

The spectrum, $G_{\text{intrinsic}}$, is inversely proportional to the Q-values of the intrinsic loss, $Q_{\text{intrinsic}}$. From Fig.9.1, G_{magnet} is about two hundreds times smaller than the result derived from the traditional mode expansion. In the traditional mode expansion, the thermal motion of the magnets is inversely proportional to the contribution to Q-values,

Q_{magnet} . Based on above discussion, Eq.(9.12) is rewritten as

$$\frac{1}{200} \times \frac{1}{Q_{\text{magnet}}} < \frac{1}{10} \times \frac{1}{Q_{\text{intrinsic}}}. \quad (9.13)$$

This expression is reduced as

$$Q_{\text{magnet}} > \frac{1}{20} Q_{\text{intrinsic}}. \quad (9.14)$$

Therefore, in future projects, Q_{magnet} must be larger than 5×10^6 .

9.2 Future works

In this thesis, the new methods, the advanced mode expansion and the direct approaches, to estimate the thermal noise were checked experimentally using the oscillators with the inhomogeneous losses. The thermal noise of the gravitational wave detectors was evaluated using the new methods. However, there are a few remaining problems in the estimation of the thermal noise induced by the inhomogeneous losses in interferometric gravitational wave detectors. These problems are considered here as future works.

9.2.1 Measurement of dissipation

In the calculation of the thermal noise of the mirror in Chapter 7, it was assumed that the frequency dependence of the dissipation is described by the structure damping model. This supposition was not based on the measurement of real mirrors although the structure damping is frequently used model when the thermal noise is calculated. Thus, the measurement of the frequency dependence of the losses is an important issue in future works.

For example, the measurement of the dissipation of the reflective coating which is a serious problem is considered here. The loss angle, ϕ_{coating} , of the coating is derived from the measured Q-values of the thin disk with and without the coating. The relationship between the loss angle and the Q-values is expressed as

$$\phi_{\text{coating}} = \frac{1}{Q_{\text{with}}} - \frac{1}{Q_{\text{without}}}, \quad (9.15)$$

where Q_{with} and Q_{without} are the Q-values of the disk with and without the coating, respectively. A merit of this experiment is that the resonant frequencies of the disk is lower than those of the mirror. It is possible to measure the loss angle, ϕ_{coating} , near the observation band which is lower than the resonance of the mirror. Another merit is that the effect of the coating loss is large. The contribution of the coating loss to the Q-values of the disk is large because the ratio of the volume of coating layer to that of the disk is large. From the requirement of the loss of the coating in Eq.(9.5), if the thickness of the disk is about a hundred times larger than that of the coating, the accuracy of the measurement is sufficient. Since the thickness of the coating of the mirror is about $10\ \mu\text{m}$, the appropriate thickness of the disk is about 1 mm. This is a reasonable size for making. This disk is useful in the measurement of the loss of the magnets and so on.

9.2.2 Experimental check

The experimental check of the estimation of the thermal noise derived from the measured frequency dependence of the loss in real mirrors is an important subject of future works. In this experimental test, the thermal noise of a real or similar mirror, not a model, should be investigated. There are two kinds of the experiments: the direct measurement of the thermal noise and the estimation from the measurement of the imaginary part of the transfer function using the fluctuation-dissipation theorem, Eq.(3.9).

When the thermal noise is measured directly, a highly sensitive sensor must be developed. There are a few projects [89, 90] for the development of the interferometer in order to observe the thermal noise directly. In addition, the interferometric gravitational wave detectors themselves are excellent sensors. For example, Fig.9.2 shows that the comparison between the estimations⁶ of the thermal noise of the mirror with magnets and the sensitivity of TAMA300. In this estimation, the intrinsic loss of the mirrors was taken into account. The dashed line represents the evaluation of the thermal noise derived from the direct approach. The long dashed line is the estimation of the traditional mode expansion. The thick and thin solid lines are the current and goal sensitivity of TAMA300, respectively. Figure 9.2 shows that the estimation of the thermal noise of the mirror with magnet can be checked when the sensitivity near 300 Hz is one hundred times better than the current sensitivity.

⁶These estimations are derived from the results in Fig.9.1.

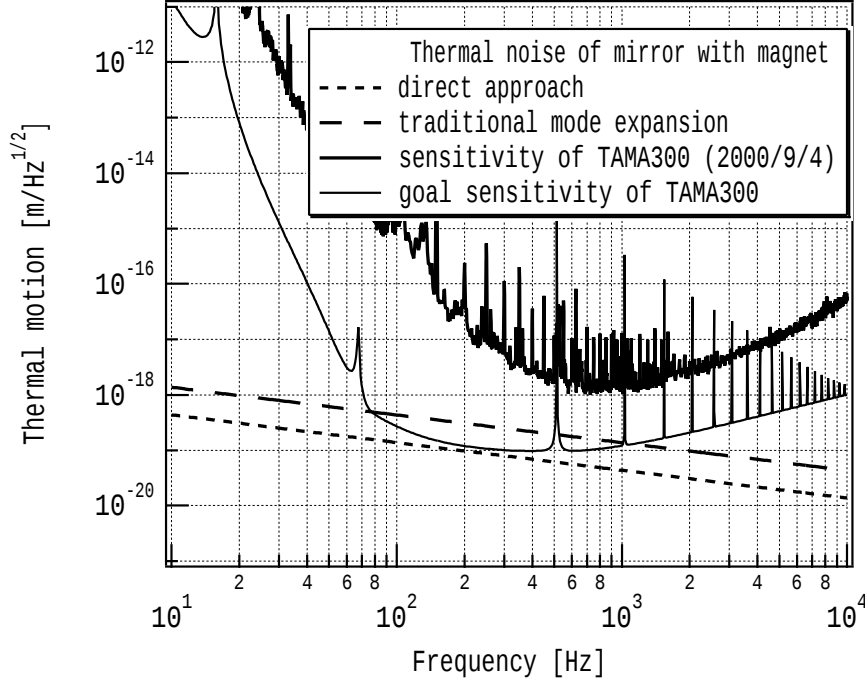


Figure 9.2: The estimation of the thermal noise of the mirrors with magnets and the sensitivity of TAMA300. The dashed line represents the evaluation of the thermal noise derived from the direct approach. The long dashed line is the estimation of the traditional mode expansion. The thick and thin solid lines are the current and goal sensitivity of TAMA300, respectively.

In the measurement of the imaginary part of the transfer function, the required sensitivity of a sensor is not as high as that of the direct measurement. However, since the imaginary part of the transfer function is much smaller than the real part, the requirement of the phase delay in the measurement system is severe. Ohishi has suggested that the imaginary part can be measured precisely at the anti-resonant frequencies because there the real part vanishes. In addition, she proved the validity of this idea experimentally [62, 63, 64]. This method is helpful in the experimental test of the estimation of the thermal noise caused by the inhomogeneous losses.

Chapter 10

Conclusion

The thermal fluctuation is one of the most important subjects in the development of the interferometric gravitational wave detectors. The normal-mode expansion commonly adopted as the method to estimate the thermal noise appears to fail. Some theoretical arguments suggested that this method is not valid in oscillators with the dissipation distributed inhomogeneously. Although the loss is generally not homogeneous, the thermal noise caused by the inhomogeneous loss has been seldom investigated. The thermal noise induced by the inhomogeneous dissipation were researched to obtain the correct estimation.

We have developed the new estimation method replacing the mode expansion. This new estimation method is called the advanced mode expansion because this method is a modification of the traditional mode expansion. The advanced mode expansion shows the clear physical interpretations of the thermal noise caused by the inhomogeneous dissipation. In the traditional mode expansion, the thermal noise is equivalent to the sum of the fluctuations in the motions of the resonant modes. The advanced mode expansion proves that the inhomogeneity of the losses causes much important correlations between the fluctuations in the motions of the modes. Since these correlations are not considered in the traditional mode expansion, this method fails when the loss is not uniform. Although the results of the direct approaches which are other new estimation methods are consistent with the evaluation of the advanced mode expansion, these approaches do not give the clear physical interpretation.

The new estimation methods, the advanced mode expansion and the direct approaches,

were tested experimentally. The thermal noise of the leaf spring with inhomogeneous loss was measured. Those results proved that the advanced mode expansion and the direct approaches are valid. On the other hand, the estimation of the traditional mode expansion does not agree with the measured thermal motions. This is the first experimental evidence which showed the failure of the traditional mode expansion.

The thermal motion of the internal modes of the mirror with inhomogeneous losses in interferometric gravitational wave detectors were evaluated using the direct approach. The results showed that the actual thermal noise is greatly different from the estimation of the traditional mode expansion. For example, the amplitude of the actual thermal noise caused by the loss in the reflective coating is two or three times larger than the old estimations. Therefore, more investigations on the losses in the dielectric coating are important. Several parts, magnets and stand-off's, are attached to mirrors for the operation of the interferometric detectors. It was thought that the losses introduced by these attachments increases greatly the thermal noise of the mirror. However, the calculation using the direct approach proved that the amplitude of the actual thermal noise is at least ten times smaller than that obtained from the traditional mode expansion. In the TAMA project, the contribution of the loss at the attachments to the thermal fluctuation is now expected to be much smaller than that of the intrinsic loss of the mirror. Thus, the loss of the attachments is a negligible problem in TAMA300.

This estimation of the thermal noise of the mirror with inhomogeneous dissipation were tested experimentally. Since the measurement of the real mirror is difficult, a mechanical model of mirrors was used. The measured values are consistent with the estimation of the advanced mode expansion and the direct approach. This results suggested that the estimation of the mirror is valid. The investigation of the frequency dependence of the loss in the real mirror and the experimental test of the estimation using the real mirror are the issues of the future study.

The studies in this thesis solved almost all the problems of the estimation method of the thermal noise caused by the inhomogeneous losses. Several important conclusions of the thermal noise of the interferometric gravitational wave detectors are obtained. This research will yield useful methods and reliable strategy in the research on thermal noise of the interferometric gravitational wave detectors.

Appendix A

Circuits

The circuits used in these experiments are shown.

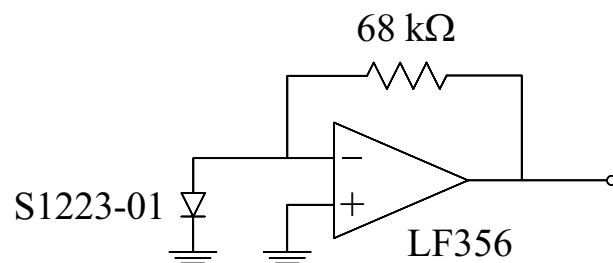


Figure A.1: The photo detector of the interferometer. This photo detector was used in the experiments of the leaf spring (Chapter 6) and the drum (Chapter 8).

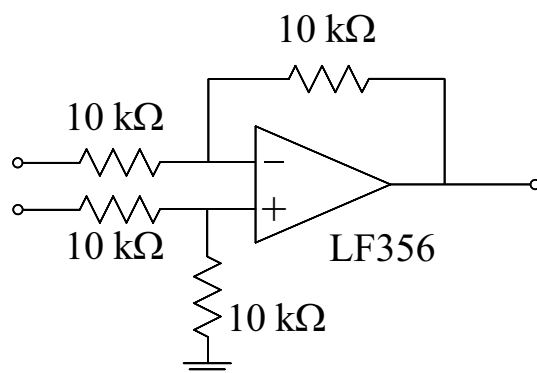


Figure A.2: The differential amplifier of the interferometer. This amplifier was used in the experiments of the leaf spring (Chapter 6) and the drum (Chapter 8).

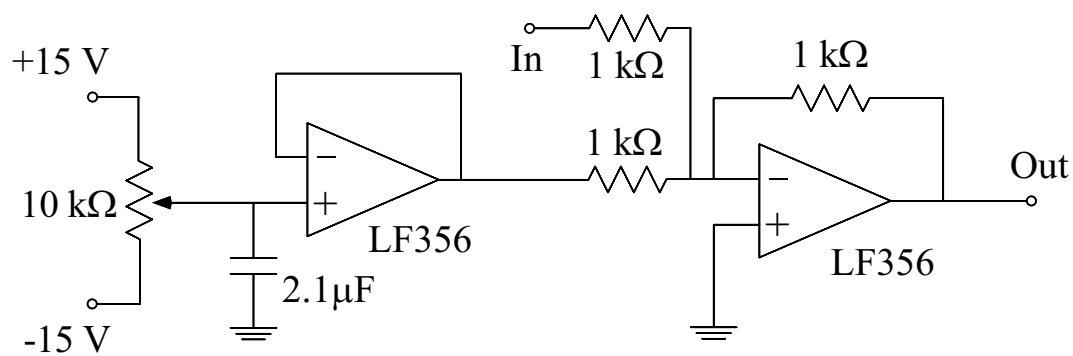


Figure A.3: The offset circuit of the interferometer. This circuit was used in the experiments of the leaf spring (Chapter 6) and the drum (Chapter 8).

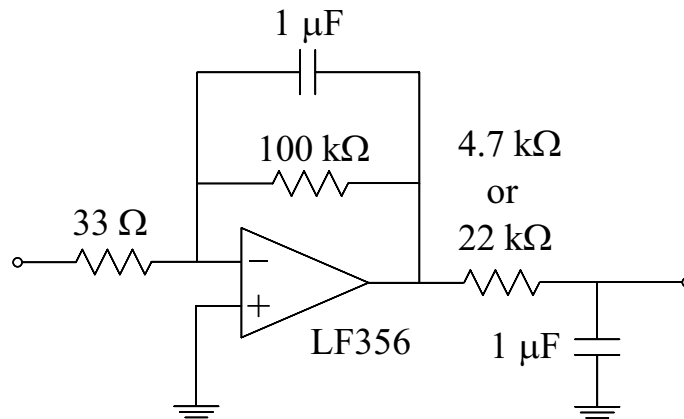


Figure A.4: The low pass filter to lock the interferometer in the experiments of the leaf spring (Chapter 6). The PZT was connected to the output of this filter.

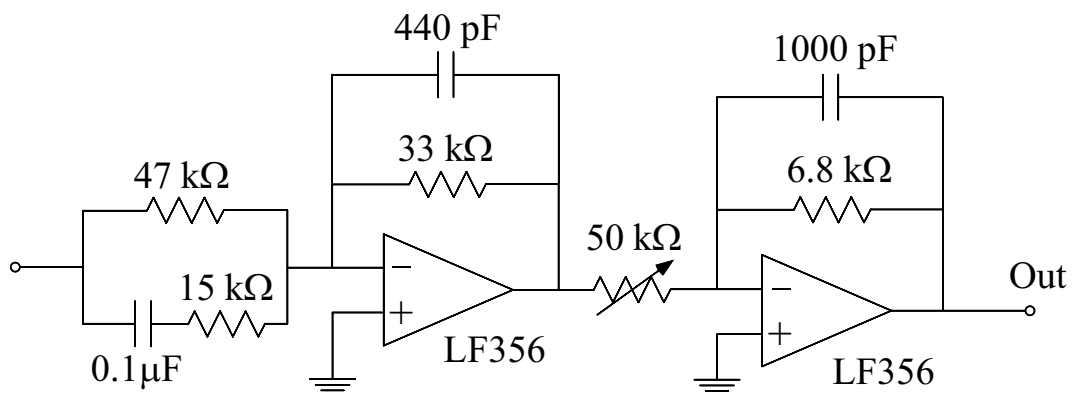


Figure A.5: The filter to lock the interferometer in the experiments of the drum (Chapter 8).

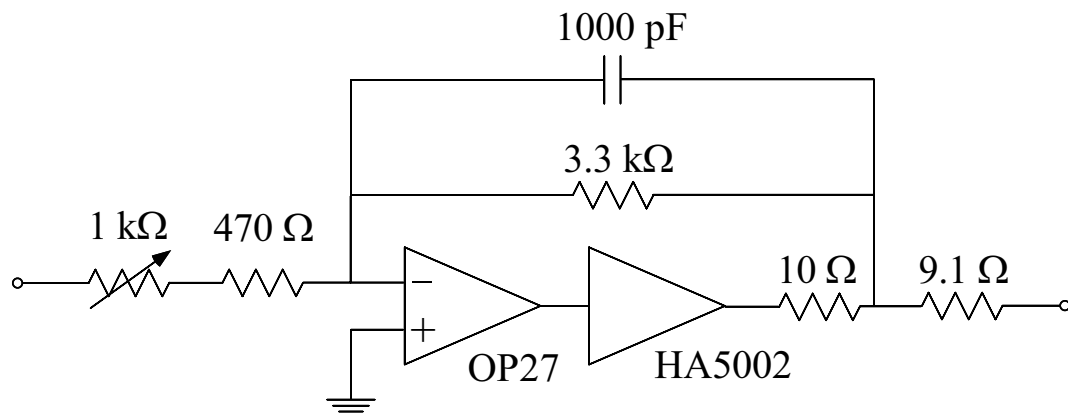


Figure A.6: The driver to lock the interferometer in the experiments of the drum (Chapter 8). The coil was connected to the output of this driver.

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