Gravitational Waves Detection via Weak Measurements

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Part 1
Quantum Measurements
Quantum Measurement

- When we execute an measurement on a quantum system, ordinarily the original state is destroyed
- An example: spin measurement, wave function collapse
- We want to measure an system without destroying quantum state
  - von Neumann’s measurement model
von Neumann measurement model

- Prepare another system called Probe or pointer: $|\psi\rangle_S, |\phi\rangle_P$
- The measured system and the probe is interacted
- The interaction Hamiltonian has the form like that

$$\hat{H} = \alpha \hat{A} \hat{X},$$

where $\hat{A}$ is an observable of the system, $\hat{X}$ is a canonical variable of the probe $[\hat{X}, \hat{P}] = i$

- Whole system $|\Psi\rangle_{SP} = |\psi\rangle_S \otimes |\phi\rangle_P$
- evolves with time as:

$$|\Psi\rangle_{SP}(t) = e^{-i \int dt \hat{H}} |\Psi\rangle_{PS}$$
Some assumptions

• For simplicity, we prepared the probe as a Gaussian in \( \hat{P} \) representation:

\[
P\langle P | \phi \rangle_P = \mathcal{N} \exp\left( - \frac{P^2}{4\Delta P^2} \right)
\]

• The system is supposed to be separable with eigenstates of \( \hat{A} \):

\[
|\psi\rangle_s = \sum c_i |a_i\rangle,
\]

\[
\hat{A} |a_i\rangle = a_i |a_i\rangle
\]

• Under such a situation, we measure \( \langle P \rangle \)
The final state

- If the interaction time $\Delta t$ is enough small,
  \[ \int dt \hat{H} = \alpha \Delta t \hat{A} \hat{X} \]

- Then,
  \[ P\langle P|\Phi\rangle_{SP}(t) = \mathcal{N} e^{-i \int dt \hat{H}} \exp \left( - \frac{P^2}{4\Delta P^2} \right) \sum c_i |a_i\rangle \]
  \[ = \mathcal{N} \sum c_i \exp \left( - \frac{(P - \alpha \Delta t a_i)^2}{4\Delta P^2} \right) |a_i\rangle \]

- The result is an mixture of Gaussians
The Probability density

- The probability density of $\langle P \rangle$ is

$$p(P) = |P\langle P|\Phi\rangle_{SP}|^2 = \sum |c_i|^2 \exp \left( -\frac{(P - \alpha \Delta t a_i)^2}{2\Delta P^2} \right)$$

- If $\Delta P$ is much smaller than the difference of $a_i$
  ➢ We will detect the value $a_i$ with probability $|c_i|^2$
- If $\Delta P$ is much bigger than all of
  ➢ The mean value is

$$\langle A \rangle = \sum |c_i|^2 a_i$$
An example

- $\Delta P$ is small ($\Delta P = 0.1$)
An example

- $\Delta P$ is big ($\Delta P = 5$)
Weak measurements

- We select the initial state and the final state of the system:
  \[ |\psi_i\rangle_S, |\psi_f\rangle_S \]
  these operations are called preselection, postselection

- Then, measurements are done:
Weak measurements

- The result:

\[ S \langle \psi_f | P | \Phi \rangle_{SP} = \mathcal{N} \langle \psi_f | e^{-i \alpha \Delta t \hat{H}} | \psi_i \rangle \exp \left( - \frac{P^2}{4 \Delta P^2} \right) \]

\[ \simeq \mathcal{N} \langle \psi_f | \psi_i \rangle \exp \left( -i \alpha \Delta t X \frac{\langle \psi_f | \psi_i \rangle}{\langle \psi_f | \hat{A} | \psi_i \rangle} \right) \times \exp \left( - \frac{P^2}{4 \Delta P^2} \right) \]
**Weak value**

- We define weak value as:

\[
A_w = \frac{\langle \psi_f | \psi_i \rangle}{\langle \psi_f | \hat{A} | \psi_i \rangle}
\]

- Then

\[
S \langle \psi_f | P \langle P | \Phi \rangle_{SP} = \simeq N \langle \psi_f | \psi_i \rangle \exp \left( - \frac{(P - \alpha \Delta t A_w)^2}{4 \Delta P^2} \right)
\]

- If we measure \( \langle P \rangle \), we know \( A_w \)

- The meaning of \( A_w \) is not trivial, controversing
Part 2

Review Of The Paper
Introduction

- This paper suggests a brand-new interferometer using weak measurement.
- As a pointer, they use polarized photons.
- The measured system is the path of the photon.
  (in detail, I will explain later)
- Note: this method executes preselection and postselection, but doesn’t use the weak value!
Notation

• Suppose we want to measure a two-level system
  \(|0\rangle, |1\rangle\)

• We use photons as the pointer:
  \(|H\rangle\) is the horizontal polarization state,
  \(|V\rangle\) is the vertical polarization state

• Other representation of polarizations:
  \(|+\rangle = (|H\rangle + |V\rangle)/\sqrt{2}, \quad |R\rangle = (|H\rangle + i|V\rangle)/\sqrt{2},\n  |\rangle = (|H\rangle - |V\rangle)/\sqrt{2}, \quad |L\rangle = (|H\rangle - i|V\rangle)/\sqrt{2}\)
Polarization states

We take the basis as:

\[ |H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

\[ |R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \]
A brand-new method

- Preparing the initial state of the pointer as
  \[ |\phi_i\rangle_P = |+\rangle = \frac{|H\rangle + |V\rangle}{\sqrt{2}} \]

- We take the initial and the final state of the system as before:
  \[ |\psi_i\rangle = \alpha |0\rangle + \beta |1\rangle, \quad |\psi_f\rangle = \gamma |0\rangle + \eta |1\rangle \]

- Now, consider a new Unitary operator
  \[ \hat{U} = |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes (|H\rangle\langle H| + e^{i\theta} |V\rangle\langle V|) \]

- $\theta$ is the signal what we want to measure and amplify by weak measurement
Assumptions

- In this case, the final state is

\[ |\Psi_f\rangle_{PS} = \hat{U} |\Psi_i\rangle_{PS} \]

\[ = \alpha |0\rangle \otimes |+\rangle + \beta |1\rangle \otimes (|H\rangle + e^{i\theta} |V\rangle) / \sqrt{2} \]

\[ |\phi_f\rangle_P = (\alpha \gamma + \beta \eta) |H\rangle + (\alpha \gamma + \beta \eta e^{i\theta}) |V\rangle \]

- Ordinary, \( \theta \ll 1 \), so in the first order

\[ \alpha \gamma + \beta \eta e^{i\theta} \simeq (\alpha \gamma + \beta \eta) e^{i\varphi}, \]

\[ \tan \varphi = \frac{\beta \eta \theta}{\beta \eta + \alpha \gamma} = \frac{\theta}{1 + \frac{\alpha \gamma}{\beta \eta}} \]
How to detect the signal

- Then, the final state of the pointer is

\[ |\phi_f\rangle_P = \frac{1}{\sqrt{2}} (|H\rangle + e^{i\varphi} |V\rangle) \]

- Using this, we extract the signal \( \varphi \) by calculating

\[ \langle \hat{\sigma}_y \rangle = P \langle \phi_f | \hat{\sigma}_y | \phi_f \rangle_P = \sin \varphi \]

- The postselection probability is

\[ P_{select} = |S \langle \psi_f | \psi_i \rangle_S|^2 = |\alpha \gamma + \beta \eta|^2 \]
Amplification factor

• Suppose $\alpha = \beta = 1/\sqrt{2}, \gamma = \cos \chi, \eta = \sin \chi$

\[
\tan \varphi = \frac{\theta}{1 + \cot \chi}
\]

• If we take $\chi = -(\pi/4 + \delta)$, $\tan \varphi \approx \frac{\theta}{\delta}$

so the amplification factor is

\[
h = \varphi \frac{1}{\theta} = \tan^{-1}(\theta/\delta)
\]

• With recently technology, we can reach $h \sim 10^3$
Trade-off of the postselection probability

- On the other hand, the postselection probability is

\[ P_{select} = \frac{1}{2} | \cos \chi + \sin \chi |^2 = \sin^2 \delta \]

- There is a trade-off between
  - The amplification of the signal
  - The postselection probability
Weak Measurements Amplification-LIGO

• Basically the same as the ordinary Fabry-Perot Michelson interferometer
• Using linearly polarized laser
• The system: $|\text{up}\rangle$ and $|\text{down}\rangle$ representing two paths
• The initial state of the system:
  $|\psi_i\rangle_S = r_1|\text{down}\rangle + t_1|\text{up}\rangle$
• The initial state of the pointer: $|+\rangle$
Polarization of photons in the MI

- PBS1: $|H\rangle$ is transmitted, $|V\rangle$ is reflected
- QWP: if photons passes it twice, it converts
  - $|H\rangle \rightarrow |V\rangle$
  - $|V\rangle \rightarrow |H\rangle$
- All photons go to the signal port
The preselection and the postselection

- The initial state is
  \[ |\Psi_i\rangle_{SP} = (r_1 |\text{down}\rangle + t_1 |\text{up}\rangle) \otimes |+\rangle \]
- If GW comes in, the phase between |H\rangle and |V\rangle is slightly change:
  \[ |\Psi_f\rangle_{SP} = r_1 |\text{down}\rangle \otimes |+\rangle + t_1 |\text{up}\rangle \otimes (|H\rangle + e^{i\theta} |V\rangle) \]
- Finally, we postselect the final states:
  \[ |\psi_f\rangle_{S} = r_2 |\text{down}\rangle + t_2 |\text{up}\rangle \]
The photons, which fly along path of up arm, enter a present path state of up arm and down arm respectively.

After passing through the beam splitter (BS1), the state of photons becomes fulfilled by the beam splitter (BS1). The state of photons, after initial state preparation, signal collection, signal amplification, and orthogonal arms. The similar configuration makes the LIGO. The WMA-LIGO, which operates on a different interferometer, is called Interferometer Gravitational-Wave Observatory (WMA-LIGO).

The WMA-LIGO consists of five parts i.e., laser source, Weak measurements amplification based gravitational-wave detector. The WMA-LIGO operates on a different interferometer. BS: beam splitter, PBS: polarizing beam splitter, PRM: power recycle mirror, HWP: half wave plate, QWP: quarter wave plate, TM: test mass, D: detector. The coefficients of reflection and transmission of BS1 and BS2 are respectively. Power recycle is realized by PRM1 and PRM2 with same reflectivity. QWPs in the two arms are fixed at 2π/4 and QWP, HWP, PBS2, D1, D2 constitute a polarization analyzer.

The initial state preparation is when we focus only on the photons comes out from the input port. The interaction of weak measurements is fulfilled such that the photons will not come out from the input port. The interaction of weak measurements is fulfilled when photons comes out of PMI with state |up⟩ and state |down⟩+ |H⟩=+|H⟩ and

\[
t_1|\text{up}\rangle \otimes (|H\rangle + e^{i\theta}|V\rangle)
\]

The post-selection is completed when photons comes out of PMI with state |H⟩+ e^{i\theta}|V⟩. The post-selection is completed when we focus only on the photons comes out from the input port. The interaction of weak measurements is fulfilled when photons comes out of PMI with state |H⟩+ e^{i\theta}|V⟩. The initial state preparation is when we focus only on the photons comes out from the input port.
Signal amplification

- This is the same as previous example:
  \[ \alpha = r_1, \beta = t_1 \]
  \[ \gamma = r_2, \eta = t_2 \]

- Then, \[
  |\phi\rangle_P = \frac{1}{\sqrt{2}}(|H\rangle + e^{i\varphi}|V\rangle) \]
  \[ \tan \varphi = \frac{\theta}{1 + \frac{r_1 r_2}{t_1 t_2}} \]

- If we choose \( r_1, r_2, t_1, t_2 \) so as to \( r_1 r_2 + t_1 t_2 \rightarrow 0 \),
  signal \( \theta \) is much amplified
Conclusion

• Using WMA-LIGO, we can amplify the signal by the factor $h \sim 10^3$
• Shot noise and radiation pressure noises are treated as usual FPMI
• All photons go to the signal port
  ➢ It means that the MI is at the bright fringe
  ➢ Shot noise is worst than ordinary interferometer?

• Quantum measurements theory is difficult
End
Setting

- Suppose the interaction Hamiltonian is
  \[\hat{H} = \alpha \hat{A} \otimes \hat{\sigma}_y,\]
  where \(\hat{A} = |0\rangle \langle 0| - |1\rangle \langle 1|,\)
  \(\hat{\sigma}_y = |R\rangle \langle R| - |L\rangle \langle L| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\)

- The initial state is
  \[|\Psi_i\rangle_{PS} = |\psi_i\rangle_P \otimes |H\rangle,\]
  \[|\psi_i\rangle_P = \alpha |0\rangle + \beta |1\rangle\]

- The final state of the system is
  \[|\psi_f\rangle_P = \gamma |0\rangle + \eta |1\rangle\]
Calculation

- If the interaction occurred between small time \( \Delta t \), the time evolution is:

\[
|\Psi_f\rangle_P = e^{-i\theta \hat{A} \hat{\sigma}_y} |\Psi_i\rangle
\]

where \( \theta = \Delta t \alpha \)

- Then, the final state of the pointer is

\[
|\phi_f\rangle_P = \langle \psi_f | \Psi_f \rangle \approx \langle \psi_f | \psi_i \rangle e^{-i\theta A_w \hat{\sigma}_y} |H\rangle
\]

if the weak value condition is satisfied.

- After some algebra, we get

\[
|\phi_f\rangle_P = \langle \psi_f | \psi_i \rangle e^{-i\theta A_w \hat{\sigma}_y} (\cos \chi |H\rangle + \sin \chi |V\rangle)
\]

\[
\approx \langle \psi_f | \psi_i \rangle (|H\rangle + \chi |V\rangle)
\]

if \( \chi = \theta A_w \ll 1 \)
• Using $|\phi_f\rangle_P$, we can calculate $A_w$ as follows:

$$\langle \hat{\sigma}_x \rangle = 2\theta \text{Re}(A_w),$$

$$\langle \hat{\sigma}_y \rangle = 2\theta \text{Im}(A_w)$$