Products of
our daily trivial discussion

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WFS: Ambition

- Initial motivation: intuitive (or, geometric) understanding of the coupled WFS (cWFS)
- During the calculation I noticed
  - ◉ Why WFS is called “wave front” sensor
  - ◉ Most people fall a pitfall in calculating WFS
- Of course I was able to explain cWFS in a geometrical way
- In this section:
  - ◉ Introduce Hermite-Gaussian (HG) modes
  - ◉ Show the pitfall in calculation of WFS
  - ◉ Explain cWFS in a geometrical way

WFS: HG Mode

Hermite Gaussian HG_{lm} mode

$$U_{lm}(x,y,z) = \left(\frac{2}{\pi w(z)}\right)^{\frac{1}{2}} e^{i(1+l+m)\eta(z)-ikz} \exp\left[-\frac{x^2+y^2}{w(z)^2} - ik\frac{x^2+y^2}{2R(z)}\right]$$

$w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$:beam radius $R(z) = \frac{z^2+{z_R}^2}{z}$: radius of curvature

$\eta(z) = \tan^{-1}\frac{z}{z_R}$: Gouy phase w_0 :beam waist $z_R = \frac{k{w_0}^2}{2}$: Rayleigh range

Divide it in each axis:

$$U_l(x,z) = \left(\frac{2}{\pi w(z)}\right)^{\frac{1}{4}} \sqrt{\frac{1}{2^l l!}} H_l\left(\frac{\sqrt{2}x}{w(z)}\right) e^{il\eta(z)} \exp\left[-\frac{x^2}{w(z)^2} - i\frac{kx^2}{2R(z)}\right]$$

$$U_{lm}(x,y,z) = e^{-ikz} U_l(x,z) U_m(y,z)$$

WFS: Basis w/o Gouy Phase

Introduce HG basis without Gouy phase:

$$u_l(x,z) = U_l(x,z)e^{-il\eta(z)}$$

i.e.

$$u_l(x,z) = \left(\frac{2}{\pi w(z)}\right)^{\frac{1}{4}}\sqrt{\frac{1}{2^l l!}}H_l\left(\frac{\sqrt{2}x}{w(z)}\right)\exp\left[-\frac{x^2}{w(z)^2} - i\frac{kx^2}{2R(z)}\right]$$

Using this, HG_{lm} mode can be written as:

$$U_{lm}(x,y,z) = e^{i(1+l+m)\eta(z)-ikz}u_l(x,z)u_m(y,z)$$

I'll show that mistaking $u_l(x,z)$ as $U_l(x,z)$ makes wrong results

WFS: Fundamental Basis

From now on I consider (z,x) plane (ignore y-axis dependency)

- Fundamental mode

$$U_0(x,z) = u_0(x,z) = \left(\frac{2}{\pi w(z)}\right)^{\frac{1}{4}} \exp\left[-\frac{x^2}{w(z)^2} - i\frac{kx^2}{2R(z)}\right]$$

- 1st mode

$$U_1(x,z) = \left(\frac{2}{\pi w(z)}\right)^{\frac{1}{4}} \frac{2x}{w(z)} e^{i\eta(z)} \exp\left[-\frac{x^2}{w(z)^2} - i\frac{kx^2}{2R(z)}\right]$$

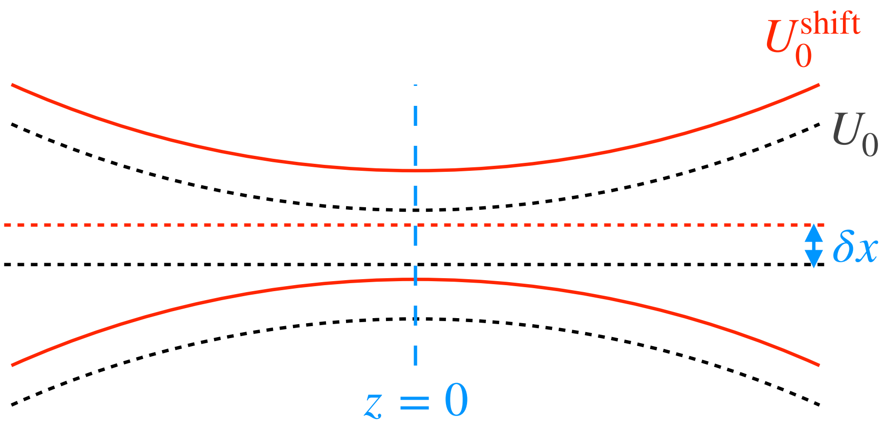
$$= \frac{2x}{w(z)} e^{i\eta(z)} U_0(x,z)$$

- When considering correspondence of wave optics to lay optics these two modes is enough

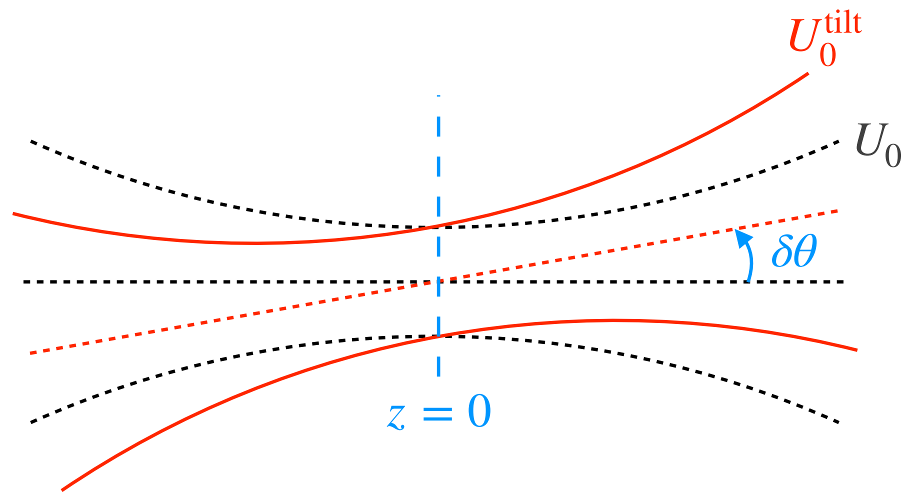
WFS: Transformation of Basis

Beam axis change at beam waist (z=0)

- shift



- tilt



$$U_0^{\text{shift}}(x,z) = U_0(x,z) + \frac{\delta x}{w_0} U_1(x,z)$$

$$U_1^{\text{shift}}(x,z) = U_1(x,z) - \frac{\delta x}{w_0} U_0(x,z)$$

$$U_0^{\text{tilt}}(x,z) = U_0(x,z) - \frac{\delta\theta}{\alpha_0} U_1(x,z)$$

$$U_1^{\text{tilt}}(x,z) = U_1(x,z) - \frac{\delta\theta}{\alpha_0} U_0(x,z)$$

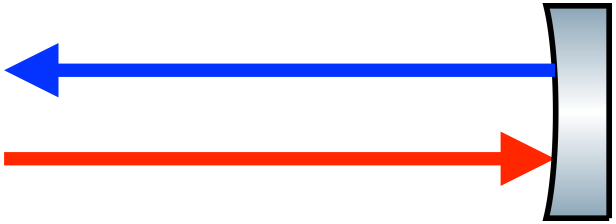
WFS: Transformation of Basis

- Backword beam

$$U^{\leftarrow}(x,z) = (U^{\rightarrow}(x,z))^*$$

- Reflection by a mode-matched mirror (i.e. a mirror with the same curvature as the beam)

$$U_l^{\leftarrow}(x,z) = (U_l^{\rightarrow}(x,z))^* \times e^{i2l\eta(z)}$$



Why?

- Phase of actual incident beam: $l\eta(z)$
- Phase of basis of outgoing beam: $-l\eta(z)$
- For continuity of phase, the reflected beam must proceed by $2l\eta(z)$ relative to the basis

WFS: Transformation of Basis

- Matrix notation

$$\gamma := \frac{\delta x}{w_0} - i\frac{\delta\theta}{\alpha_0}, \quad M_\gamma := \begin{pmatrix} 1 & -\gamma^* \\ \gamma & 1 \end{pmatrix}$$

$$\begin{pmatrix} U_0' & U_1' \end{pmatrix} = \begin{pmatrix} U_0 & U_1 \end{pmatrix} \begin{pmatrix} 1 & -\gamma^* \\ \gamma & 1 \end{pmatrix} = \begin{pmatrix} U_0 & U_1 \end{pmatrix} M_\gamma$$

- Inverse transformation

$$M_\gamma^{-1} = \frac{1}{|\gamma|^2} \begin{pmatrix} 1 & \gamma^* \\ -\gamma & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & \gamma^* \\ -\gamma & 1 \end{pmatrix}$$

for small γ $(|\gamma| \ll 1)$

WFS: Measurement by Detector

- QPD signal = beat between u_0 & u_1
- Consider

$$U_1'(x,z) = \left(\frac{\delta x}{w_0} - i\frac{\delta\theta}{\alpha_0}\right) U_1(x,z) = \left(\frac{\delta x}{w_0} - i\frac{\delta\theta}{\alpha_0}\right) e^{i\eta(z)} u_1(x,z)$$

- After some math, we get

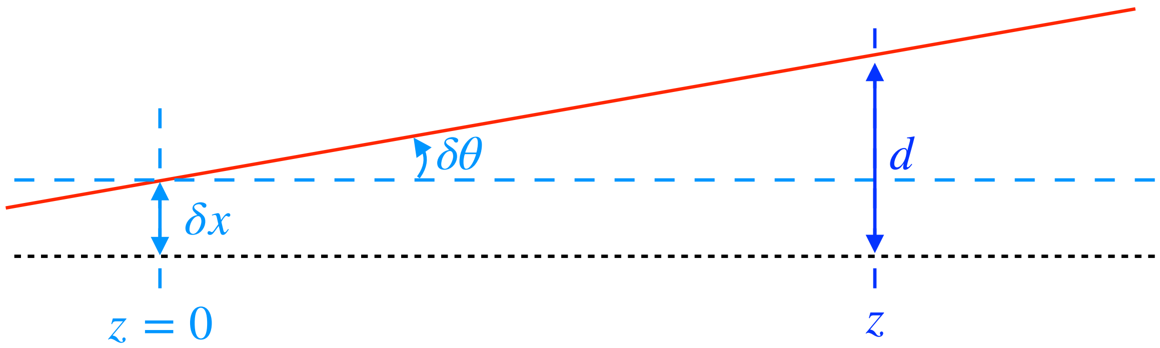
$$\left(\frac{\delta x}{w_0} - i\frac{\delta\theta}{\alpha_0}\right) e^{i\eta(z)} = \left(\frac{\delta x}{w_0}\cos\eta(z) + \frac{\delta\theta}{\alpha_0}\sin\eta(z)\right) + i\left(\frac{\delta x}{w_0}\sin\eta(z) - \frac{\delta\theta}{\alpha_0}\cos\eta(z)\right)$$

$$= \frac{\delta x + z\delta\theta}{w(z)} - i\frac{kw(z)}{2}\frac{(R(x) - z)\delta\theta - \delta x}{R(z)}$$

- real part: beam position shift at z
- imaginary part: crossing angle of wavefronts at x=0

WFS: Optical Lever

- Consider the beam center shift d at z



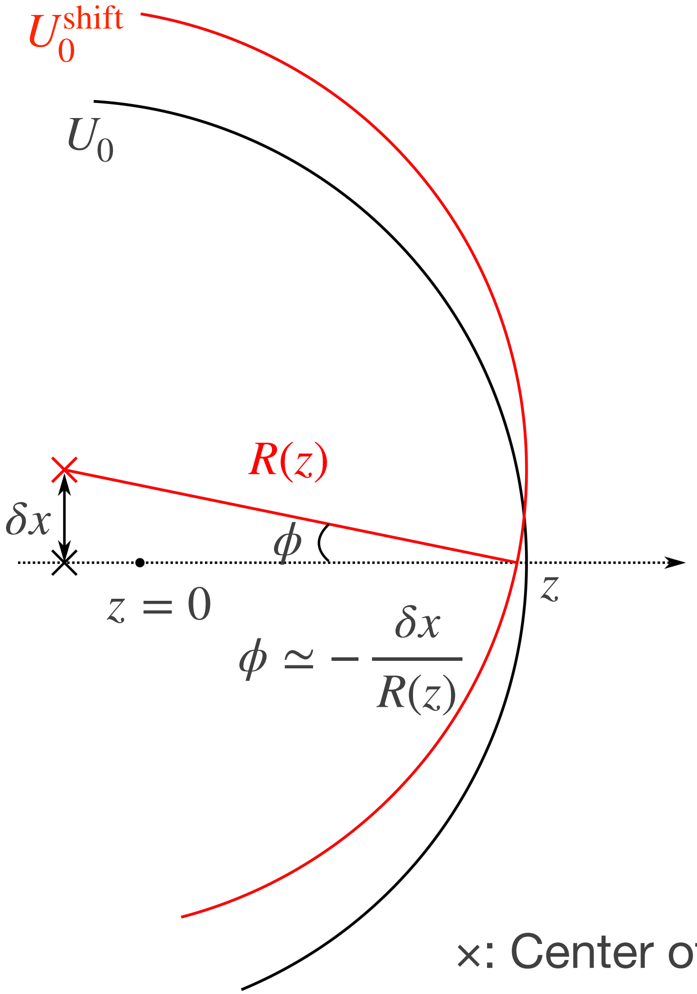
- For a beam shifted by δx and tilted by $\delta\theta$ at the beam waist
- Beam center shift $d = \delta x + z\delta\theta$
- Real part of the beat signal is

$$\frac{d}{w(z)}$$

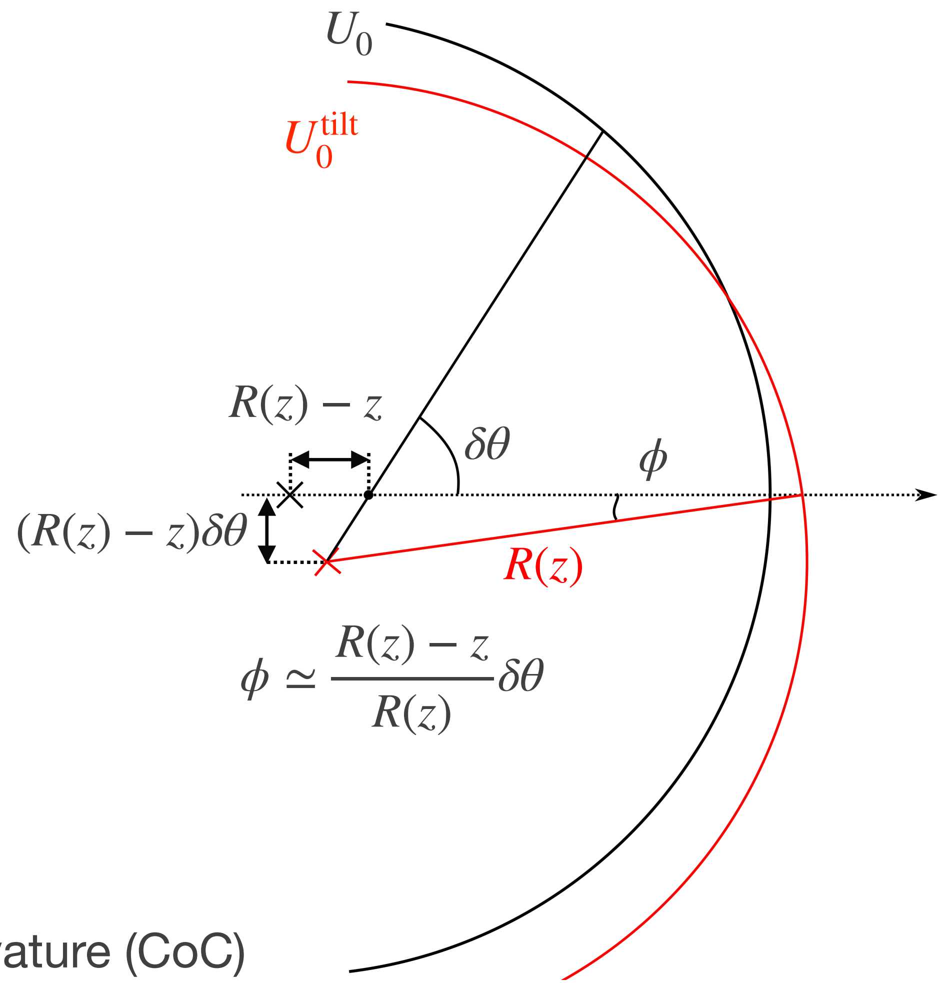
  - Sensing beam spot shift

WFS: Crossing Angle of Wave Fronts

- Consider crossing angle ϕ of wave fronts at (x=0,z):
- shift



- tilt



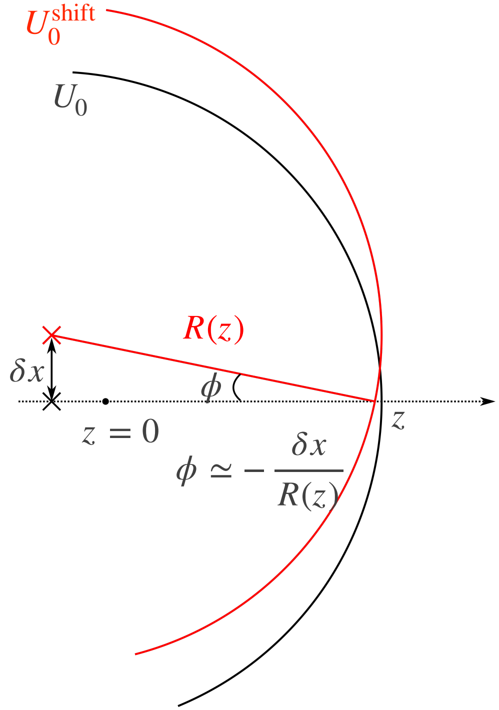
×: Center of Curvature (CoC)

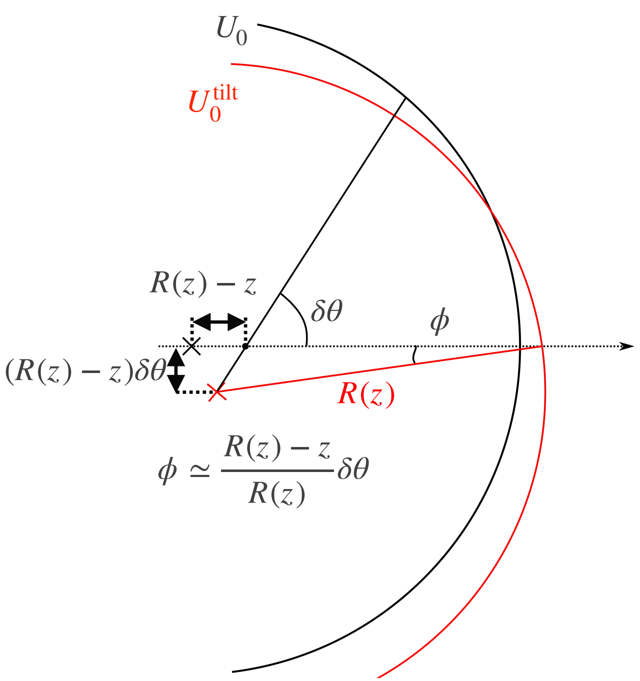
WFS: Crossing Angle of Wave Fronts

- Consider crossing angle ϕ of wave fronts at (x=0,z):

$$\phi = \frac{(R(z) - z)\delta\theta - \delta x}{R(z)}$$

- Imaginary part of the beat signal $\propto \phi$
  - Sensing crossing angle of wave front





WFS: Signal Detection

Beat signal:

- Real = beam center shift
- Imag = crossing angle of wave front

How to extract these signals?

- Real: DC signal → optical lever

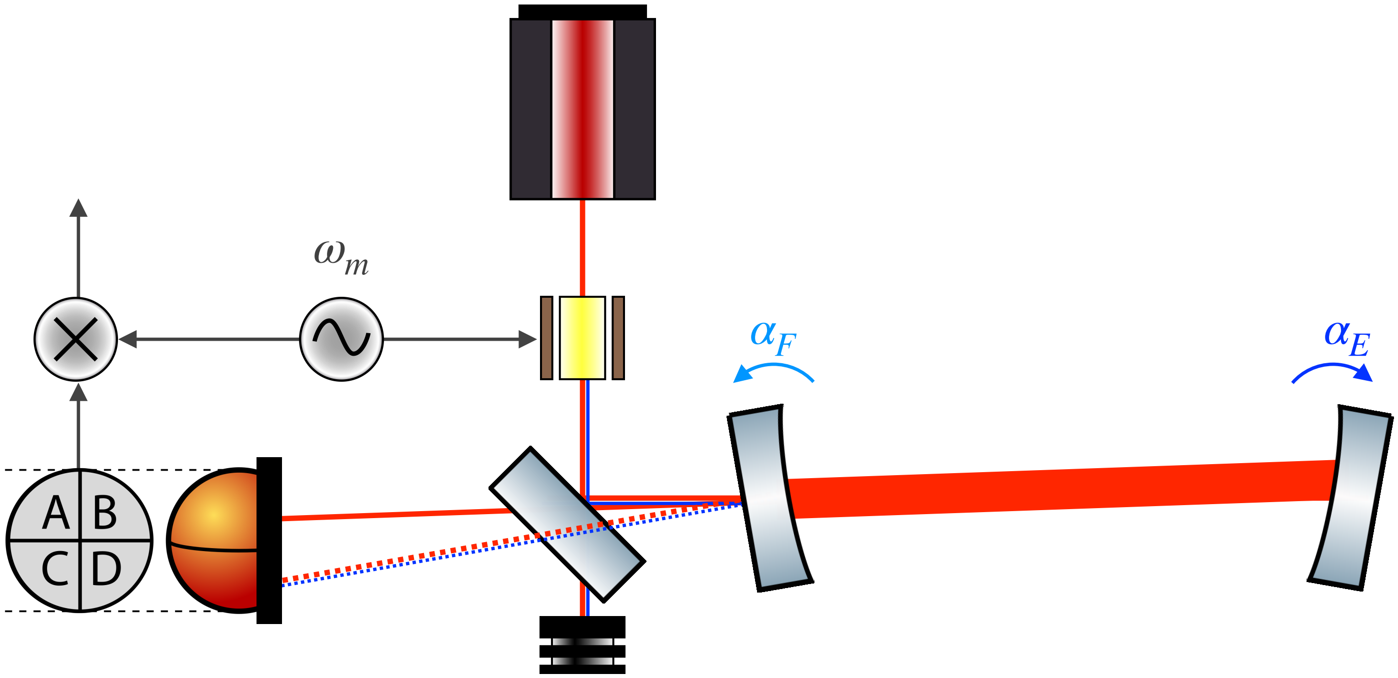
$$\int_{\mathrm{QPD}} dS \left| u_0 + \gamma u_1 \right|^2 = \int_{\mathrm{QPD}} dS\, 2u_0u_1 \mathrm{Re}[\gamma] \propto \mathrm{Re}[\gamma]$$

- Imag: phase modulation/demodulation → WFS

$$\int_{\mathrm{QPD}} dS \left| \gamma u_1 + u_0 e^{i\omega_m t} - u_0 e^{-i\omega_m t} \right|^2 = \int_{\mathrm{QPD}} dS\, 2u_0u_1 \mathrm{Im}[\gamma] \sin \omega_m t$$

$$\propto \mathrm{Im}[\gamma] \sin \omega_m t$$

WFS: Brief Introduction



- Front/End mirror tilt → change cavity axis
- Phase modulation → off resonance, reflected by the front mirror
- Measure beat signal between cavity-leakage and phase-modulated beam

WFS: Mathematics

- γ : axis difference between incident and cavity
- γ_r : axis difference between cavity and front reflected
- Now incident beam is

$$E_{\mathrm{in}} = \begin{pmatrix} U_0 & U_1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

- Then, reflected beam is

$$E_{\mathrm{refl}} = \begin{pmatrix} U_0^{\mathrm{c}} & U_1^{\mathrm{c}} \end{pmatrix}^* e^{-2i\eta_{\mathrm{F}}} \begin{pmatrix} r_0 & 0 \\ 0 & r_1 \end{pmatrix} M_\gamma \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

$$= \begin{pmatrix} U_0^{\mathrm{r}} & U_1^{\mathrm{r}} \end{pmatrix}^* M_{\gamma_r}^{-1} \begin{pmatrix} r_0 & 0 \\ 0 & r_1 \end{pmatrix} M_\gamma \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

Pitfalls

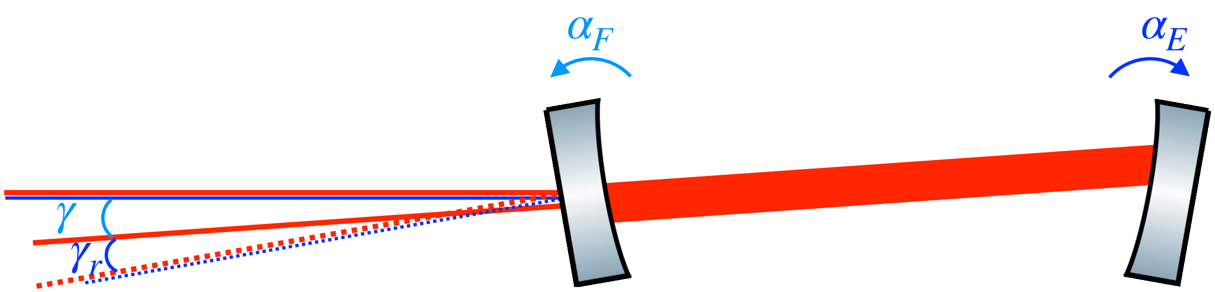
r_i : cavity reflectivity of each mode

U_i : incident axis

U_i^{c} : cavity axis

U_i^{r} : front refl. axis

η_{F} : Gouy phase at front mirror



WFS: Why Pitfalls

- When considering resonant condition of higher order mode, Gouy phase diff. from TEM_{00} is taken into account
  - We calculate coefficients of u_l, not U_l
  - If applying transformation of the basis, we have to compensate Gouy phase at the position of the mirrors!
- In many literatures authors forgot this correction and obtained wrong results…
- WFS signal

$$W_{\mathrm{WFS}} \propto \frac{\delta x_r}{w_0} \sin \eta - \frac{\delta \theta_r}{\alpha_0} \cos \eta$$

- In Michimura-san's document

$$W_{\mathrm{WFS}} \propto \frac{\delta x}{w_0} \sin \eta - \frac{\delta \theta}{\alpha_0} \cos \eta$$

- The mistake comes from this Gouy phase difference

WFS: Why WFS

- Again, WFS signal

$$W_{\mathrm{WFS}} \propto \frac{\delta x_r}{w_0}\sin\eta - \frac{\delta\theta_r}{\alpha_0}\cos\eta$$

- Beat between u_0 and u_1

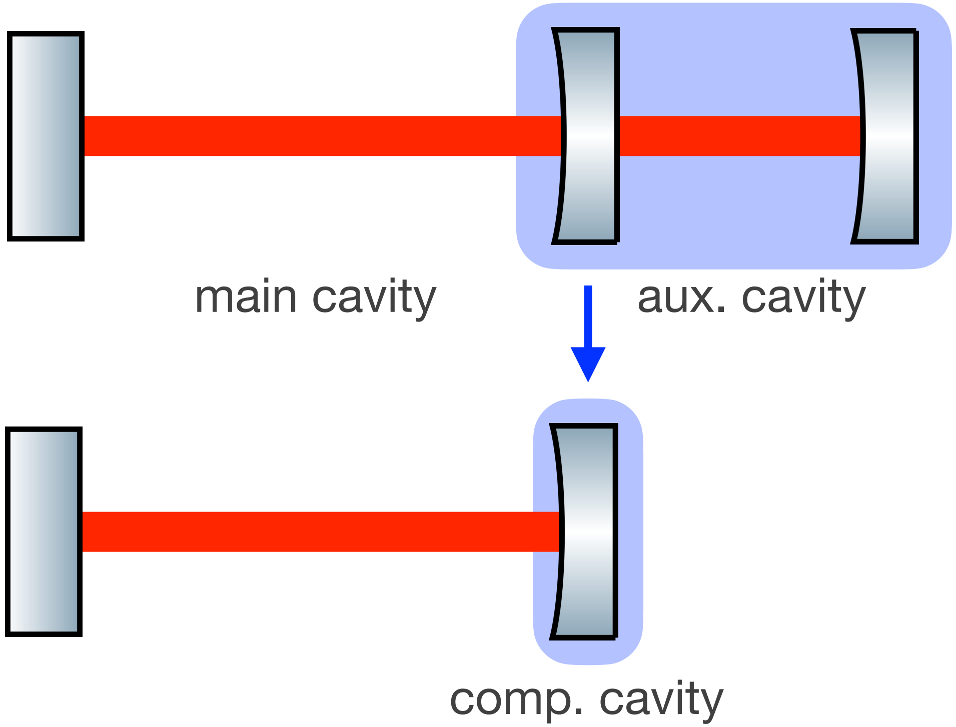
$$\left(\frac{\delta x}{w_0} - i\frac{\delta\theta}{\alpha_0}\right)e^{i\eta(z)} = \left(\frac{\delta x}{w_0}\cos\eta(z) + \frac{\delta\theta}{\alpha_0}\sin\eta(z)\right) + i\left(\frac{\delta x}{w_0}\sin\eta(z) - \frac{\delta\theta}{\alpha_0}\cos\eta(z)\right)$$

beam spot shift | wave front

- ▸ WFS actually sees crossing angle of wavefronts
- ▸ between cavity leakage & front refl. beam

WFS: Coupled WFS

- Ordinary WFS: HG_{10} is off resonance
- Coupled WFS: HG_{10} also on resonance thanks to the aux. cavity



- How can we understand cWFS if we take the aux.cavity a a compound mirror?

WFS Reflection at Mid Mirror

- Reflected beam: $U_0^{\text{refl}} = e^{2i\eta}U_0,\ U_1^{\text{refl}} = e^{4i\eta}U_1$
- When incident beam is

$$U_0 + \left(\frac{\delta x}{w_0} - i\frac{\delta\theta}{\alpha_0}\right)U_1$$

- then reflected beam is

$$e^{2i\eta}\left[U_0 + \left(\frac{\delta x}{w_0} - i\frac{\delta\theta}{\alpha_0}\right)e^{2i\eta}U_1\right]$$

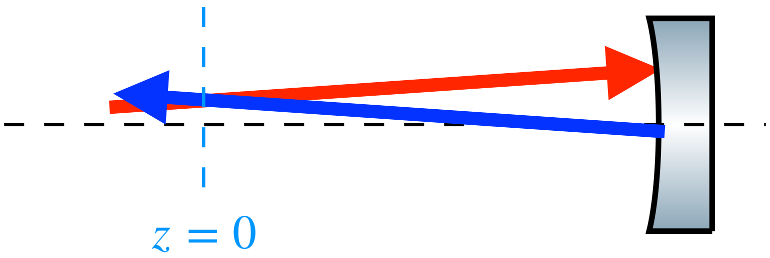
$$= e^{2i\eta}\left[U_0 + \left(\frac{(1-2\frac{z}{R})\delta x + 2(1-\frac{z}{R})z\delta\theta}{w_0} - i\frac{(1-\frac{2z}{R})\delta\theta - \frac{2}{R}\delta x}{\alpha_0}\right)U_1\right]$$

- The result is coincident with that of ray optics

WFS: Reflection by Compound Mirror

- For the compound mirror, Gouy phase difference of HG_{00} & HG_{10} is compensated
  - Reflection beam is

$$e^{2i\eta}\left[U_0+\left(\frac{\delta x}{w_0}-i\frac{\delta\theta}{\alpha_0}\right)e^{2i\eta}U_1\times e^{-2i\eta}\right]=e^{2i\eta}\left[U_0+\left(\frac{\delta x}{w_0}-i\frac{\delta\theta}{\alpha_0}\right)U_1\right]$$

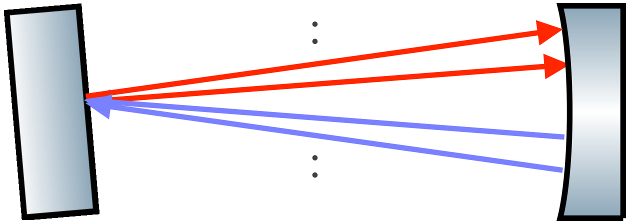


- What this means?
  - Reflected beam goes to the same position at the beam waist

WFS: Understanding cWFS

- Every time reflected by front mirror tilted α_F, the beam tilt increase by $2\alpha_F$
- After reflected by comp. mirror beam goes to the same position at the waist = front mirror without changing beam tilt
- The beam tilt again increase by front mirror reflection
- …
- Finally, the beam tilt is increased by number of cavity reflection

$\propto$ cavity Finesse !



Statistics: Motivation

- At start: question about Brown motion experiments
- How should we consider statistical error exactly in these experiments?
- Is what we've told B3 students right or not?
- Furthermore, the statistical analysis of Brown motion and axion search seems to have something in common
- In this section:

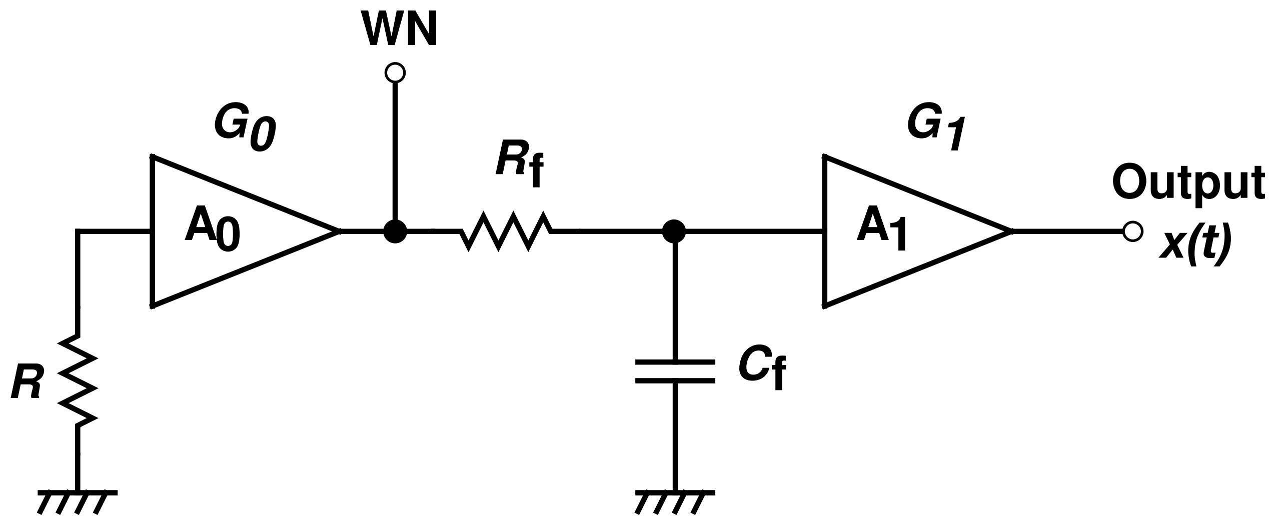
◉ Statistical error of thermal noise of resistor

◉ Statistical error of Brown motion

◉ Common point of Brown motion and axion search

Statistics: Thermal Noise of Resistors

- Measure thermal noise of resistors and estimate temperature of the resistor
- We impose students to take measurements so that statistical error is within 5%
- Output is low-passed
  - Effective number of measurements decreases by a factor of τ_f
  - τ_f : time constant of autocorrelation (= that of the low-pass filter)



Statistics: Question for Thermal Noise

Questions

- How should we consider the statistical error of variance?
  - $T \propto \mathrm{Var}[x]$ → error of T is prop. to error of variance of x
  - Need to consider 4th order moment?
- What is the exact factor to be divided for calculating the statistical error?
  - Exactly τ_{f} ? Is there any factor?

Answers

- Calculate 4th order moment
  - Gaussian → written by autocorrelation
- The factor is EXACTLY τ_{f}

Statistics: No Filter Case

- First, consider the output without a low-pass filter
- $\hat{x}(t)$: white Gaussian with σ_x^2, $\mathrm{E}[\hat{x}(t)] = 0$, $\mathrm{Var}[\hat{x}(t)] = \sigma_x^2$
- Autocorrelation $R_x(\tau) := \mathrm{E}[\hat{x}(t)\hat{x}(t+\tau)] = \lim_{\Delta t \to 0} \sigma_x^2 \delta(\tau) \Delta t$
- Estimated variance

$$\hat{\sigma}_x^2 := \frac{1}{T} \int_{-T/2}^{T/2} dt\, \hat{x}(t)$$

- Mean of $\hat{\sigma}_x^2$

$$\mathrm{E}[\hat{\sigma}_x^2] = \frac{1}{T} \int_{-T/2}^{T/2} dt\, \mathrm{E}[\hat{x}(t)^2] = \sigma_x^2$$

  - unbiased estimator
- Variance of $\hat{\sigma}_x^2$

$$\mathrm{Var}[\hat{\sigma}_x^2] = \mathrm{E}[(\hat{\sigma}_x^2)^2] - \mathrm{E}[\hat{\sigma}_x^2]^2 = \dots$$

$$= \frac{2}{T} \sigma_x^4$$

Statistics: With Filter

- Next, consider with filter
- $\hat{y}(t)$: noise filtered by $h(t)$: $\hat{y}(t) = \hat{x}(t) * h(t)$, $\mathrm{E}[\hat{y}(t)] = 0$,

$$\mathrm{Var}[\hat{y}(t)] = \sigma_x^2 \int_{-\infty}^{\infty} df \left| \tilde{h}(f) \right|^2 =: \sigma_y^2$$

- Estimated variance

$$\hat{\sigma}_y^2 := \frac{1}{N} \sum_{n=0}^{N-1} \hat{y}(n)^2$$

- Mean of $\hat{\sigma}_y^2$

$$\mathrm{E}[\hat{\sigma}_y^2] = \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{E}[\hat{y}(n)^2] = \sigma_y^2$$

  - unbiased estimator

Statistics: With Filter

$$\mathrm{E}[(\hat{\sigma}_y^2)^2] = \frac{1}{T^2}\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 \mathrm{E}[\hat{y}(t_1)^2 \hat{y}(t_2)^2] = \dots$$

$$= \sigma_y^4 + \frac{4}{T}\int_0^T d\tau\,(T - |\tau|)R(\tau)^2$$

- Therefore

$$\mathrm{Var}[\hat{\sigma}_y^2] = \frac{4}{T^2}\int_0^T d\tau\,(T - |\tau|)R(\tau)^2$$

- For enough large T

$$\mathrm{Var}[\hat{\sigma}_y^2] \simeq \frac{4}{T}\int_0^T d\tau\,R(\tau)^2$$

Statistics: With Filter

- In our case

$$\sigma_y^2 = \frac{\sigma_x^2}{R_\mathrm{f}C_\mathrm{f}} = \frac{\sigma_x^2}{\tau_\mathrm{f}}, \quad R(\tau) = \frac{\sigma_x^2}{2R_\mathrm{f}C_\mathrm{f}} e^{-\frac{\tau}{R_\mathrm{f}C_\mathrm{f}}} = \frac{\sigma_x^2}{2\tau_\mathrm{f}} e^{-\frac{\tau}{\tau_\mathrm{f}}} = \sigma_y^2 e^{-\frac{\tau}{\tau_\mathrm{f}}}$$

- Then, for $T \gg \tau_\mathrm{f}$

$$\mathrm{Var}[\hat{\sigma}_y^2] \simeq \frac{2\tau_\mathrm{f}}{T}\sigma_y^2$$

- Comparing to no filter case:

$$\mathrm{Var}[\hat{\sigma}_x^2] = \frac{2}{T}\sigma_x^2 \qquad \mathrm{Var}[\hat{\sigma}_y^2] = \frac{2\tau_\mathrm{f}}{T}\sigma_y^2$$

- Due to the low-pass filter, statistical error is bigger by τ_f
  - Fortunately, what we've told to B3 student is right!

Statistics: SNR of Brown Motion

- Signal: $\hat{x}[n]$
- Spectral density estimator

$$\hat{S}_j^T := \frac{2}{T}\left|\sum_{n=0}^{N-1}\hat{x}[n]e^{-i2\pi\frac{nj}{N}}\right|^2$$

- Hypotheses

$$\begin{aligned} H_0 &: \ \hat{x}[n] = \hat{w}[n] \\ H_1 &: \ \hat{x}[n] = \hat{s}[n] + \hat{w}[n] \end{aligned}$$

- For each hypothesis, the probability density is:

$$p\left(\hat{S}_j^T \middle| H_0\right) = \frac{1}{S_w(f_j)}\exp\left[-\frac{\hat{S}_j^T}{S_w(f_j)}\right] \qquad \mathrm{E}\left[\hat{S}_j^T \middle| H_0\right] = S_w(f_j)$$

$$p\left(\hat{S}_j^T \middle| H_1\right) = \frac{1}{S_w(f_j)}\exp\left[-\frac{\hat{S}_j^T}{S_s(f_j) + S_w(f_j)}\right] \qquad \mathrm{E}\left[\hat{S}_j^T \middle| H_1\right] = S_s(f_j) + S_w(f_j)$$

Statistics: Likelihood Ratio

- Next, define likelihood ratio:

$$\Lambda\left(\hat{x}\,\middle|\,H_1\right) := \frac{p\left(\hat{x}\,\middle|\,H_1\right)}{p\left(\hat{x}\,\middle|\,H_0\right)}$$

$$= \prod_{j=1}^{N-1} \frac{p\left(\hat{S}_j^T\,\middle|\,H_1\right)}{p\left(\hat{S}_j^T\,\middle|\,H_0\right)}$$

$$l\left(\hat{x}\,\middle|\,H_1\right) := \ln \Lambda\left(\hat{x}\,\middle|\,H_1\right)$$

$$= \sum_{j=1}^{N-1} \left[\ln p\left(\hat{S}_j^T\,\middle|\,H_1\right) - \ln p\left(\hat{S}_j^T\,\middle|\,H_0\right)\right]$$

$$= \sum_{j=1}^{N-1} \left[-\frac{\hat{S}_j^T}{S_s(f_j) + S_w(f_j)} + \frac{\hat{S}_j^T}{S_w(f_j)} - \ln \frac{S_w(f_j)}{S_s(f_j) + S_w(f_j)}\right]$$

Statistics: Fisher Information

- Using likelihood ratio, calculate fisher information

$$I(\lambda) := \mathrm{E}\left[\left(\frac{\partial}{\partial\lambda} l\left(\hat{x} \,|\, H_1(\lambda)\right)\right)^2 \middle| H_1(\lambda)\right]$$

- Then, the lower band of statistical error is given by Cramer-Rao inequality:

$$\mathrm{Var}[\lambda] \geq \frac{1}{I(\lambda)}$$

- SNR is then defined as

$$\frac{S^2}{N^2} = \frac{\lambda^2}{1/I(\lambda)}$$

Statistics: SNR

- In Brown motion case, $\lambda = T_{\mathrm{room}}$
- After some calculation we get

$$I(\lambda) = \lambda^2 \sum_{j=1}^{N-1} \left(\frac{S_s(f_j)}{S_s(f_j) + S_w(f_j)} \right)^2$$

- Around resonance peak $S_s(f_j) \gg S_w(f_j)$
  - $I(\lambda) \simeq \lambda^2 \times$ number of bins within bandwidth of resonance peak

$$\simeq \lambda^2 \frac{2\pi f_0 T}{Q}$$

$$\frac{S^2}{N^2} \simeq \frac{2\pi f_0 T}{Q}$$

→ SNR is proportional to √T

Statistics: Axion Search

- How about axion search?
- I think same procedure is applicable

◉ Coherent case:

$$I(\lambda) = \lambda^2 \left(\frac{S_s(f_j)}{S_s(f_j) + S_w(f_j)} \right)^2$$

◉ Incoherent case:

$$I(\lambda) = \lambda^2 \sum_{j=1}^{N-1} \left(\frac{S_s(f_j)}{S_s(f_j) + S_w(f_j)} \right)^2$$

$$\simeq \lambda^2 T \int_0^\infty df \left(\frac{S_s(f)}{S_s(f) + S_w(f)} \right)^2$$

- That may make our understanding further?