Cavity response

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1 Scope of this paper

In the aLIGO Arm Length Stabilization (ALS) technique, the transmitted light from a Fabry-Perot (FP) cavity is picked off and used as a part of the main sensor. One of the important issue associated with such a scheme is that one might need to estimate how much frequency noise can be added in the transmitted light due to displacement in the cavity length.

This report summarizes the derivation of the transfer function from the displacement to the frequency, which turns out to be a high pass filter [1].

2 Gedanken Experiment

To discuss the issue further, we perform a Gedanken experiment in which an FP cavity is illuminated by a laser source, wavelength of which is at λ as shown in figure 1. To discuss the transmitted field E_t, hereafter we consider only the intra-cavity field E because they are essentially equivalent.

The intra-cavity field propagating toward the end mirror $E(t)$ satisfies,

$$E(t) = t_1 E_{\mathrm{in}}(t) + r_1 r_2 e^{-2ik(L+\xi)} E\left(t - 2T\right), \tag{1}$$

where T is the one-way-trip time defined by $T = L/c$. Since we are interested in the case where the FP cavity is on resonance, we assume $2ikL = 2\pi n$. Thus the equation simply becomes

$$E(t) = t_1 E_{\mathrm{in}}(t) + r_1 r_2 e^{-2ik\xi(t)} E\left(t - 2T\right). \tag{2}$$

Here ξ represents the amount of displacement from the resonance point at a given time t [2].

[1] Most of the derivation argued in this report follow the formalization done by M. Rakhmanov in his Ph.D. thesis.

[2] Actually ξ can be expressed by $\xi(t) = x_2(t - T) - x_1(t)$. However this form is unnecessary for now as what we care is only the displacement in the cavity length.

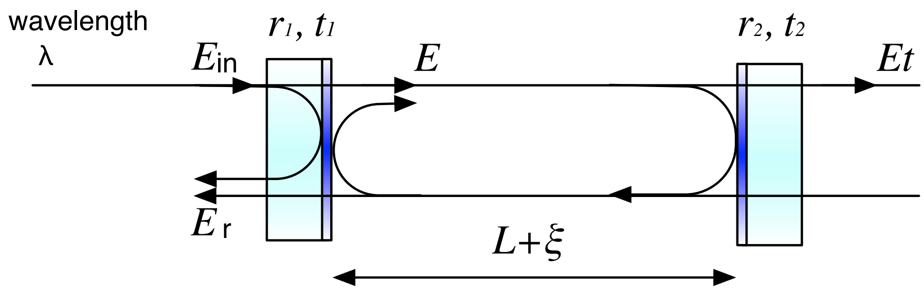

Figure 1: An FP cavity

3 The transfer function

Consider a small displacement so that

$$e^{-2ik\xi(t)} \approx 1 - 2ik\xi(t). \tag{3}$$

Due to the small deviation ξ, the intra-cavity field is perturbed and can be expressed by

$$E(t) = \bar{E} + \delta E(t). \tag{4}$$

Here $\bar{E}$ is a constant, representing the static field. On the other hand the second term $\delta E(t)$ is a small field induced by the displacement and hence a function of time.

Since we don't care any variations in the incident field, let us assume that the incident field is also a constant, so that

$$E_{\rm in} = A. \tag{5}$$

Plunging equation (3),(4) and (5) into equation (2), one can obtain

$$\bar{E} + \delta E(t) = t_1 A + r_1 r_2 \bar{E} - 2ik\xi(t) r_1 r_2 \bar{E} + r_1 r_2 \delta E(t - 2T). \tag{6}$$

During the algebra we neglect a higher order term of $2ikr_1r_2\xi\delta E(t-2T)$. The static solution (when no perturbation is applied) is found to be

$$\bar{E} = t_1 A + r_1 r_2 \bar{E}. \tag{7}$$

Using this relation we remove some of the terms from equation (6),

$$\delta E(t) - r_1 r_2 \delta E(t - 2T) = -2ikr_1 r_2 \xi \bar{E} \tag{8}$$

Applying the Laplace transform, we obtain

$$\tilde{\delta E} - r_1 r_2 \tilde{\delta E} e^{-2ST} = -2ikr_1 r_2 \bar{E} \tilde{\xi}. \tag{9}$$

Therefore

$$\frac{\tilde{\delta E}}{\bar{E}} = -2ik \frac{r_1 r_2}{1 - r_1 r_2 e^{-2ST}} \tilde{\xi}. \tag{10}$$

This form is very convenient since a small phase rotation $\Delta\phi$ in a general field E can be written as

$$\begin{aligned} E(\phi + \Delta\phi) &\approx E(\phi)\left(1 + i\Delta\phi\right), & (11)\\ &= E\left(1 + \frac{\delta E}{E}\right). & (12) \end{aligned}$$

Therefore it is clear that equation (10) represents the amount of the phase rotation induced by the displacement ξ.

Now let us convert equation (10) into the frequency of the field by applying the $\phi \to f$ formula : $\nu = S\phi/2\pi$,

$$H_x(s) \equiv \frac{\nu}{\xi} = -i\frac{k}{\pi} \frac{r_1 r_2 S}{1 - r_1 r_2 e^{-2ST}} \tilde{\xi}. \tag{13}$$

This is the transfer function from the displacement to the frequency of the transmitted light.

For a practical use, it is useful to consider the displacement as a frequency variation in the eigen frequency of the cavity. The eigen frequency of the cavity and displacement is connected via

$$\frac{\delta\nu}{\mathrm{FSR}} = \frac{\xi}{\lambda/2}. \tag{14}$$

Using this, one can obtain another form of the transfer function,

$$H_\nu(s) = -2ST \frac{r_1 r_2}{1 - r_1 r_2 e^{-2ST}}. \tag{15}$$

When a frequency of interest (or S) is sufficiently large, where $1 << r_1 r_2 e^{-2ST}$, the transfer function becomes $H_\nu(s) \to 1$.

Figure 2 shows an example of the transfer function, As shown in the figure, the transfer function is a high pass filter, and becomes flat above the

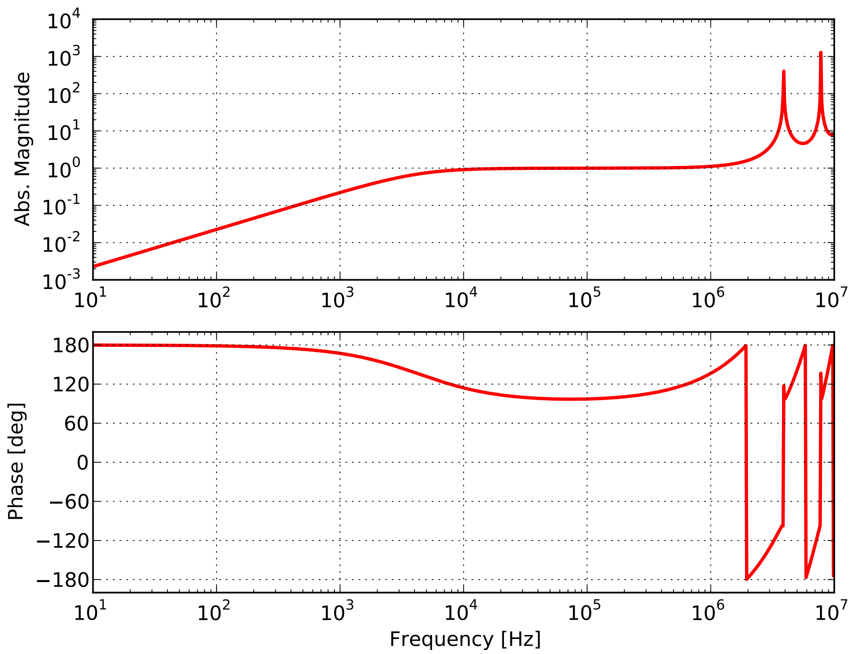

Figure 2: Bode plot of the transfer function

cavity pole. One the frequency reaches the FSR, it exhibits some peaks. For most of modeling works, it is often approximated by

$$H_\nu(s) = \frac{S}{\omega_c + S}, \tag{16}$$

where ω_c is a cavity pole.