観測天文学特別講義I @ 理1-1043

Observational Astronomy Advanced Course I

October 31, 2025

レーザー干渉計型重力波検出器

Laser interferometric gravitational wave detectors

3. Spectral Analysis and Linear Systems

Yuta Michimura

RESCEU, University of Tokyo

michimura@resceu.s.u-tokyo.ac.jp

https://yutamich.gitlab.io/lectures.html#lectures

Assignment for Oct 31

- In laser interferometric gravitational wave detectors, identify a component that should *not* be treated as a linear system, and explain why.
- You may also answer from the Google Form below
  https://forms.gle/6AwJ48XcpWQXqMon9

Don’t forget to put your name and student#

You may answer in any language

FORM

Plan of the Lecture Today

- Basics of spectral analysis
  - Power spectral density
- Basics of linear system
  - Transfer function
- Examples of linear systems
  - Electronics
  - Hamonic oscillators
- Basics of controls
  - Open loop transfer function
  - Optical spring and damping
- **Goal**: Understand the basics for treating noises and feedback loops

Important for
seismic
thermal
quantum noise
calculations

Forbes

Thirsty Ravens' Tap-Tap-Tapping Creates Data Glitch At LIGO

By GrrlScientist, Senior Contributor. ⓘ GrrlScientist writes about evolution, ec...

Follow Author

Published May 15, 2018, 12:21pm EDT, Updated May 18, 2018, 03:14pm EDT

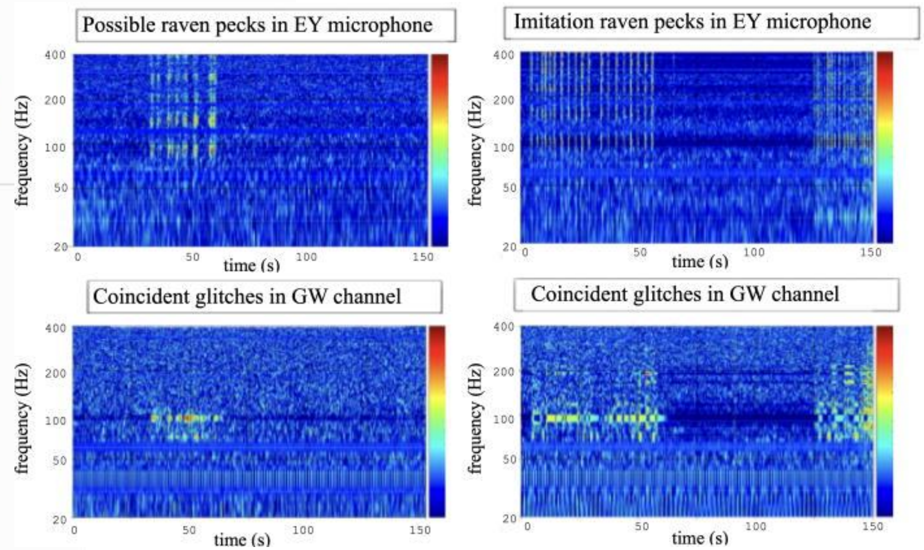
Share Save

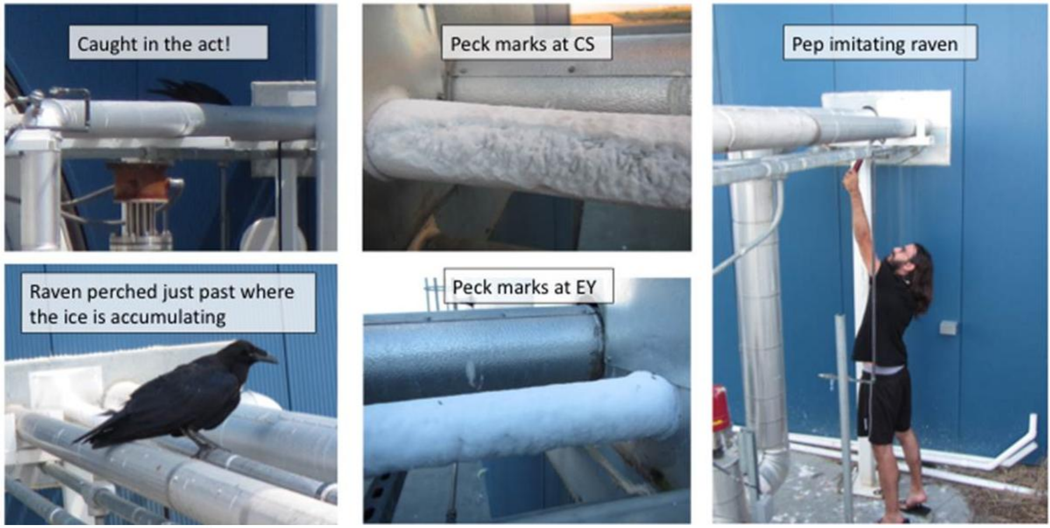
This article is more than 7 years old.

Rascally ravens are rapping ice off some frozen pipes at the Hanford LIGO detection site to create their own snow cones on hot days -- and they're creating a world of trouble for astrophysicists

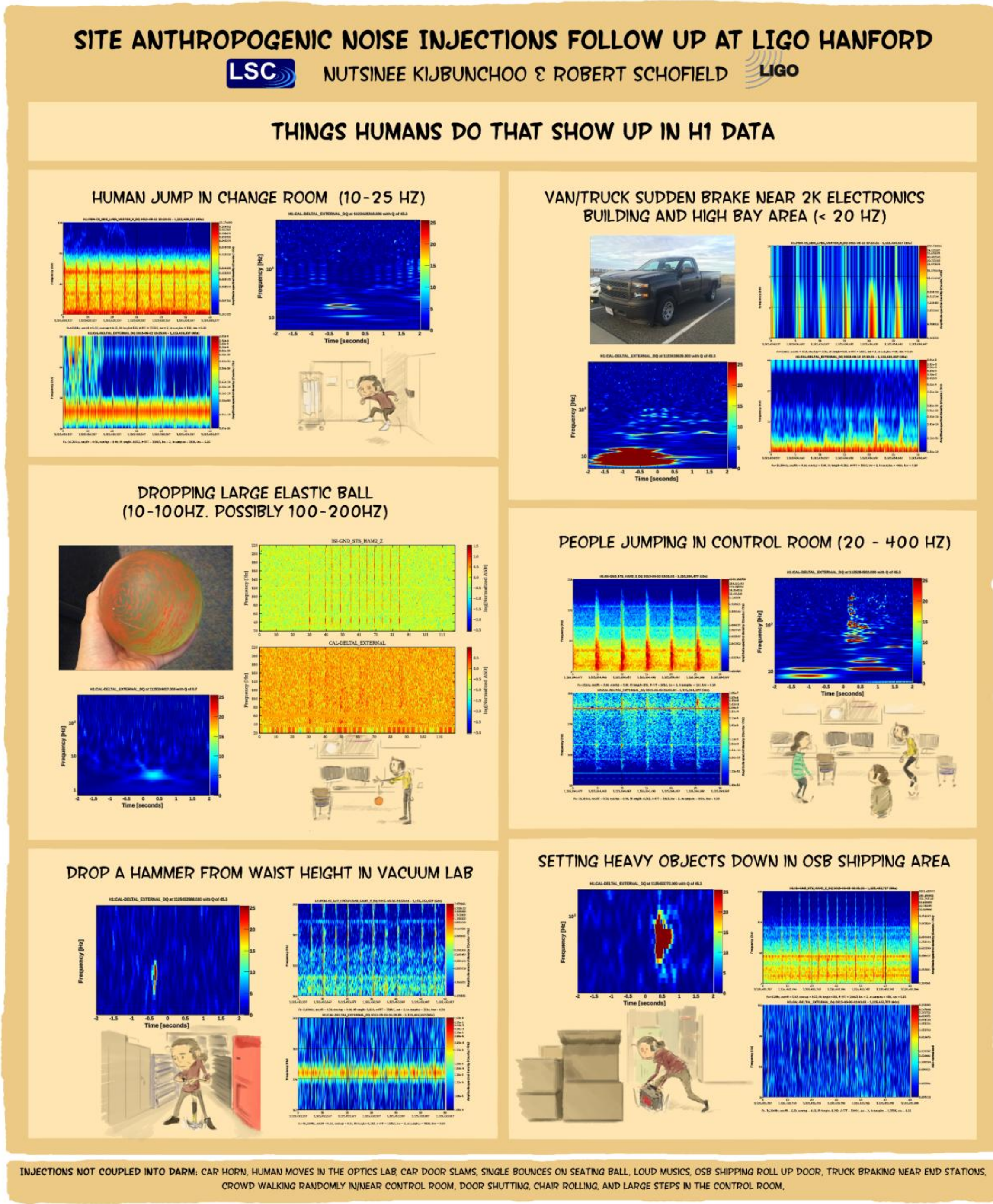
Adult common raven (Corvus corax). (Credit: Minette Layne / CC BY-NC 2.0)
MINETTE LAYNE VIA A CREATIVE COMMONS LICENSE

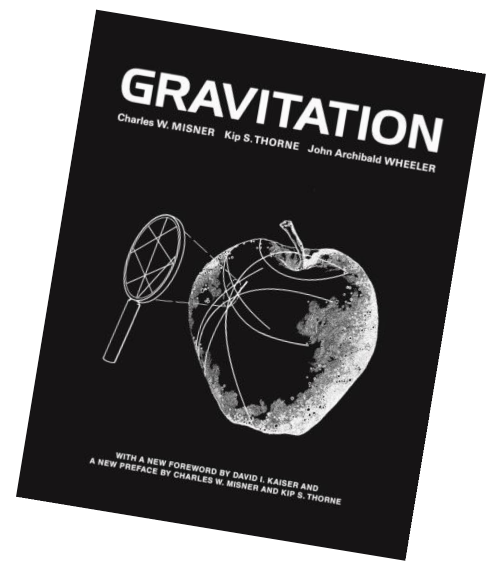
The Advanced Laser Interferometer Gravitational-Wave Observatory (aLIGO) is a pair of vibration detectors, one is located in the desert in Washington state whilst its twin is located in Louisiana. aLIGO was designed to directly observe and measure gravitational waves





Berger+, *Appl. Phys. Lett.* 122, 184101 (2023)
https://www.forbes.com/sites/grrlscientist/2018/05/15/thirsty-ravens-tap-tap-tapping-creates-data-glitch-at-ligo/
https://meetings.aps.org/Meeting/APR18/Event/326669

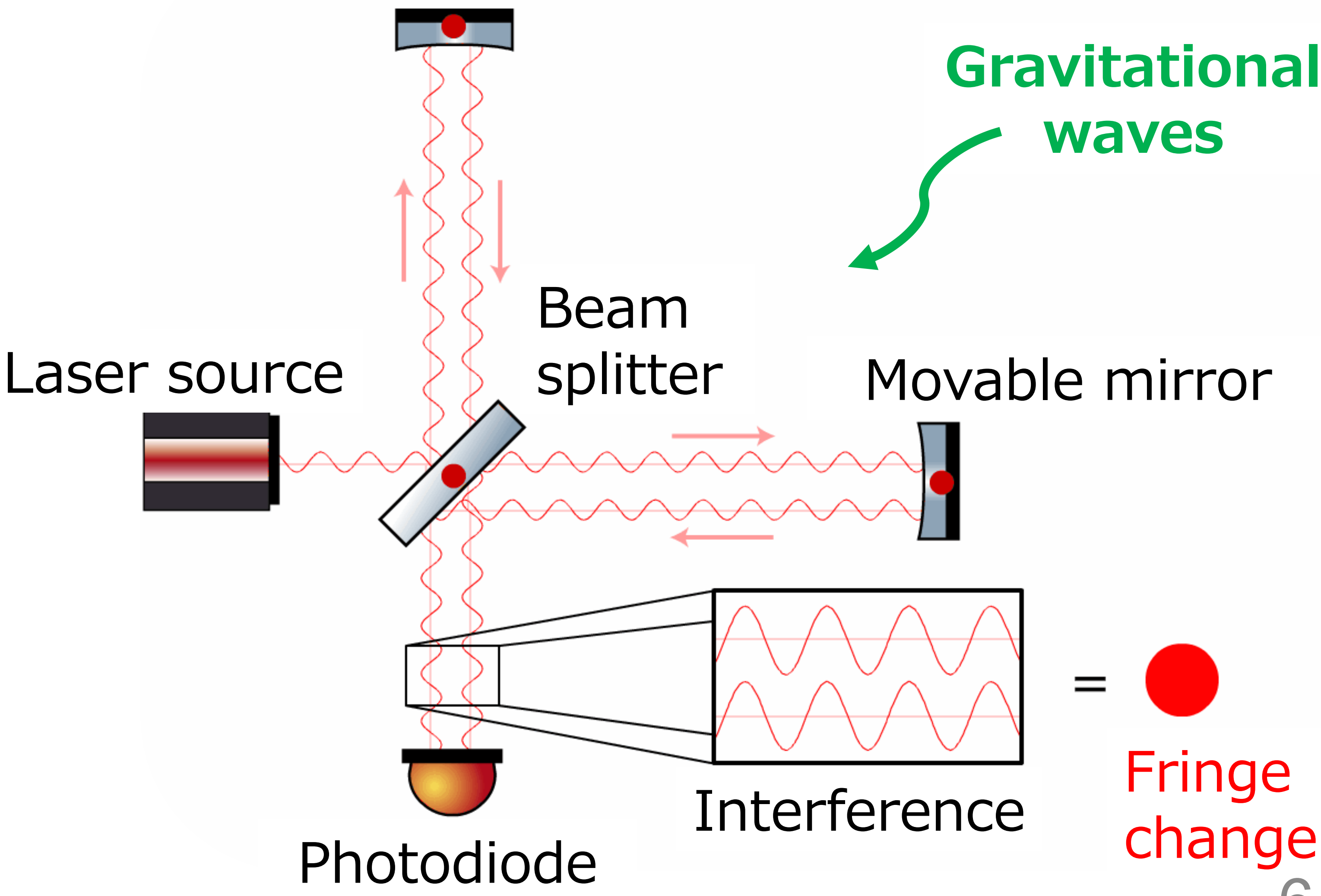




LIGO-G1600547

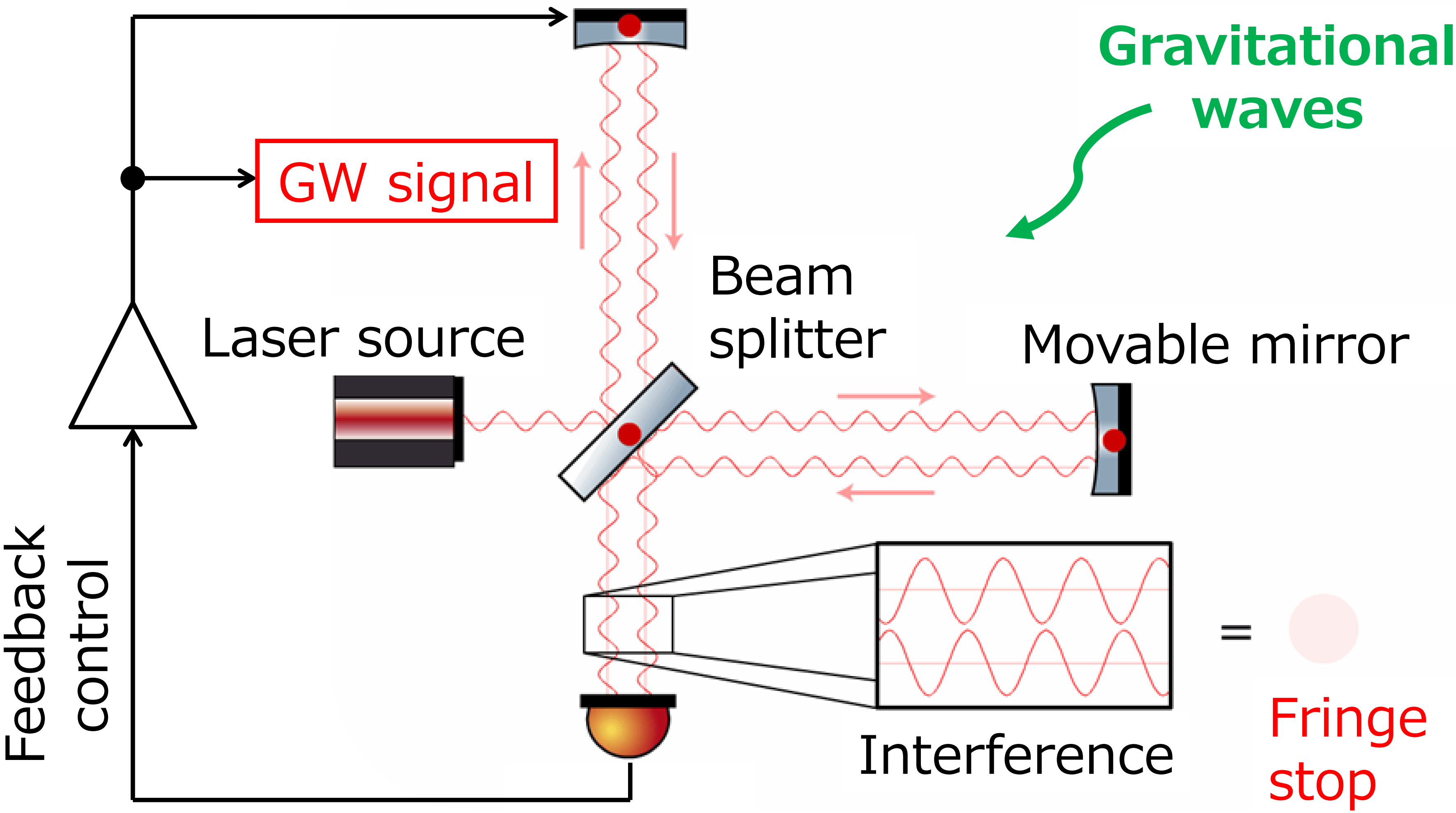
Laser Interferometric GW Detectors

- Measures differential arm length change



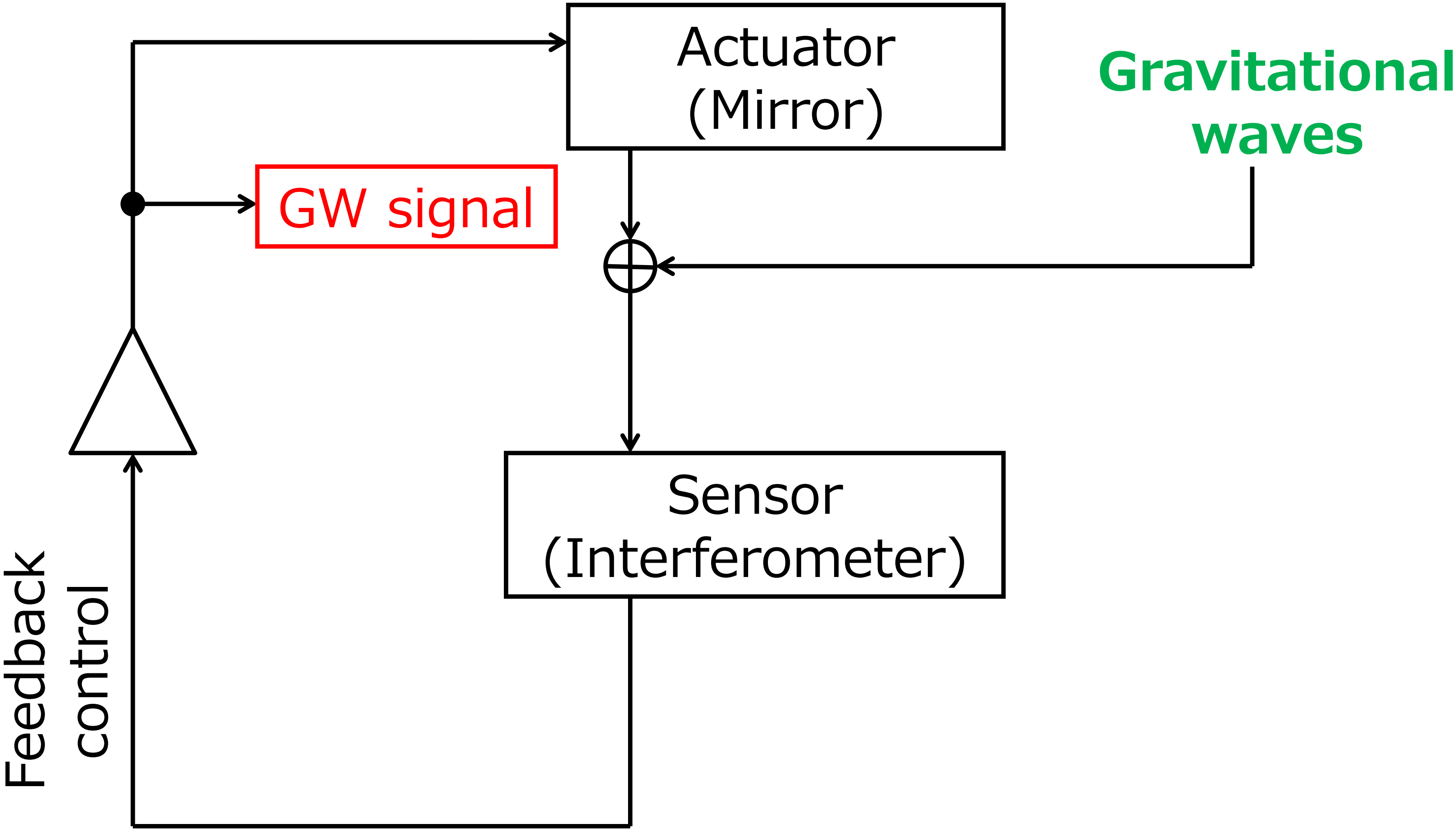
Interferometer Control

- Control mirror positions to “lock” the fringe



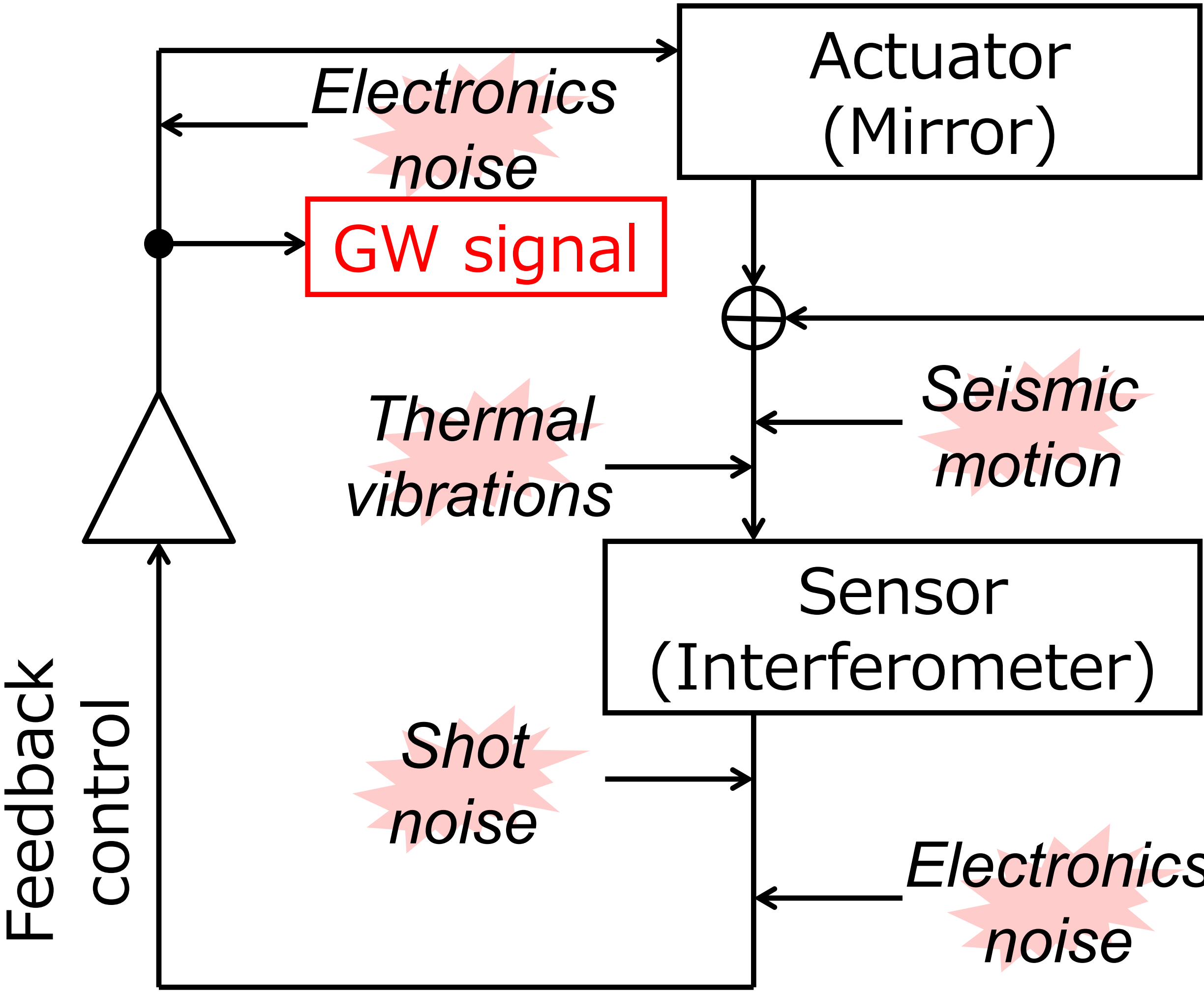
Linear System

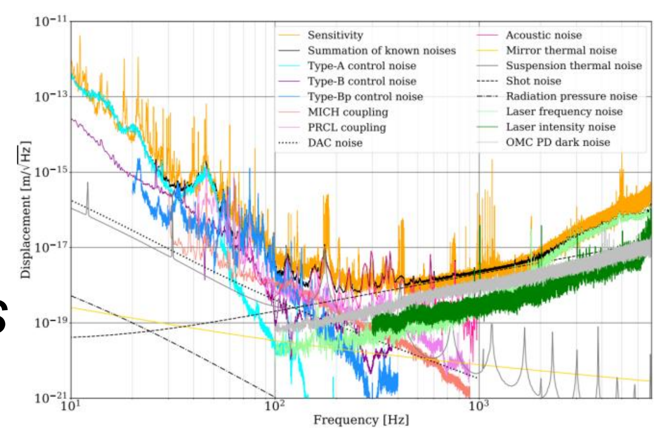
- Control makes the system linear



Noises and Sensitivity

- Noises in the loop limits the sensitivity





Fourier Transformation

- It is useful to understand the system in frequency domain
- Fourier transformation $X(\omega)$ of time series data $x(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} \mathrm{d}t$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} \mathrm{d}\omega$$

- Parseval's Theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \mathrm{d}\omega$$

Fourier Transformation of Calculus

- Differential

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} X(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$$

$$= \int_{-\infty}^{\infty} \underline{i\omega X(\omega)}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$$

- So, Fourier transformation of

Differential of $x(t)$ is $i\omega X(\omega)$

Integral of $x(t)$ is $\frac{1}{i\omega}X(\omega)$

Fourier Transform of Finite Data

- Actually, we can only take data in finite time

$$\begin{aligned} x_T(t) &= x(t) \quad |t| \le T/2 \\ &= 0 \quad\quad |t| > T/2 \end{aligned}$$

$$X_T(\omega) = \int_{-T/2}^{T/2} x(t) e^{-i\omega t} \mathrm{d}t$$

- Note that in real analysis, we usually use proper window functions (Hanning, Hamming, etc.)

Power Spectral Density

- Power spectral density (PSD, or simply spectrum)

$$S(\omega) = \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T}$$

- From Perseval's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \mathrm{d}\omega$$

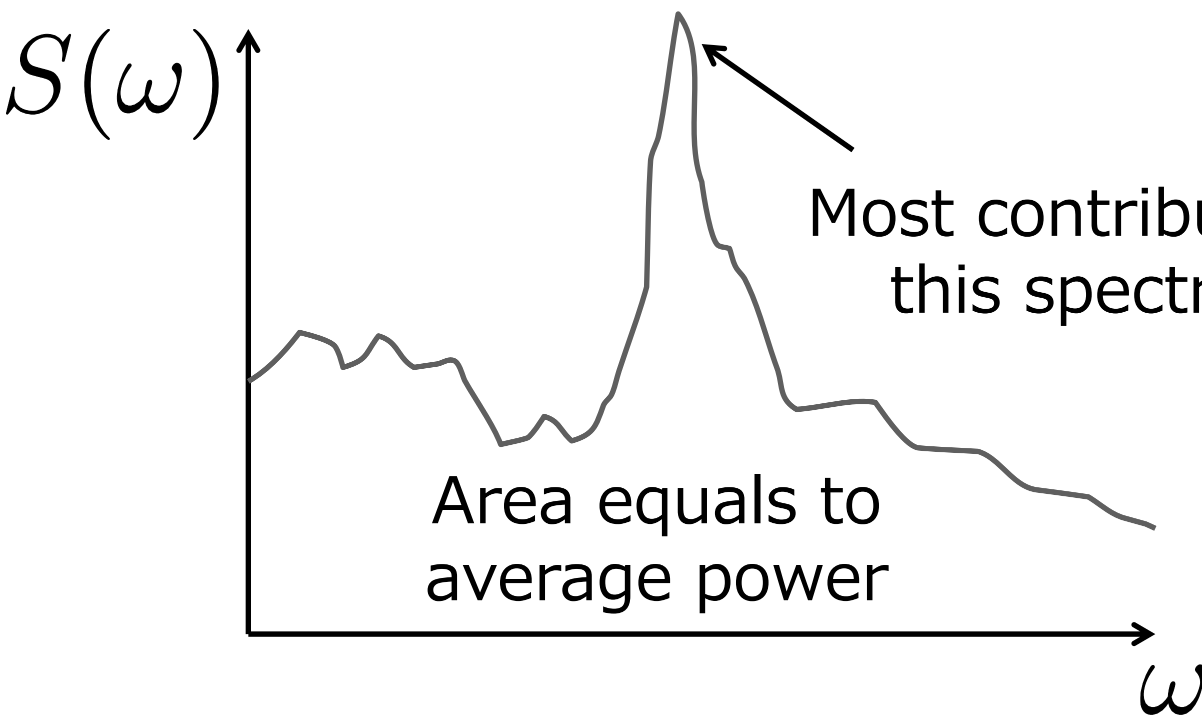
- Therefore

$$\boxed{\overline{x^2(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \mathrm{d}\omega}$$

Power Spectral Density

- Contribution to average power from each frequency bin

$$\boxed{\overline{x^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\mathrm{d}\omega}$$



If x(t) is [m]
S(ω) has [m²/Hz]

Autocorrelation Function

- Autocorrelation function

$$C(\tau) = \langle x(t)x(t+\tau) \rangle$$

- Wiener-Khinchin theorem

$$S(\omega) = \int_{-\infty}^{\infty} C(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\mathrm{d}\tau$$

$$C(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\omega$$

Not used for today, but will be used in the next lecture

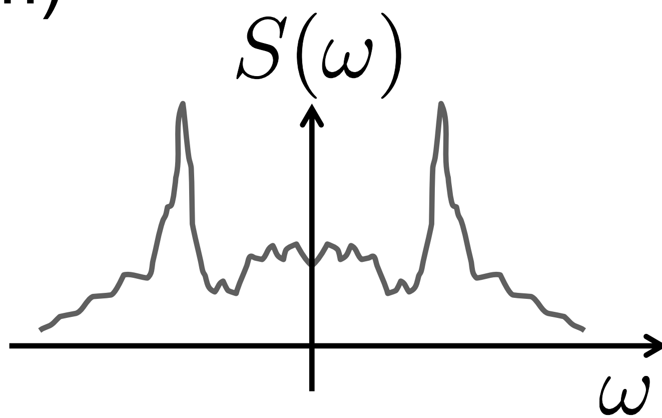
- Power spectral density is a Fourier transform of autocorrelation function

Amplitude Spectral Density

- We usually plot one-sided amplitude spectral density (ASD, or simply spectrum)

$$\overline{x^2(t)} = \int_{\underline{0}}^{\infty} G(f)\mathrm{d}f$$

$$G(f) = \underline{2}S(\omega)$$



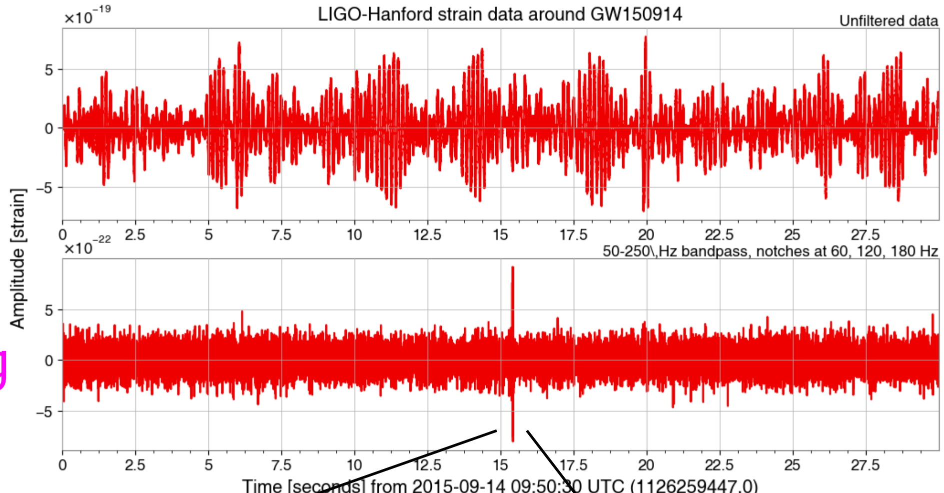
$$S(-\omega) = S(\omega)$$

$\sqrt{G(f)}$ is ASD

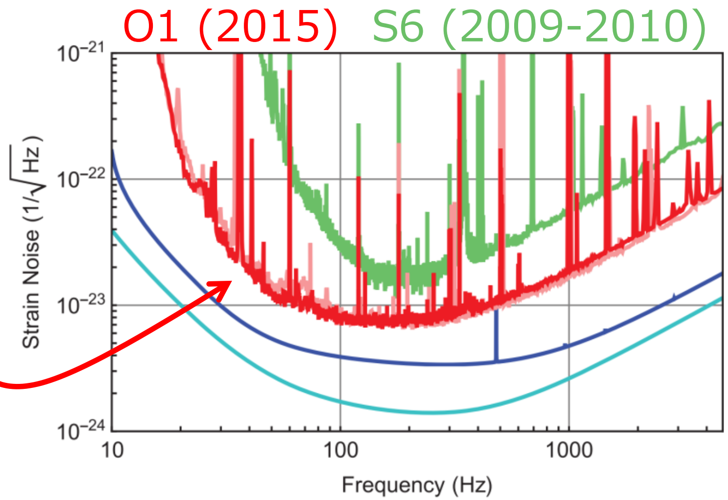
If x(t) is [m]
√G(f) has [m/√Hz]

GW150914 Example

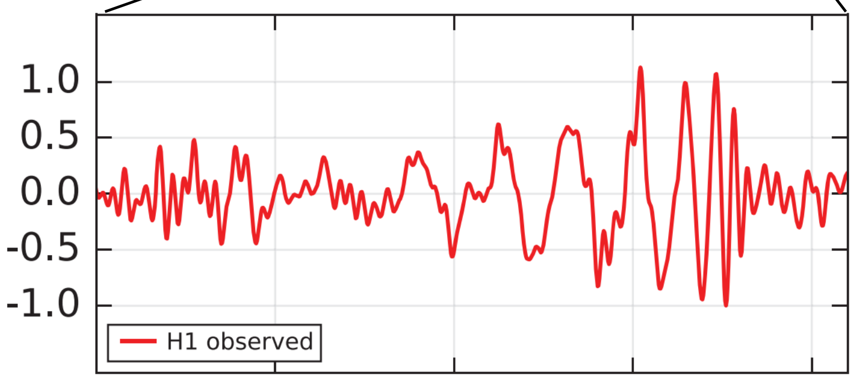
Time series data x(t) in the unit of strain [1]



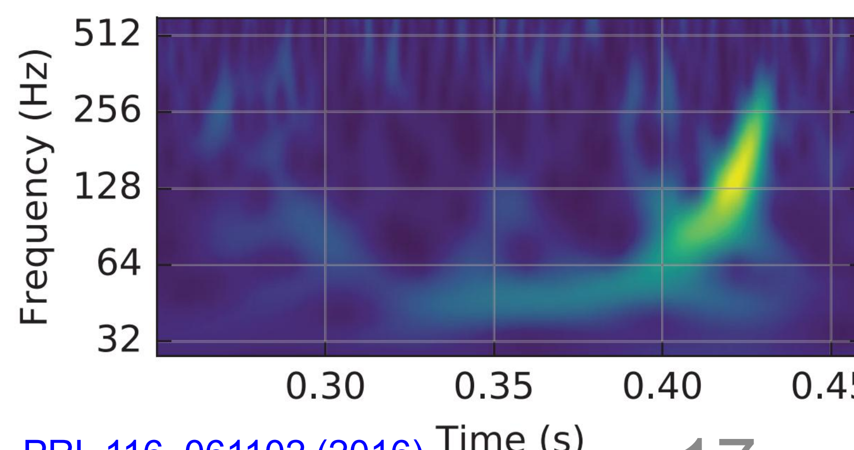
filtering



Amplitude Spectral Density [1/√Hz]



Spectrogram √G(f) of t

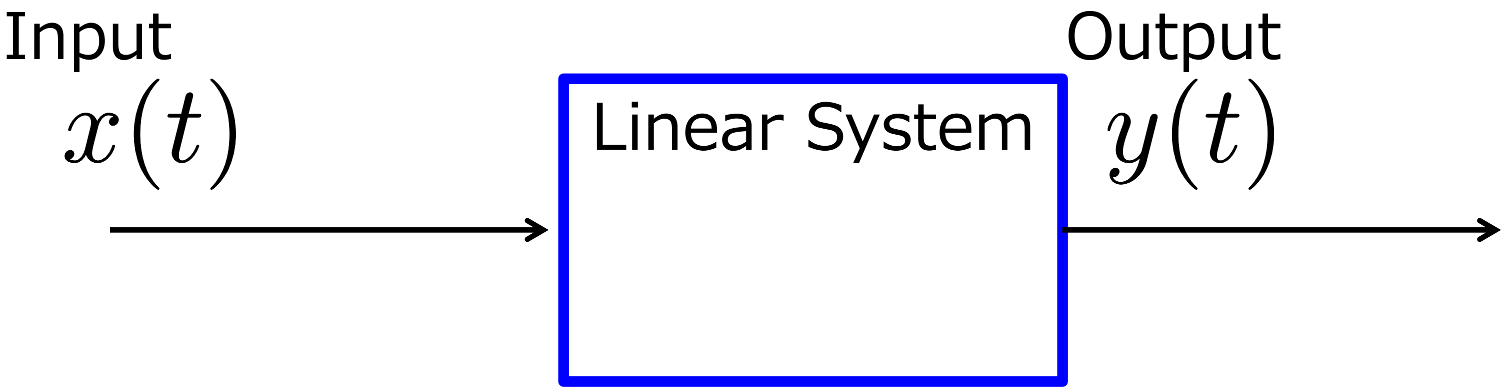


[PRL 116, 061102 (2016)](PRL 116, 061102 (2016))
[PRL 116, 131103 (2016)](PRL 116, 131103 (2016))

https://gwpy.github.io/docs/latest/examples/signal/gw150914/

Linear System

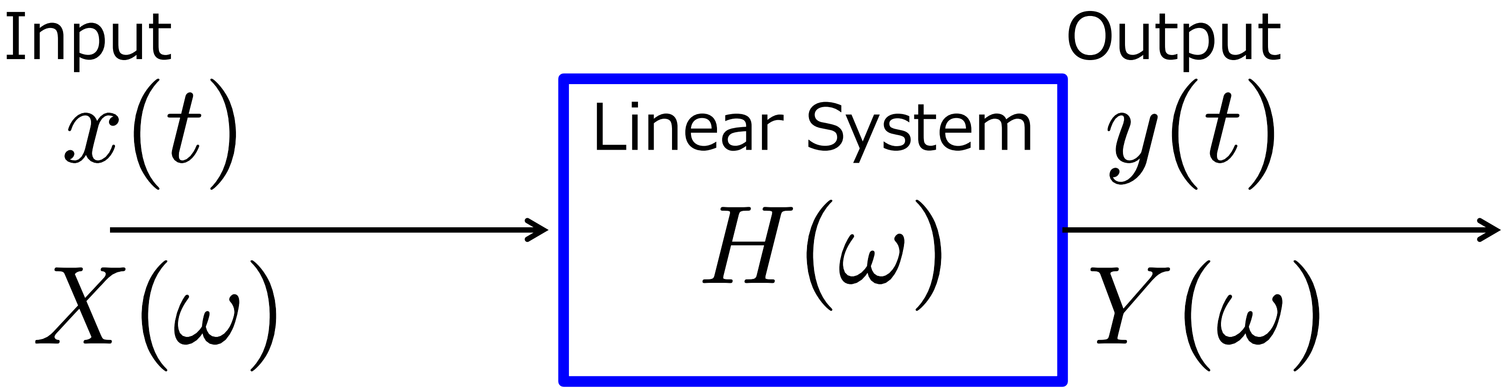
- System that has linearity in input output relation



- Linearity
  - Additivity (加法性)
    Input $x_1(t) + x_2(t)$ Ouput $y_1(t) + y_2(t)$
  - Homogeneity (斉次性)
    Input $ax(t)$ Ouput $ay(t)$

Transfer Function

- Frequency response function of a linear system



- Transfer function (TF)

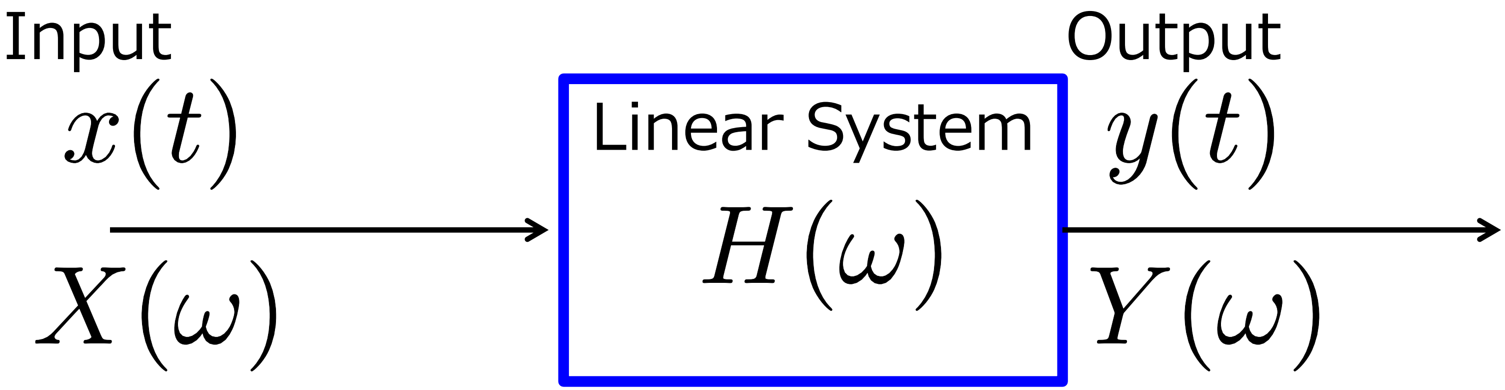
$$H(\omega) = \frac{Y(\omega)\ \text{Output}}{X(\omega)\ \text{Input}}$$

- PSD of the output will be

$$G_y(f) = |H(f)|^2 G_x(f)$$

Impulse Response

- Frequency response function of a linear system



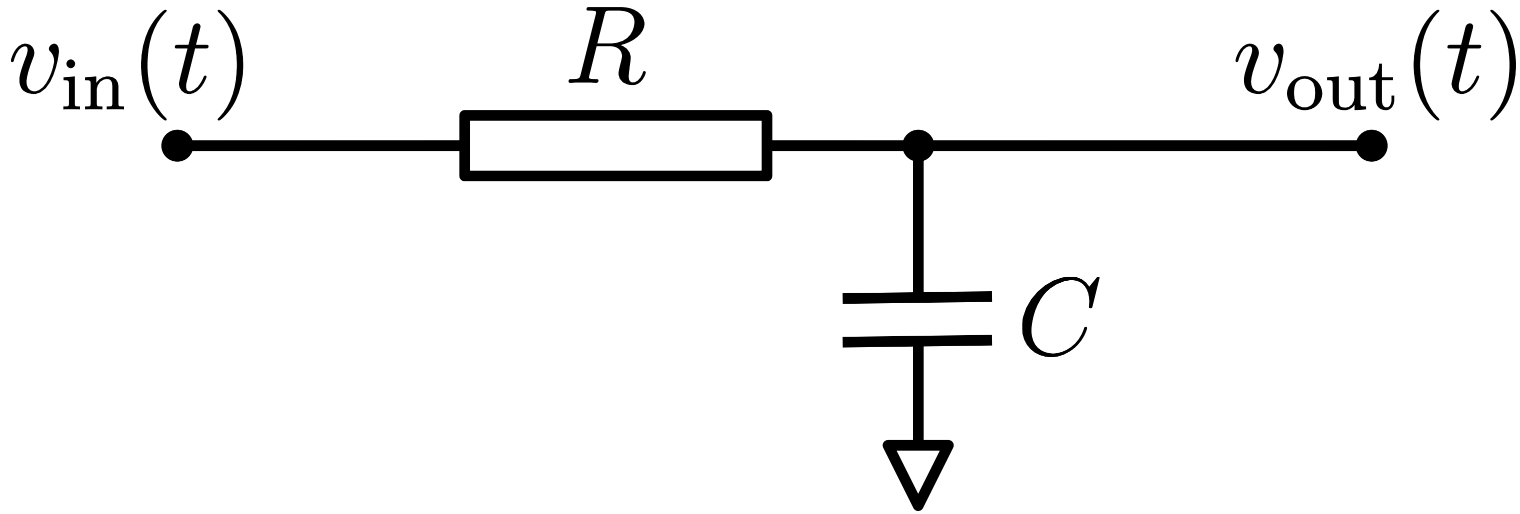
- When $x(t) = 2\pi\delta(x)$, $X(\omega) = 1$, so $H(\omega) = Y(\omega)$

$$y(t) = \int_{-\infty}^{\infty} H(\omega) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega$$

- Impulse response will be a Fourier transformation of the system

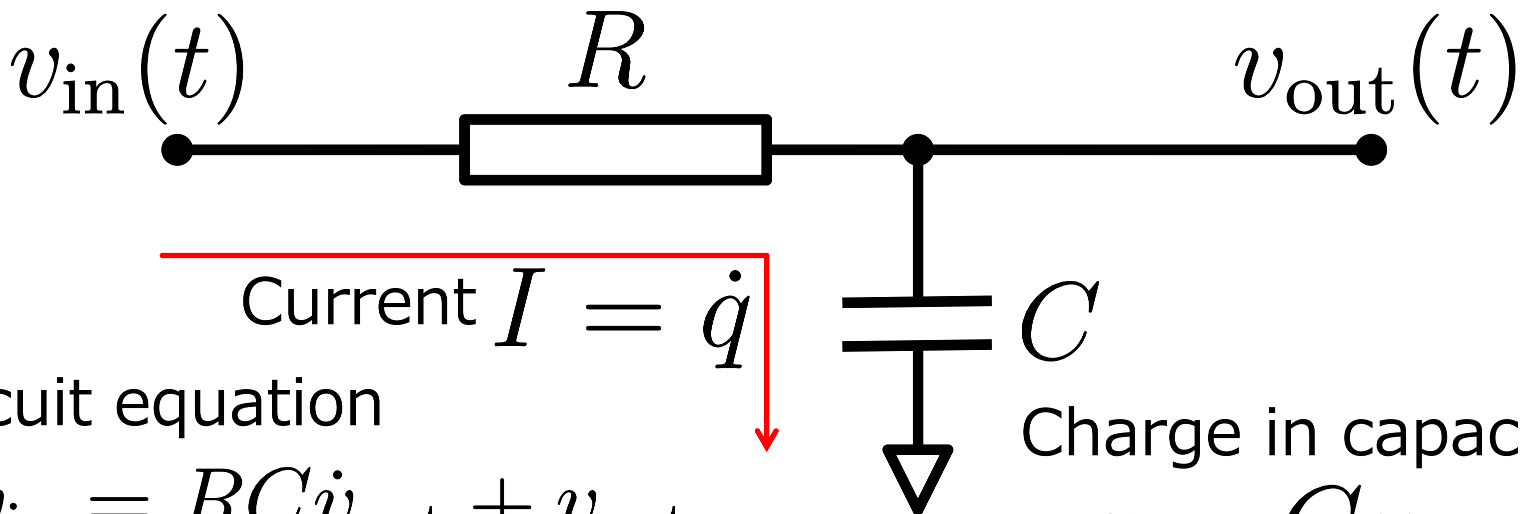
Example 1: Electric Circuit

- What is the transfer function?



Example 1: Electric Circuit

- What is the transfer function?



Charge in capacitor

$$q = C v_{\mathrm{out}}(t)$$

Circuit equation

$$v_{\mathrm{in}} = RC\dot{v}_{\mathrm{out}} + v_{\mathrm{out}}$$

Fourier transform

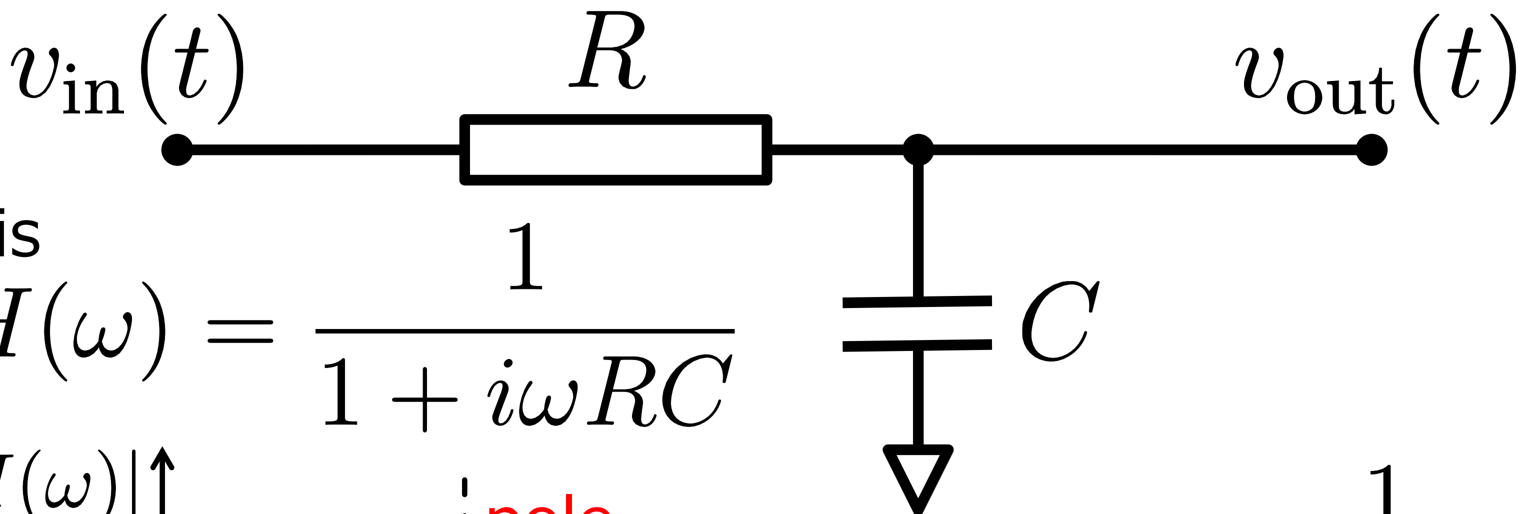
$$V_{\mathrm{in}} = i\omega RC V_{\mathrm{out}} + V_{\mathrm{out}}$$

TF will be

$$H(\omega) = \frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \frac{1}{1 + i\omega RC}$$

Example 1: Electric Circuit

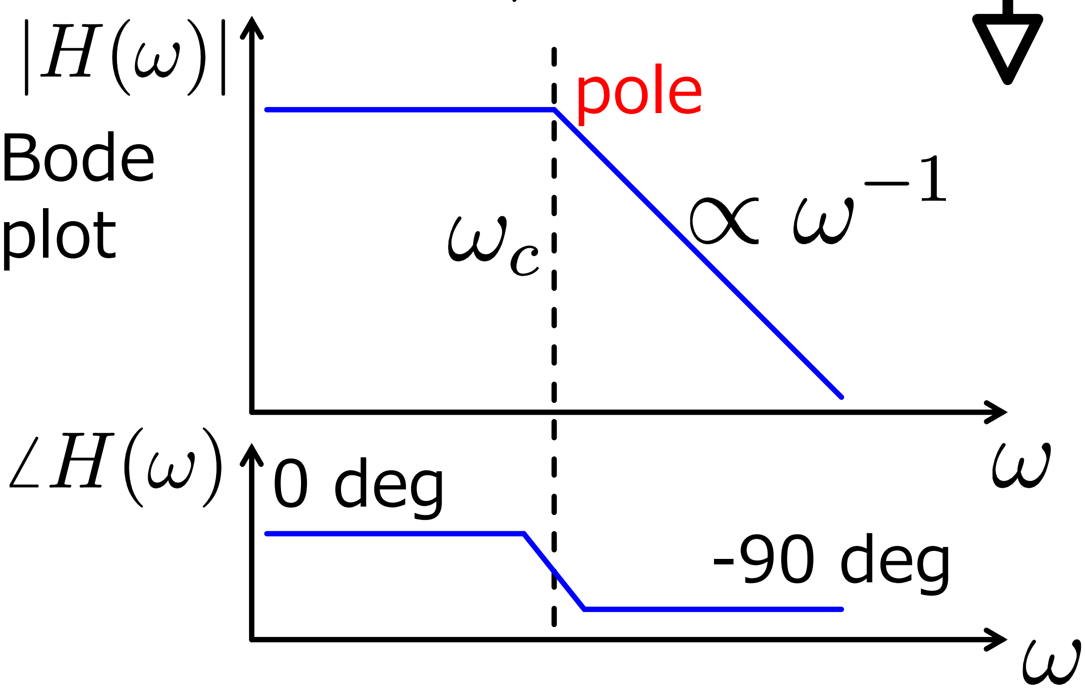
- What is the transfer function?



TF is

$$H(\omega) = \frac{1}{1 + i\omega RC}$$

Bode plot

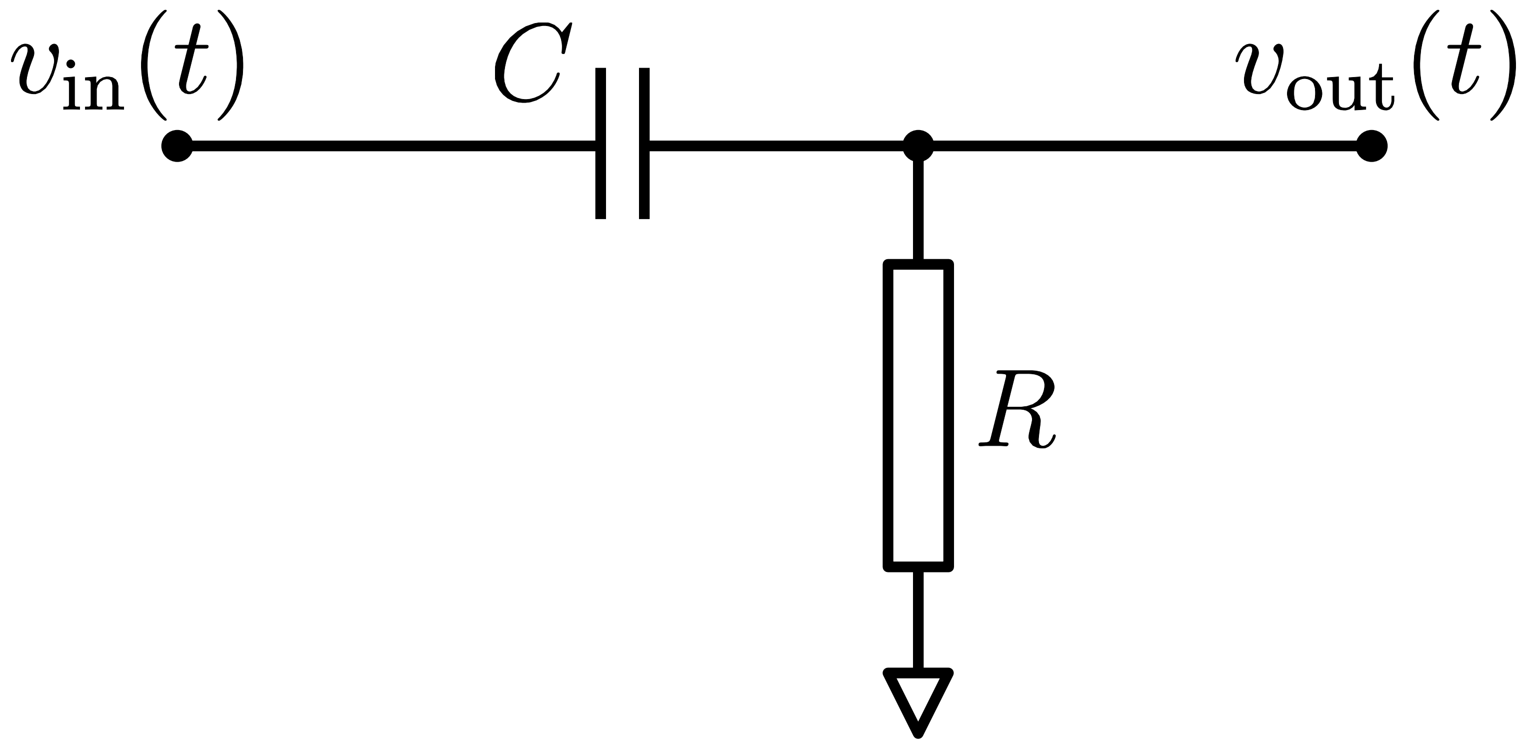


$$\omega_c = \frac{1}{RC}$$

Low pass filter
Integrator circuit

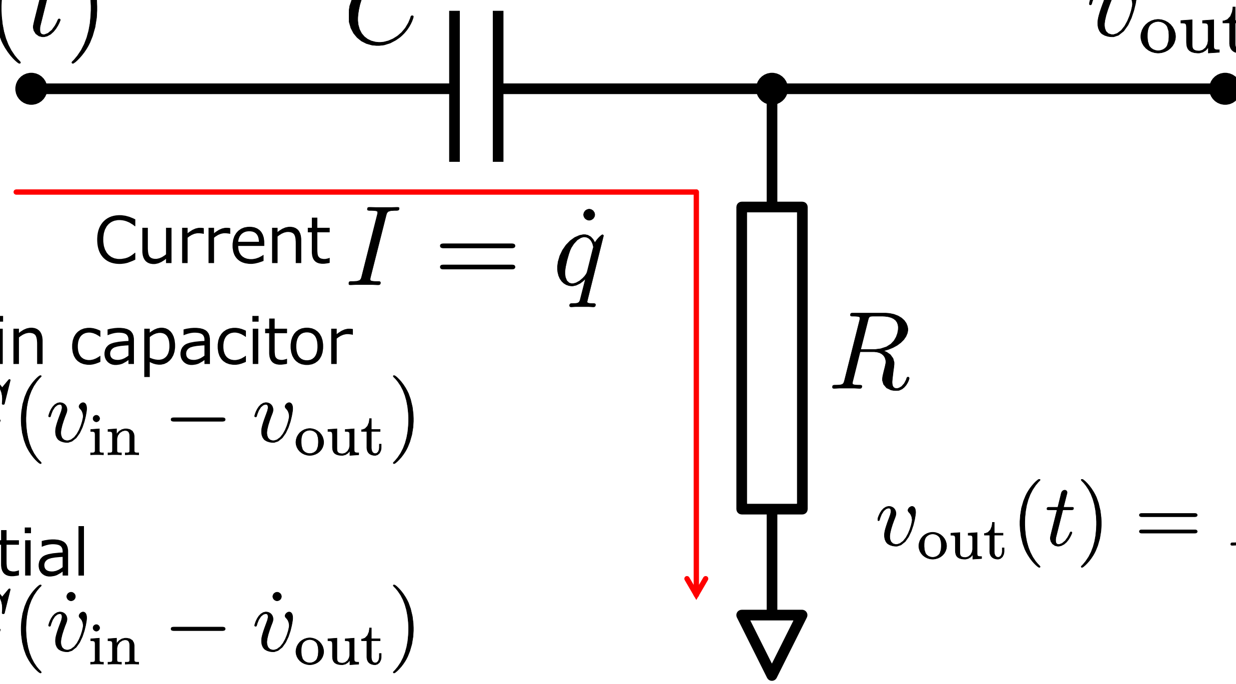
Example 2: Electric Circuit

- What is the transfer function?



Example 2: Electric Circuit

- What is the transfer function?



Charge in capacitor

$$q = C(v_{\text{in}} - v_{\text{out}})$$

$$v_{\text{out}}(t) = RI = R\dot{q}$$

Differential

$$\dot{q} = C(\dot{v}_{\text{in}} - \dot{v}_{\text{out}})$$

$$v_{\text{out}} = RC(\dot{v}_{\text{in}} - \dot{v}_{\text{out}})$$

Fourier transform

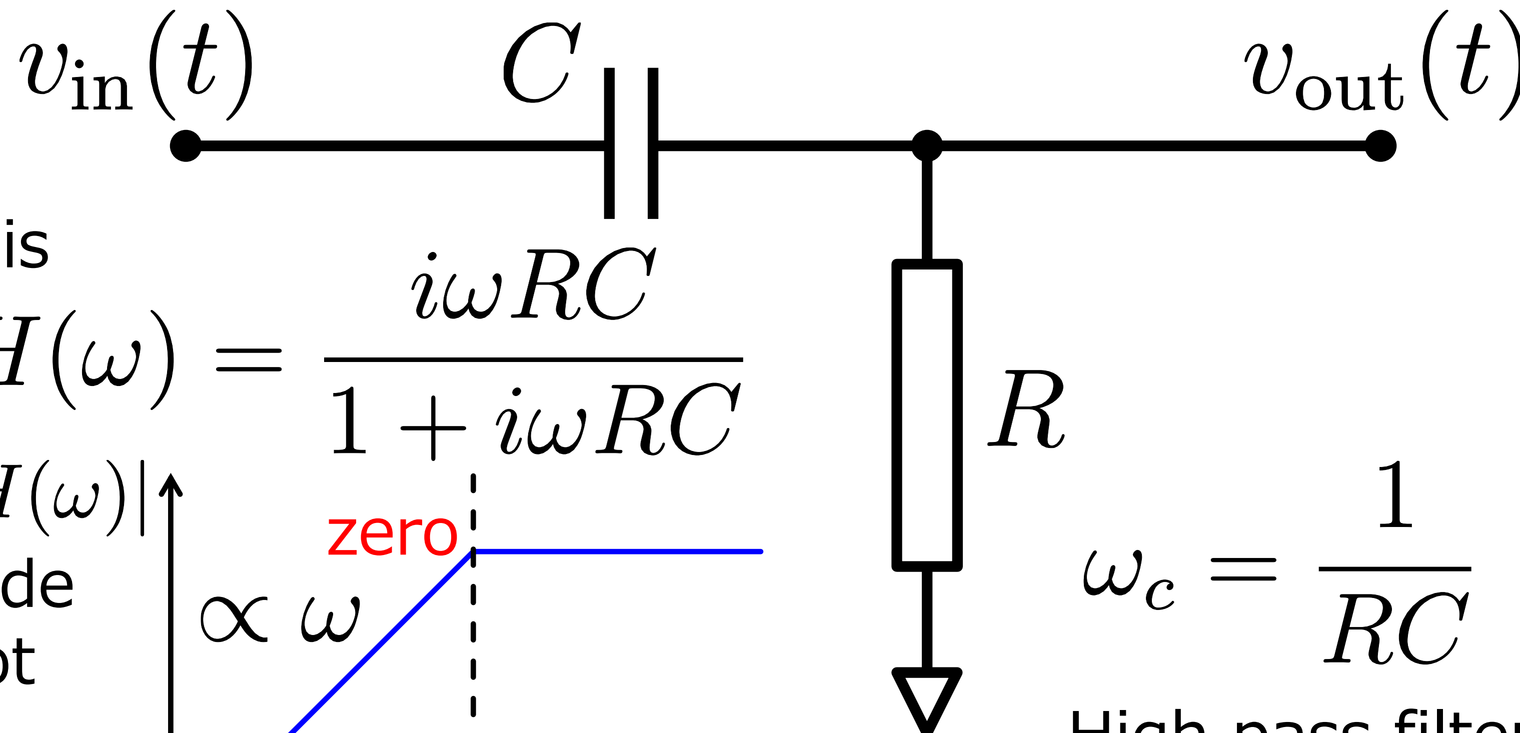
$$V_{\text{out}} = i\omega RC(V_{\text{in}} - V_{\text{out}})$$

TF is

$$H(\omega) = \frac{i\omega RC}{1 + i\omega RC}$$

Example 2: Electric Circuit

- What is the transfer function?



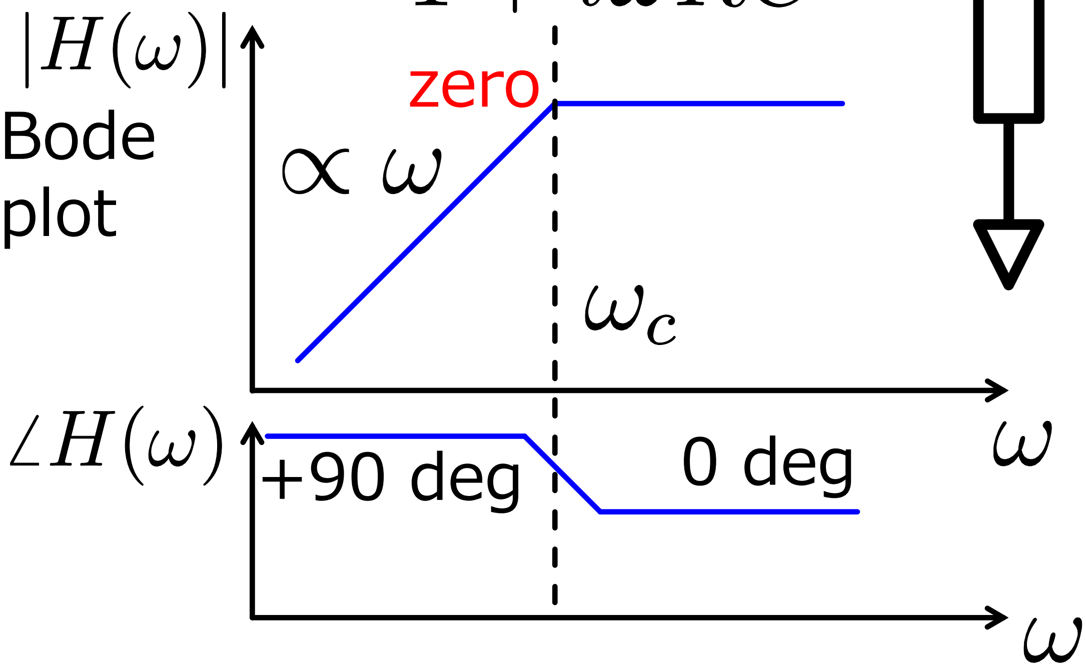
TF is

$$H(\omega) = \frac{i\omega RC}{1 + i\omega RC}$$

$$\omega_c = \frac{1}{RC}$$

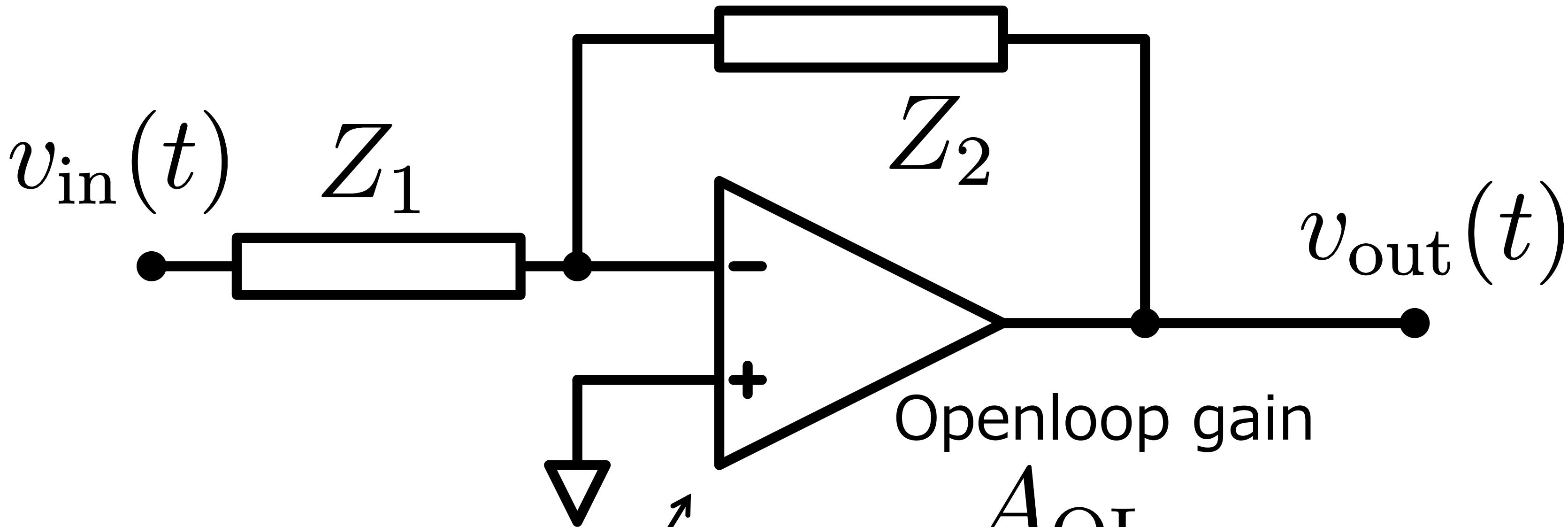
High pass filter
Differentiator circuit

Bode plot



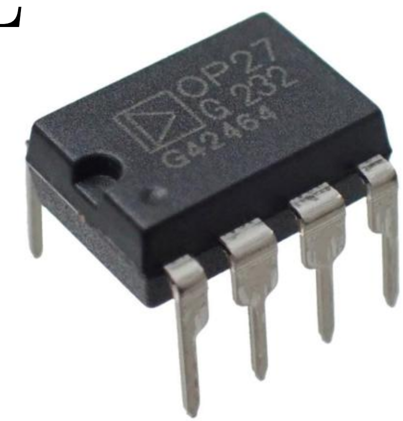
Example 3: Electric Circuit

- What is the transfer function?



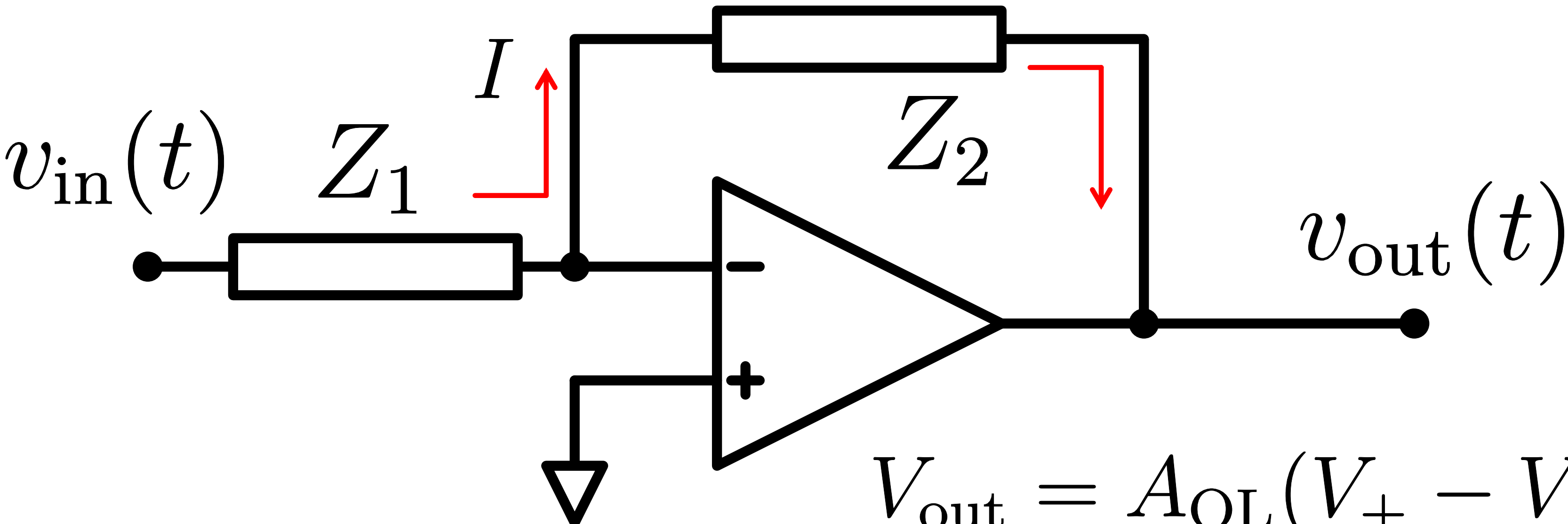
Input impedance is very high (~MΩ)

Very high (~10^5)



Example 3: Electric Circuit

- What is the transfer function?



$$V_{\text{out}} = A_{\text{OL}}(V_+ - V_-)$$

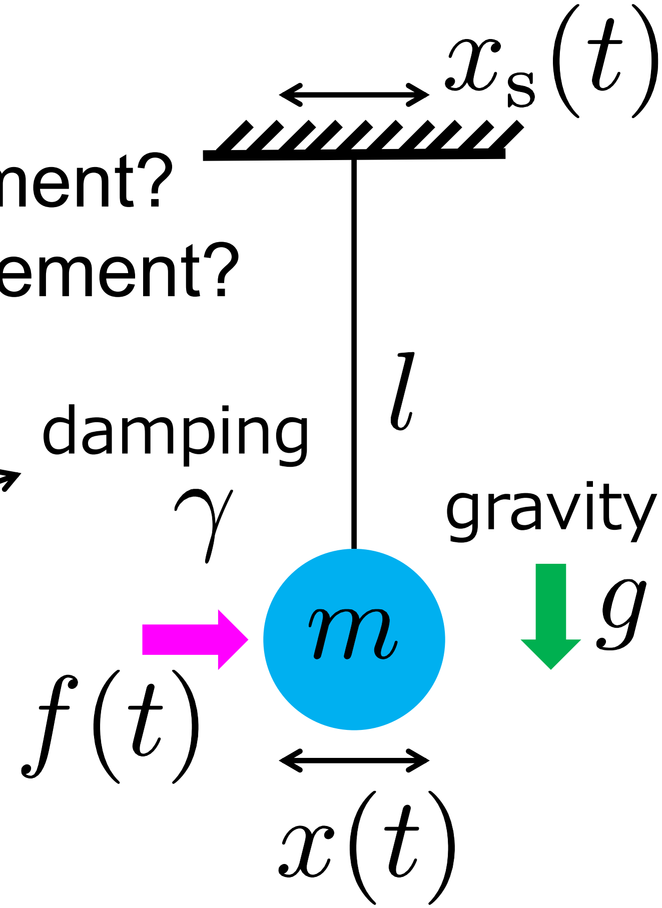
$$I = (V_- - V_{\text{in}})/Z_1 = (V_{\text{out}} - V_-)/Z_2$$

Since A_{OL} >> 1, V_1 ~ V_2 = 0. TF is

$$H(\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{Z_2}{Z_1}$$

Example 4: Harmonic Oscillator

- What are the transfer functions
  - from applied force to displacement?
  - from seismic motion to displacement?



Energy damping rate [rad/s]
Dissipation related to Fluctuation dissipation theorem (next lecture)

Also, phonon number of initially ground-state-cooled oscillator increases as

$$\bar{n} = \frac{k_\mathrm{B} T_\mathrm{th}}{\hbar \omega_\mathrm{m}} (1 - e^{-\gamma t})$$

※ Sometimes amplitude damping rate [rad/s] is defined as γ.
In this case, 2γ is the energy damping rate.

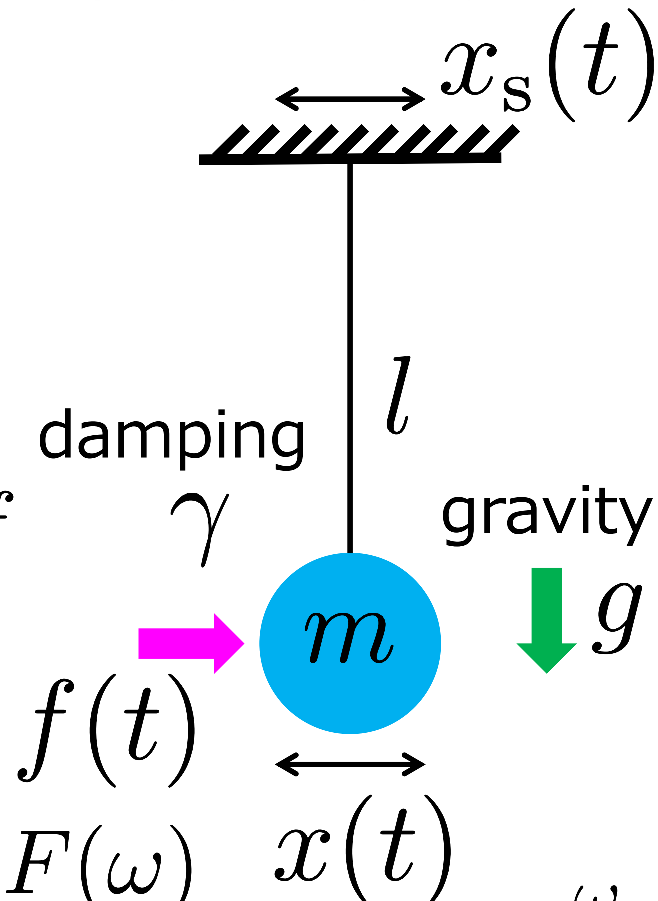
Example 4: Harmonic Oscillator

- Spring constant

$$k = \frac{mg}{l} = m\omega_{\rm m}^2$$

- Equation of motion

$$m\left(\ddot{x} + \gamma\dot{x} + \omega_{\rm m}^2(x - x_s)\right) = f$$



- Fourier transform

$$X(\omega) = H_s(\omega)X_s(\omega) + H_F(\omega)F(\omega)$$

$$Q \equiv \frac{\omega_{\rm m}}{\gamma}$$

- Here, TFs are

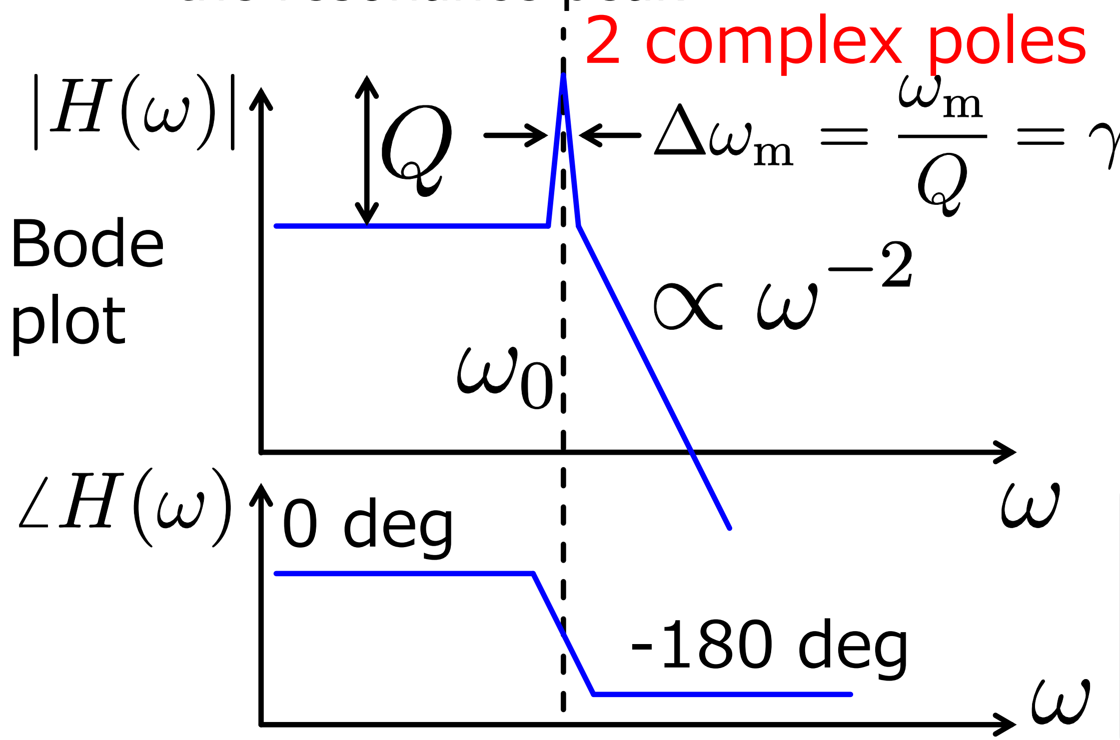
$$H_s(\omega) = \frac{\omega_{\rm m}^2}{\omega_{\rm m}^2 - \omega^2 + i\frac{\omega\omega_{\rm m}}{Q}} \qquad H_F(\omega) = \frac{1}{m}\frac{1}{\omega_{\rm m}^2 - \omega^2 + i\frac{\omega\omega_{\rm m}}{Q}}$$

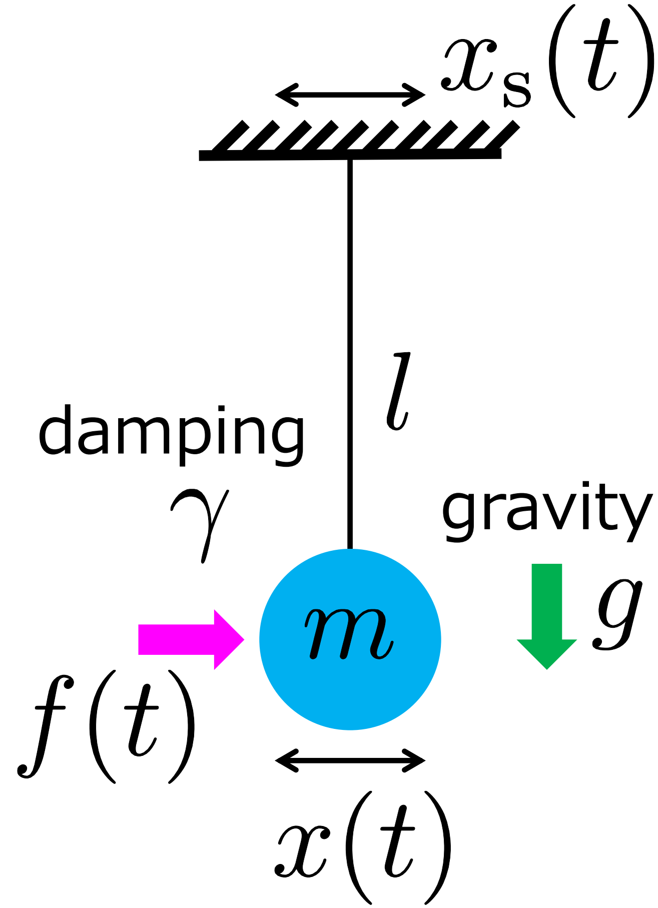
Example 4: Harmonic Oscillator

- Vibration attenuation factor

$$H_s(\omega) = \frac{\omega_{\mathrm{m}}^2}{\omega_{\mathrm{m}}^2 - \omega^2 + i\frac{\omega\omega_{\mathrm{m}}}{Q}}$$

Higher the Q, higher and narrower the resonance peak





Pendulum ↔ Cavity

$$Q = \frac{\omega_{\mathrm{L}}}{2\kappa} = \frac{\omega_{\mathrm{L}}}{\omega_{\mathrm{FSR}}/\mathcal{F}}$$

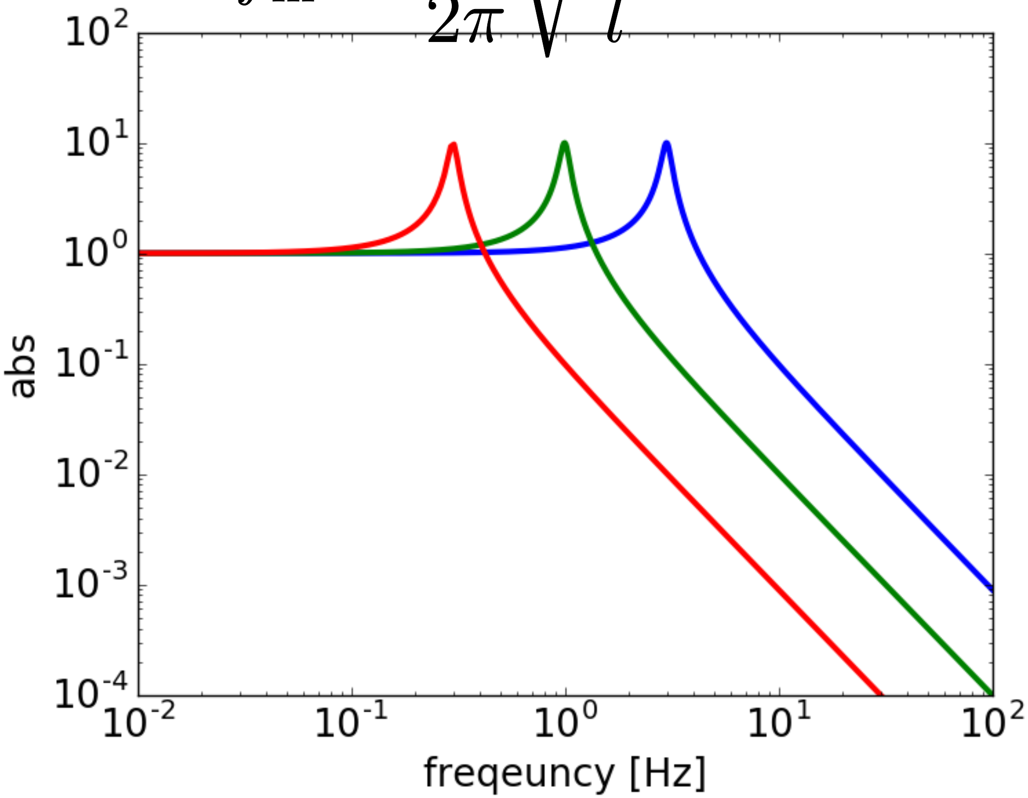
Cavity decay rate

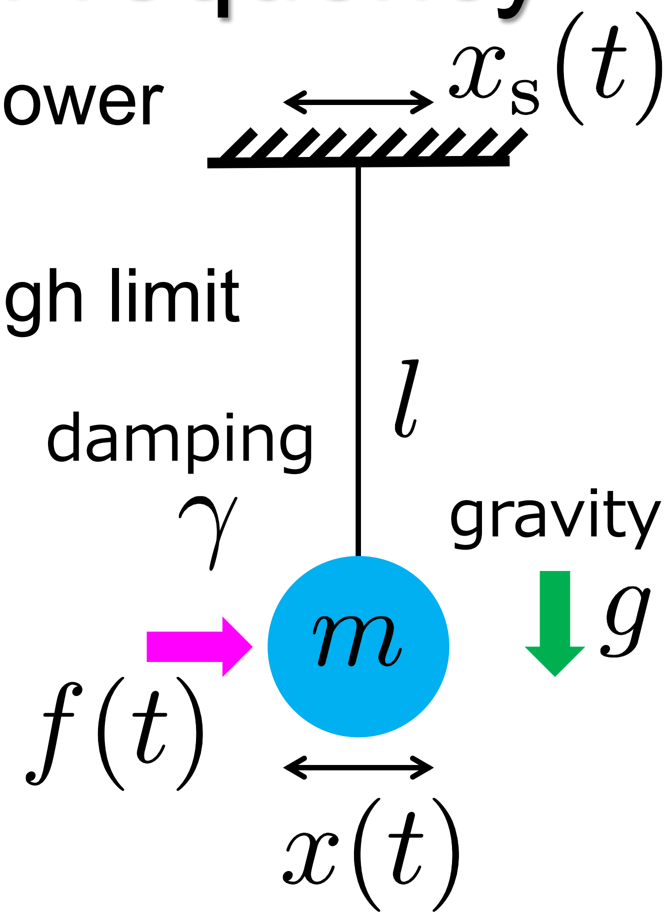
$2\kappa = T f_{\mathrm{FSR}}$

Pendulum Resonant Frequency

- Better vibration isolation with a lower resonant frequency
- But on Earth, $l \sim 1\,\mathrm{m}$ is a rough limit

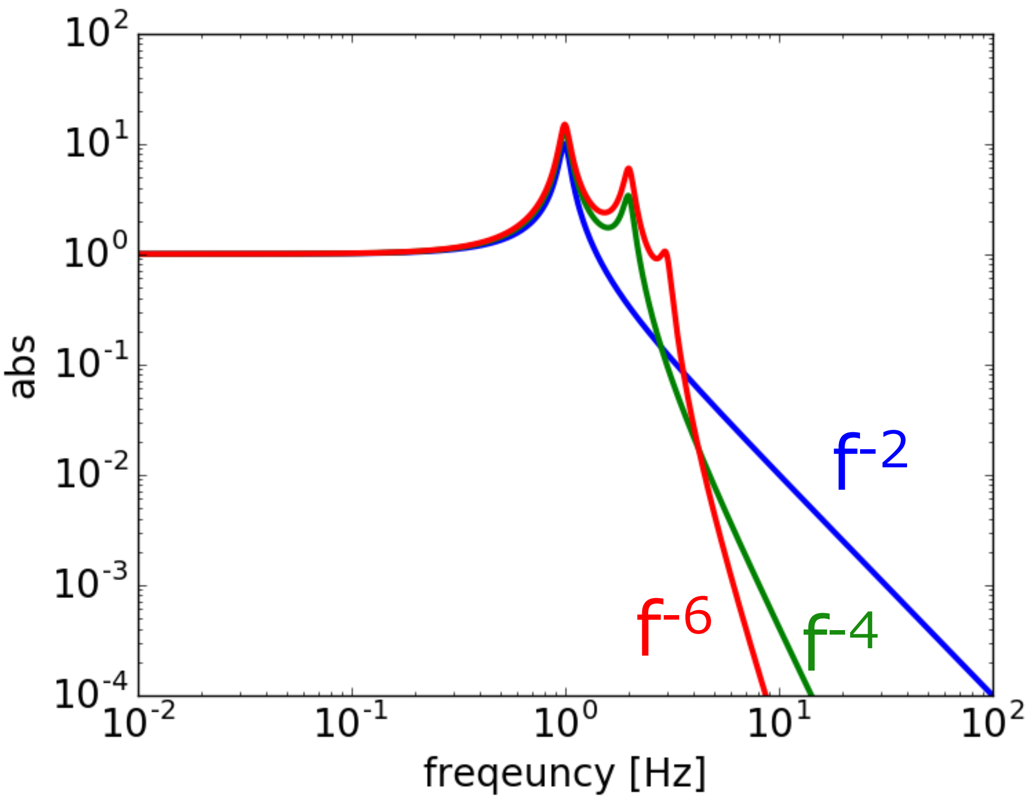
$$f_{\mathrm{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \simeq 0.5\ \mathrm{Hz}$$

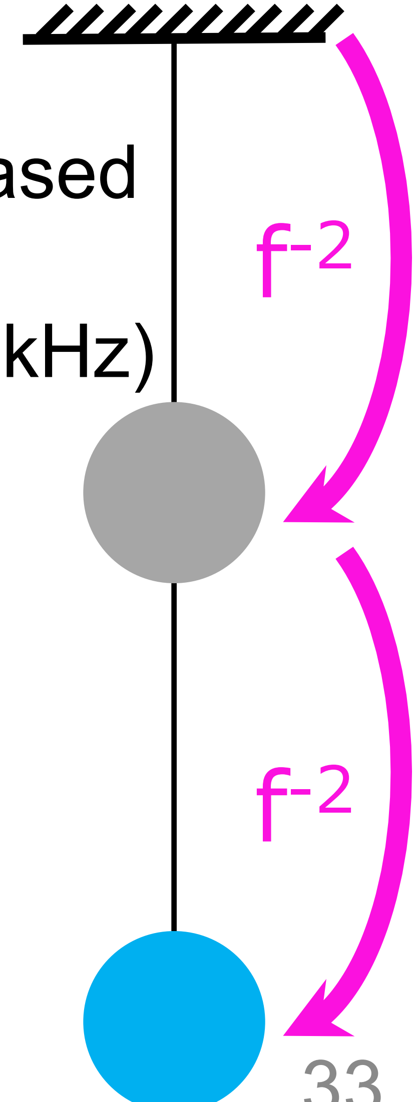




Multi-Stage Vibration Isolation

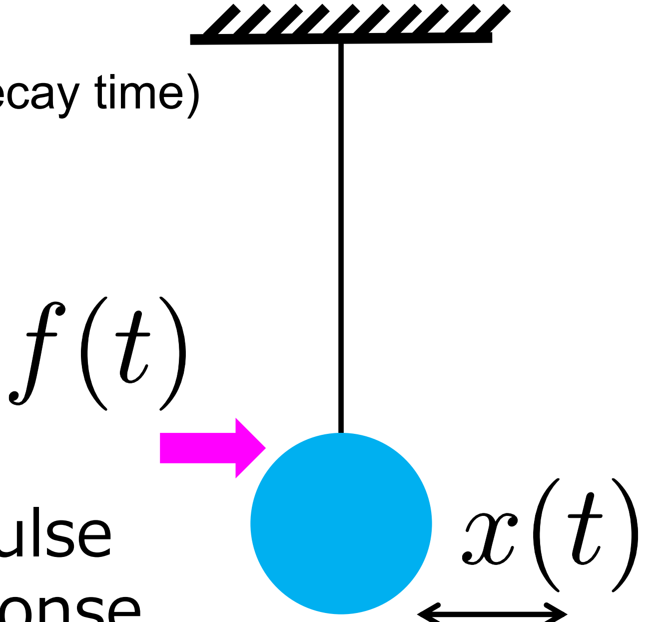
- Multi-stage pendulum for better isolation at higher frequencies
- Good vibration isolation for ground-based gravitational-wave detector band (10 Hz ~ 1 kHz)

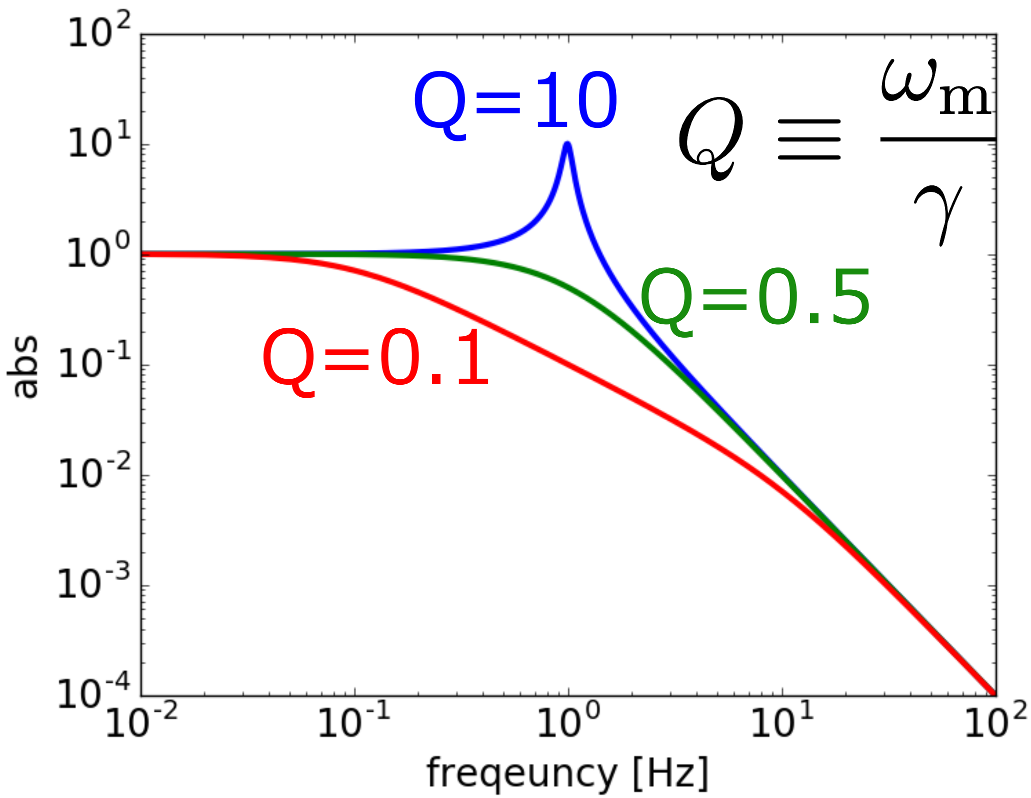




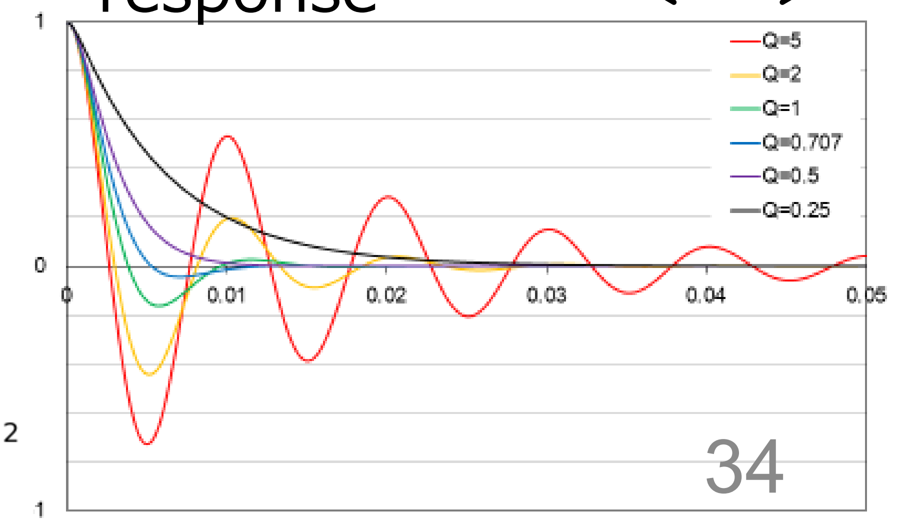
Q-factor

- Damped oscillation $\omega_{\mathrm{m}} > 2\gamma$
- Critical damping $\omega_{\mathrm{m}} = 2\gamma$ (least decay time)
- Over damping $\omega_{\mathrm{m}} < 2\gamma$



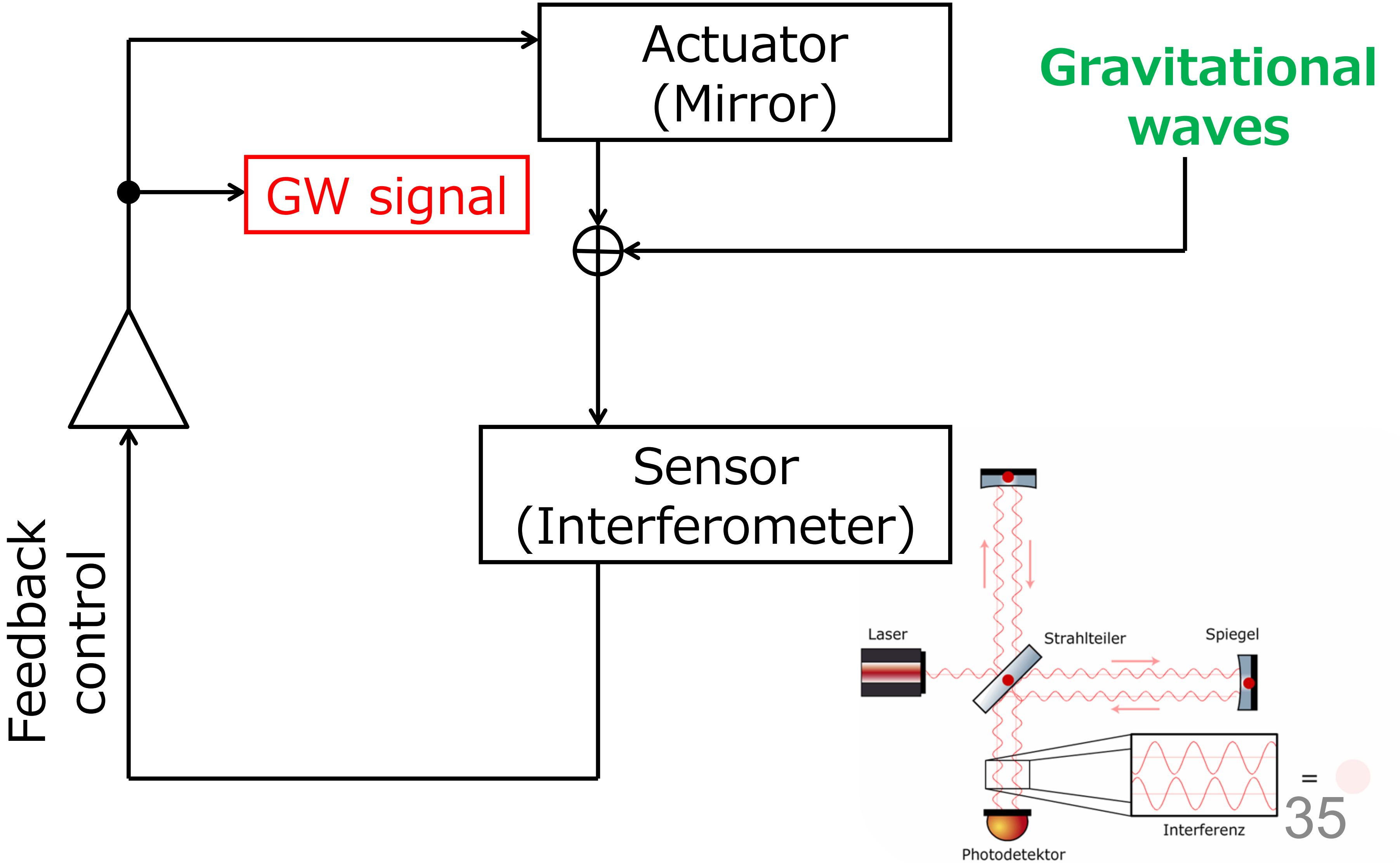


Impulse response



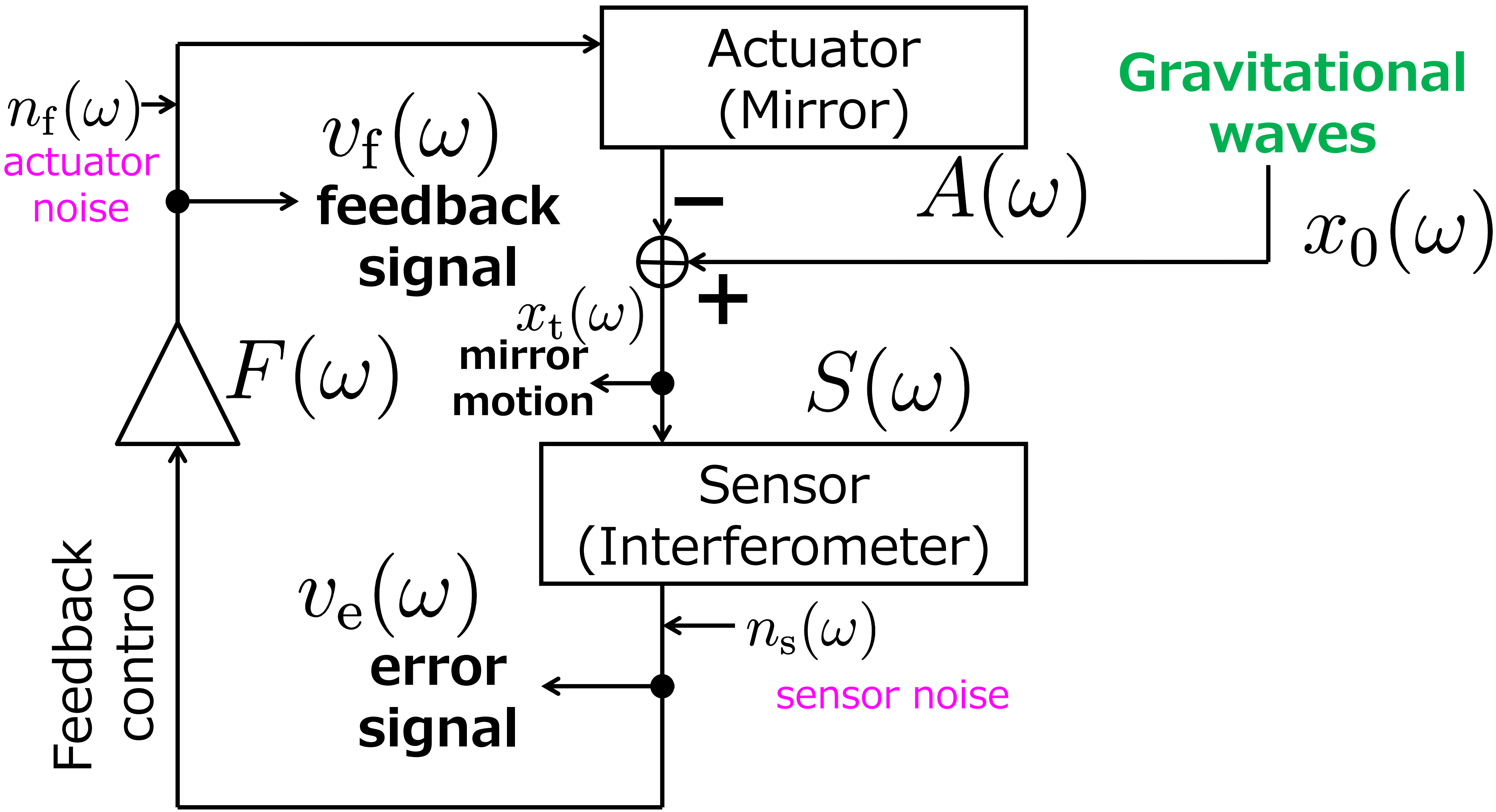
Feedback Control Loop Analysis

- Let us consider a control loop



Feedback Control Loop Analysis

- Let us consider a control loop



Loop Analysis and Calibration

- Open loop transfer function $G \equiv SFA$
- Error signal

$$v_{\mathrm{e}} = S\left(x_0 - A(n_{\mathrm{f}} + Fv_{\mathrm{e}})\right) + n_{\mathrm{s}}$$

$$v_{\mathrm{e}} = \frac{S}{1+G}x_0 - \frac{SA}{1+G}n_{\mathrm{f}} + \frac{1}{1+G}n_{\mathrm{s}}$$

- Feedback signal

$$v_{\mathrm{f}} = \frac{SF}{1+G}x_0 - \frac{G}{1+G}n_{\mathrm{f}} + \frac{F}{1+G}n_{\mathrm{s}}$$

- When $|G| \gg 1$, $v_{\mathrm{e}} \sim 0$, and x_0 can be estimated from Av_{f}
When $|G| \ll 1$, x_0 can be estimated from v_{e}/S

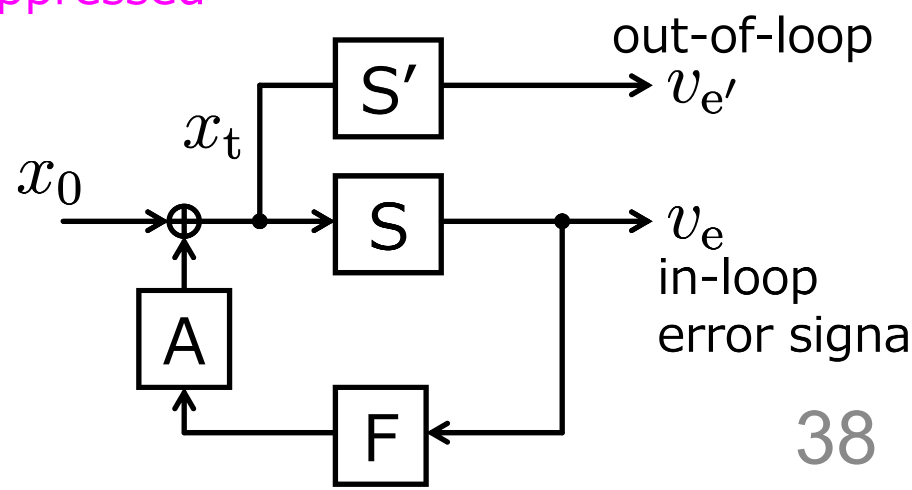
In-loop and Out-of-loop

- Even if $|G| \gg 1$ and $v_\mathrm{e} \sim 0$, this does not mean that mirror motion is actually zero

$$x_t = \frac{v_\mathrm{e} - n_\mathrm{s}}{S}$$

$$= \underbrace{\frac{1}{1+G}x_0 - \frac{A}{1+G}n_\mathrm{f}}_{\text{external disturbance and actuation noise are suppressed}} - \underbrace{\frac{G}{1+G}\frac{1}{S}n_\mathrm{s}}_{\text{sensor noise remains}}$$

- To evaluate the actual mirror motion, out-of-loop sensor is necessary



Interferometer Control Loop

- Sensor: Interferometer (recall Lecture 2)

$$S(\omega) \propto \frac{\partial P_{\mathrm{PD}}}{\partial L_-} = \frac{2\pi P_0}{\lambda} \sin \frac{4\pi L_-}{\lambda}$$

  - Constant at low frequencies (f << 2L/c)
- Actuator: Suspended mirror (pendulum)

$$A(\omega) = H_F(\omega) = \frac{1}{m} \frac{1}{\omega_{\mathrm{m}}^2 - \omega^2 + i\frac{\omega\omega_{\mathrm{m}}}{Q}}$$

- Filter: Electronics (we can design as we want)

$$F(\omega)$$

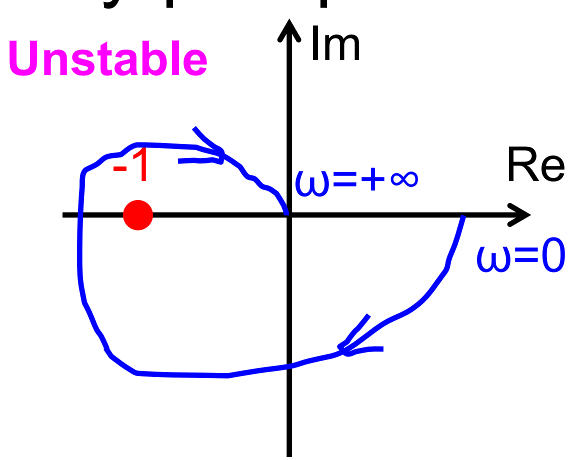
- We need to design $F(\omega)$ to have $|G| \gg 1$ as much as possible, without the system being unstable

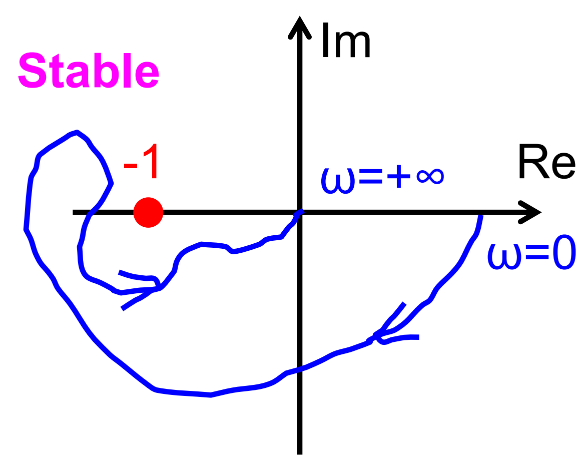
Stability of Control Loops

- Nyquist stability criterion:
  For the closed-loop system to be stable, the Nyquist plot of G must encircle the point -1 in the complex plane a number of times equal to the number of poles of G in the right-half plane, in the counter-clockwise direction.

  Poles in the right-half plane are the poles that have a positive real part, and are unstable poles. Usually, we don't have unstable poles in a system.

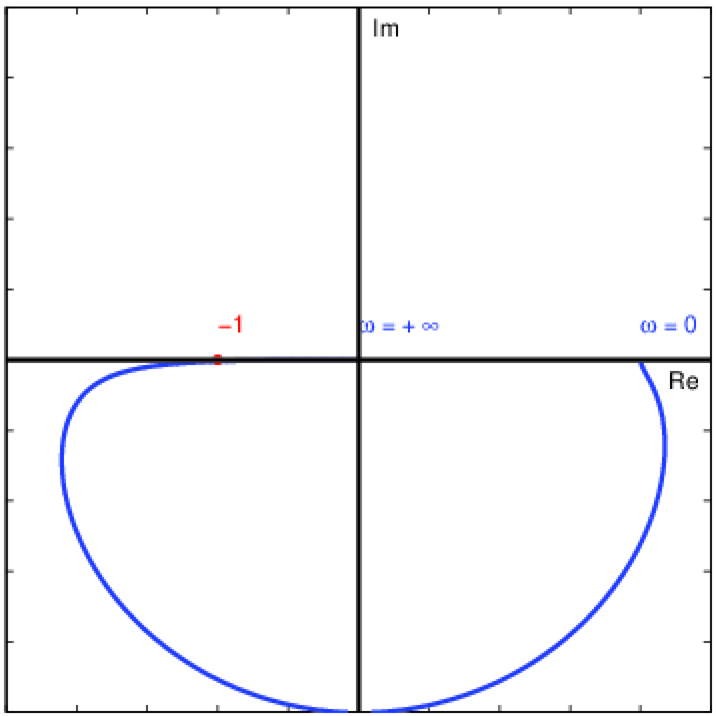
- Nyquist plot

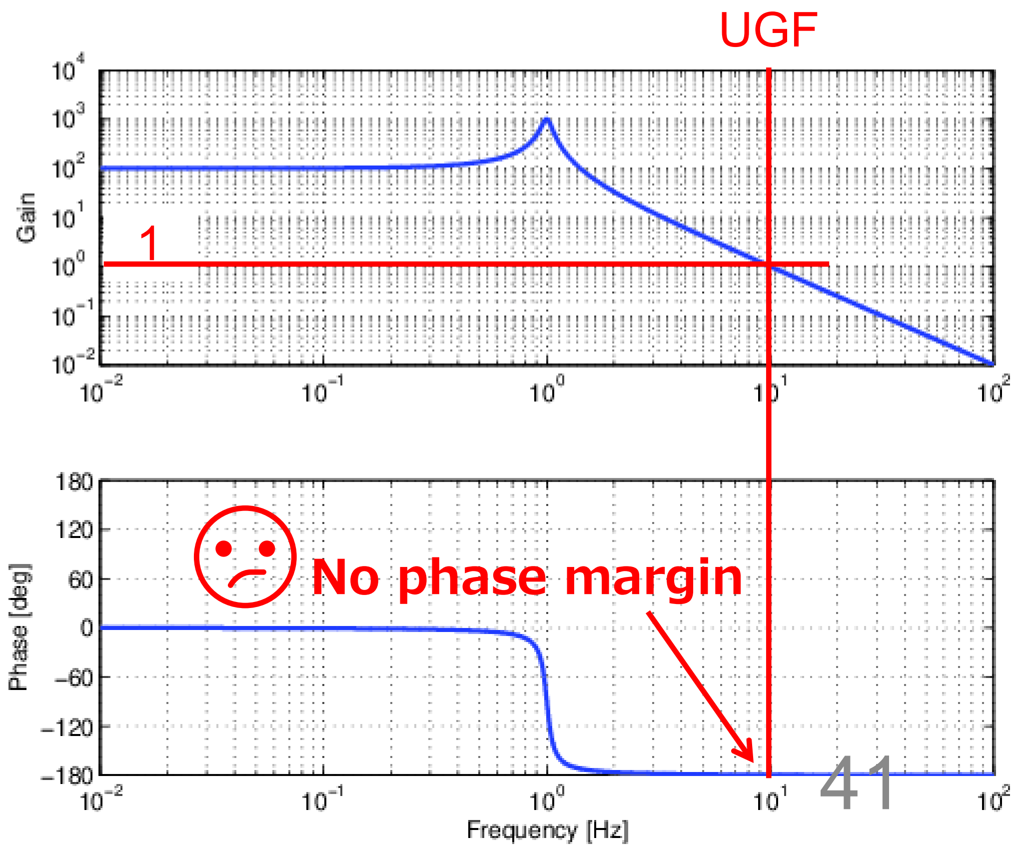




Filter Design

- If $F(\omega) = 1$, the system is close to unstable
- Phase margin at unity gain frequency (UGF) is 0

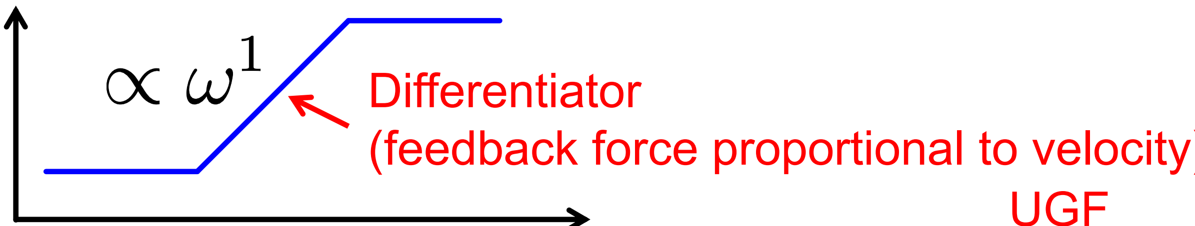


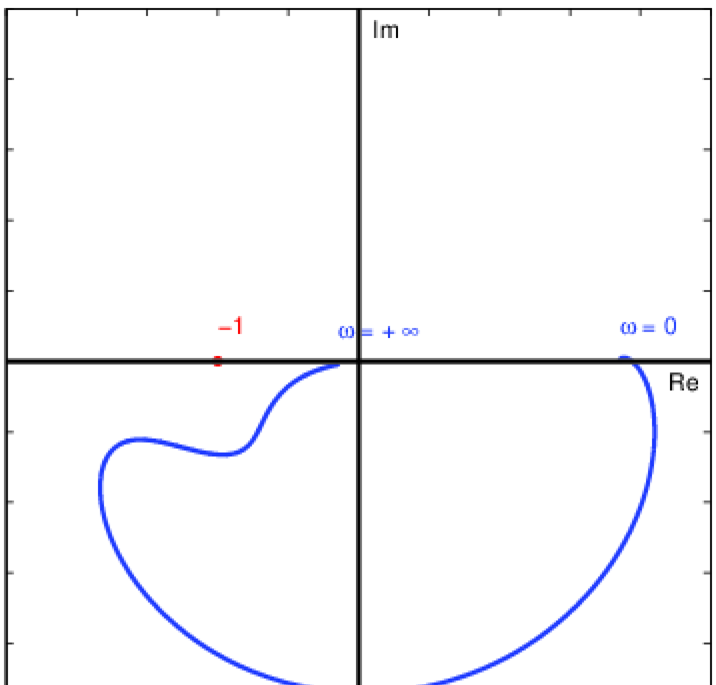


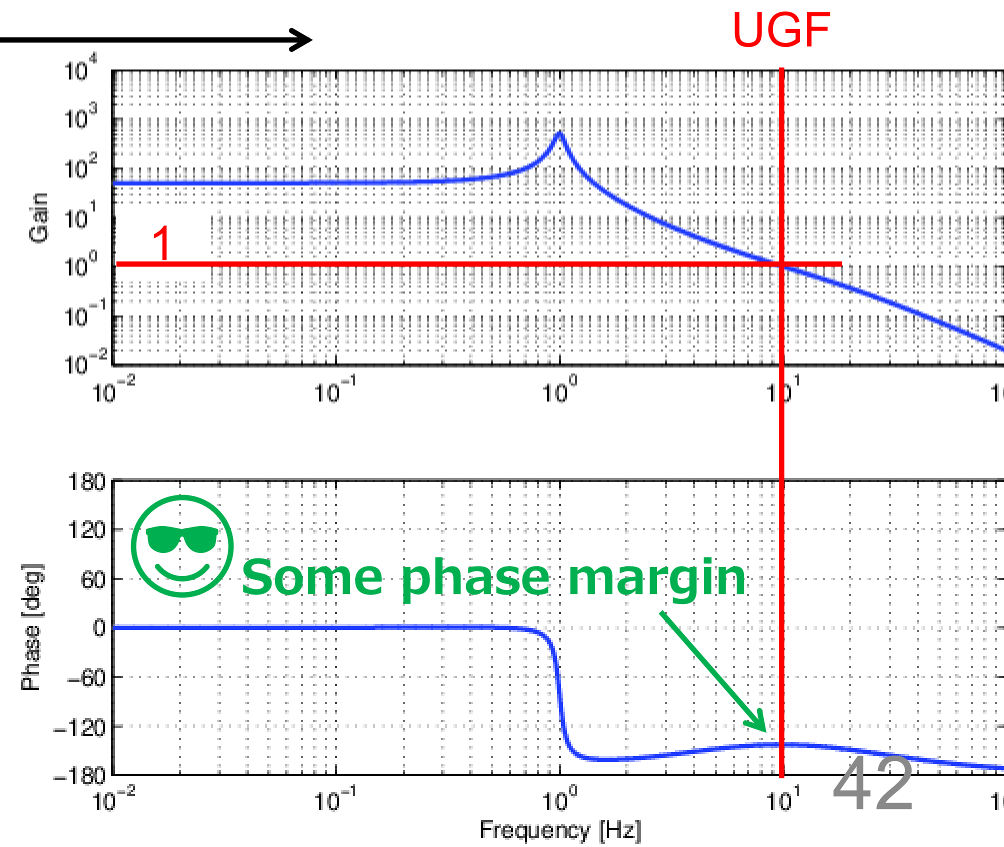
Filter Design

- With a lead-lag filter, the closed loop will be stable with enough phase margin

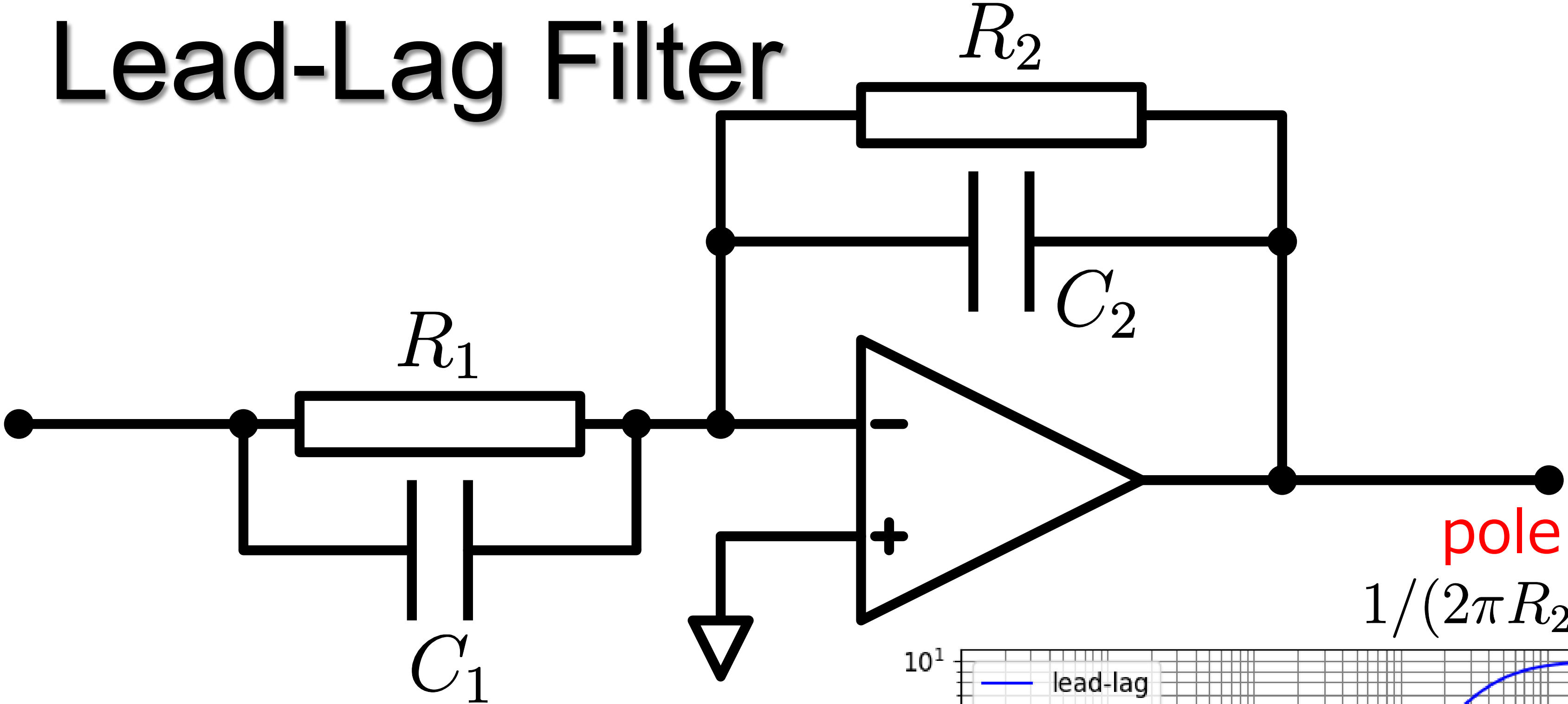
$$F(\omega) =$$



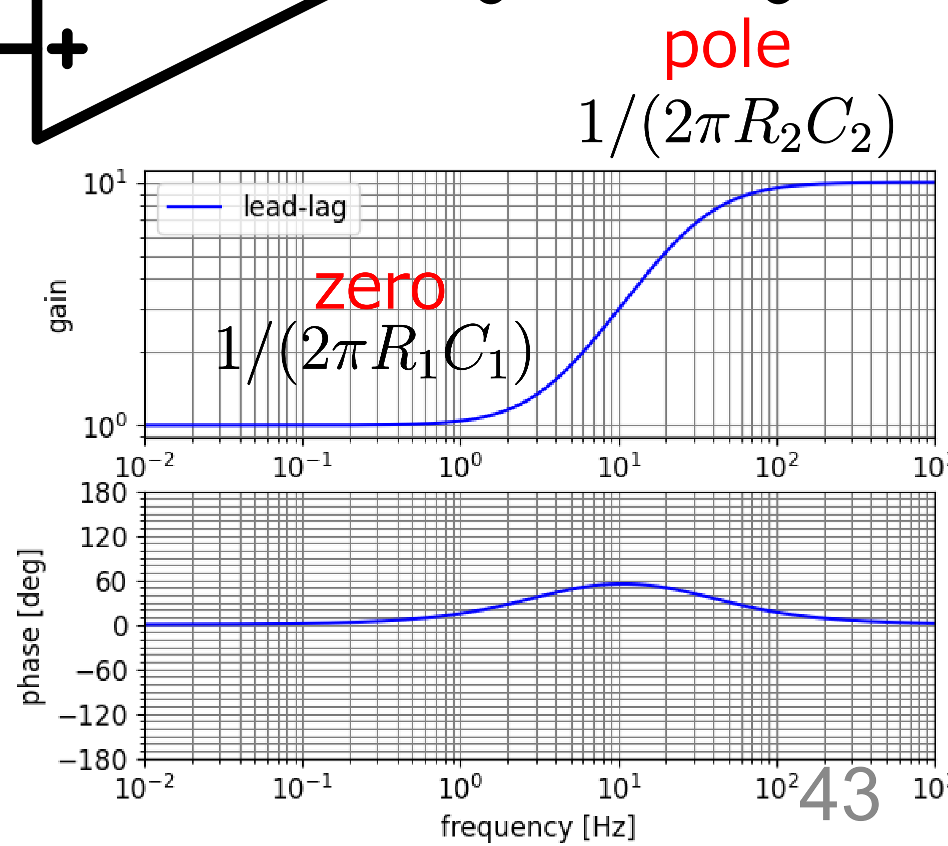




Lead-Lag Filter

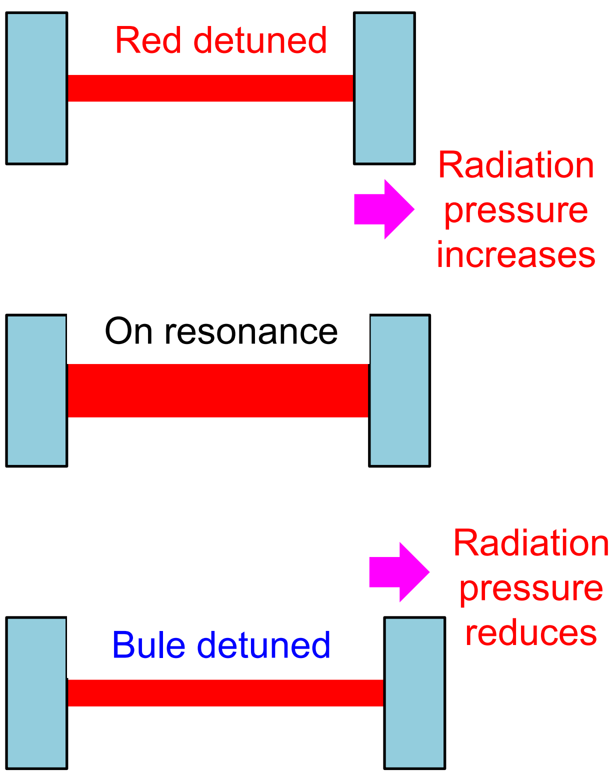


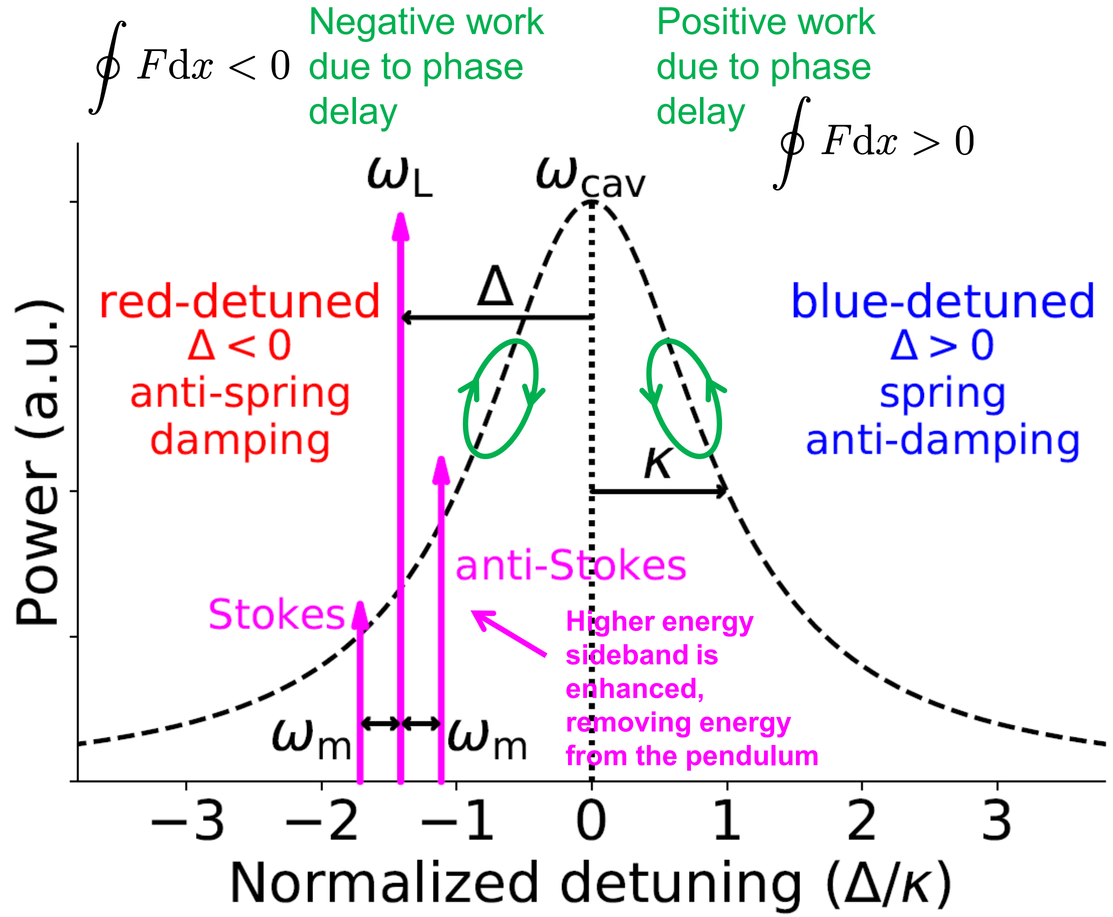
$$F(\omega) = -\frac{R_2(1 + i\omega R_1 C_1)}{R_1(1 + i\omega R_2 C_2)}$$



Optical Spring and Damping

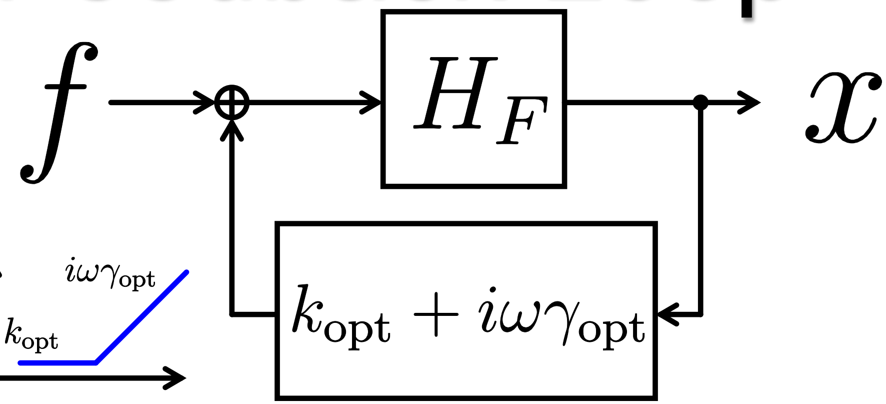
- Radiation pressure force inside optical cavity works as a spring and damping





Y. Michimura, K. Komori, [EPJD 74, 126 (2020)]
M. Aspelmeyer, T. J. Kippenberg, F. Marquardt, [Rev. Mod. Phys. 86, 1391 (2014)]

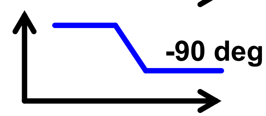
Optical Spring as Feedback Loop



- Mechanical force to displacement transfer function (susceptibility)

感受率

$$H_F(\omega) = \frac{1}{m}\frac{1}{\omega_{\rm m}^2 - \omega^2 + i\frac{\omega\omega_{\rm m}}{Q}}$$



$$\omega_{\rm m} = \sqrt{\frac{k}{m}}$$

$$Q = \frac{\omega_{\rm m}}{\gamma}$$

- When optical spring is considered

$$H'_F = \frac{H_F}{1 + H_F(k_{\rm opt} + i\omega\gamma_{\rm opt})}$$

$$= \frac{1}{m}\frac{1}{\omega_{\rm eff}^2 - \omega^2 + i\frac{\omega\omega_{\rm m}}{Q_{\rm eff}}}$$

$$\omega_{\rm eff} \equiv \sqrt{\frac{k + k_{\rm opt}}{m}}$$

$$Q_{\rm eff} \equiv \frac{\omega_{\rm m}}{\gamma + \gamma_{\rm opt}}$$

- We can effectively change the susceptibility without introducing additional dissipation (optical dilution)

希薄化

Summary

- Power spectral density
- Transfer function
- Open loop transfer function

- We’ve learned everything to understand noises and control loops

Assignment for Oct 31

- In laser interferometric gravitational wave detectors, identify a component that should *not* be treated as a linear system, and explain why.
- You may also answer from the Google Form below https://forms.gle/6AwJ48XcpWQXqMon9

Don't forget to put your name and student#

You may answer in any language