観測天文学特別講義I @ 理1-1043

Observational Astronomy Advanced Course I

October 24, 2025

レーザー干渉計型重力波検出器 Laser interferometric gravitational wave detectors

2. Laser Interferometers and Optical Cavities

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https://yutamich.gitlab.io/lectures.html#lectures

Assignment for Oct 24

- What are the possible noise sources for laser interferometers? List up the most ridiculous noise sources you can imagine.
- You may also answer from the Google Form below https://forms.gle/6AwJ48XcpWQXqMon9

Don’t forget to put your name and student#

You may answer in any language

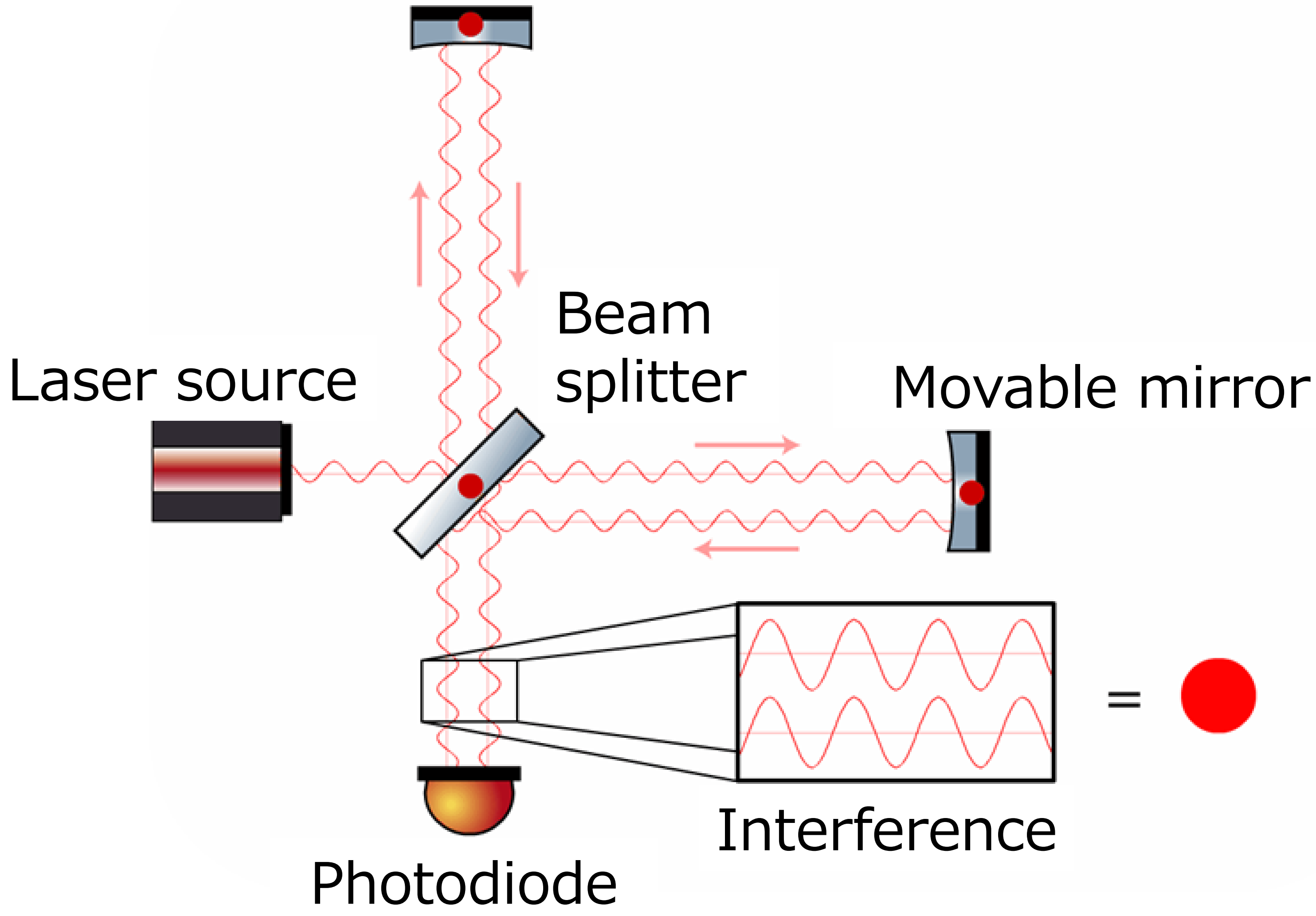
Plan of the Lecture Today

- Optical response of Michelson Interferometer
- Optical response of Fabry-Pérot Cavity
- Phasor diagram and sideband picture
- Modulation-demodulation method

- **Goal**: Understand how laser interferometer works and how to extract the signal intuitively

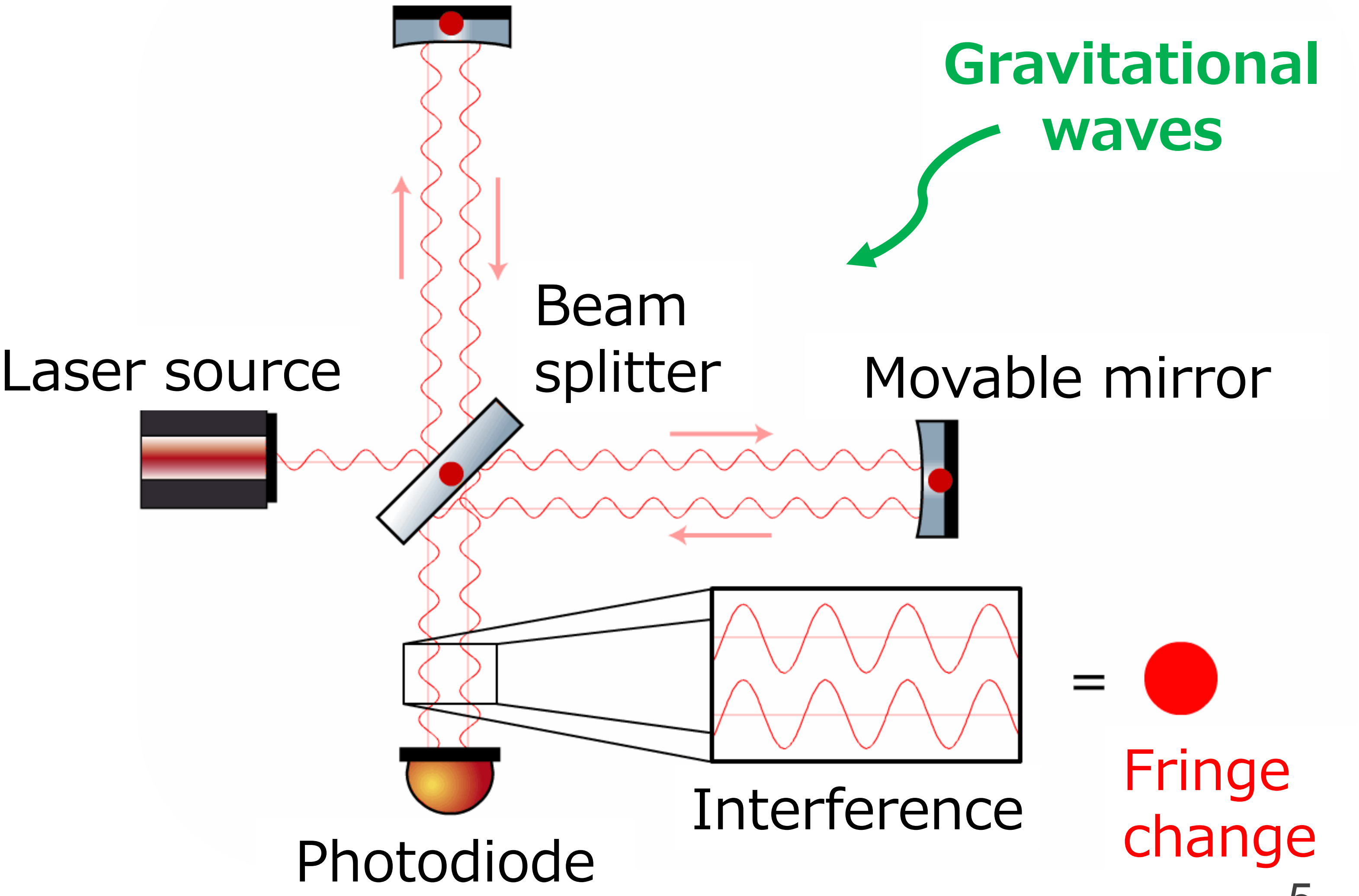
Laser Interferometric GW Detectors

- Measures differential arm length change



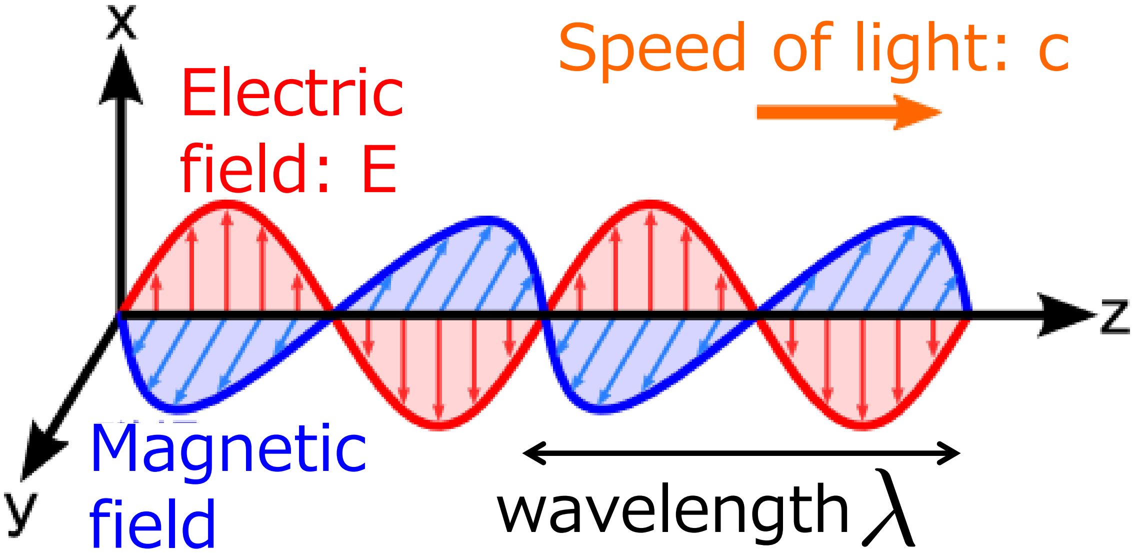
Laser Interferometric GW Detectors

- Measures differential arm length change



Laser Beam

- Electro-magnetic waves



- Electric field can be written as

$$E = E_0 e^{i(\omega t - \phi)}$$

amplitude

angular frequency of laser

phase

$$\omega = \frac{2\pi c}{\lambda}$$

$$\phi = \frac{2\pi L}{\lambda}$$

phase at distance L

Photodiodes

- Photodiodes (PDs)

  Convert photons into electrons

  Detects light power (square of amplitude)

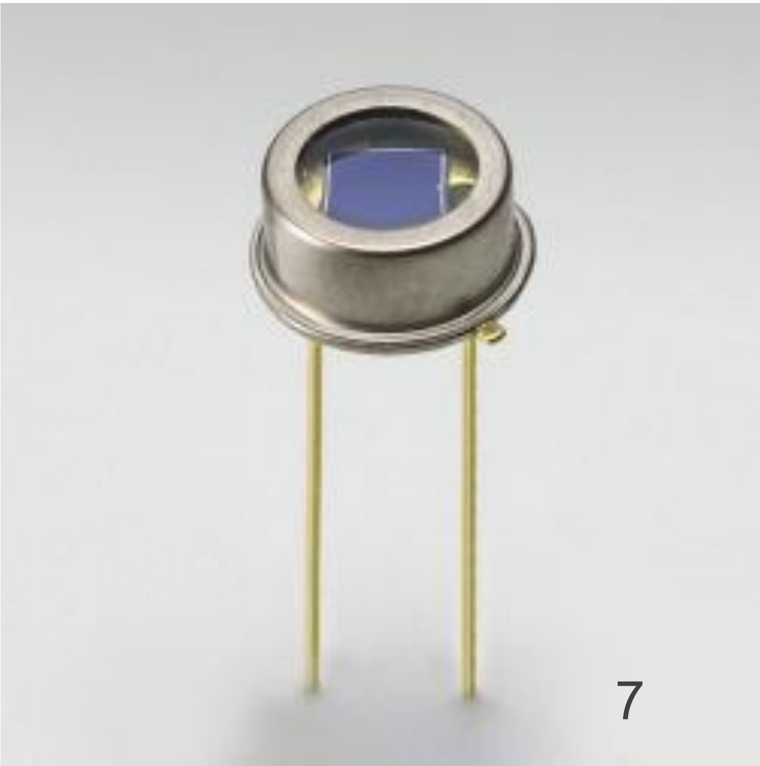
$$P \propto |E|^2 = E_0^2$$

To make it simple, we will just say

$$P = |E|^2 = E_0^2$$

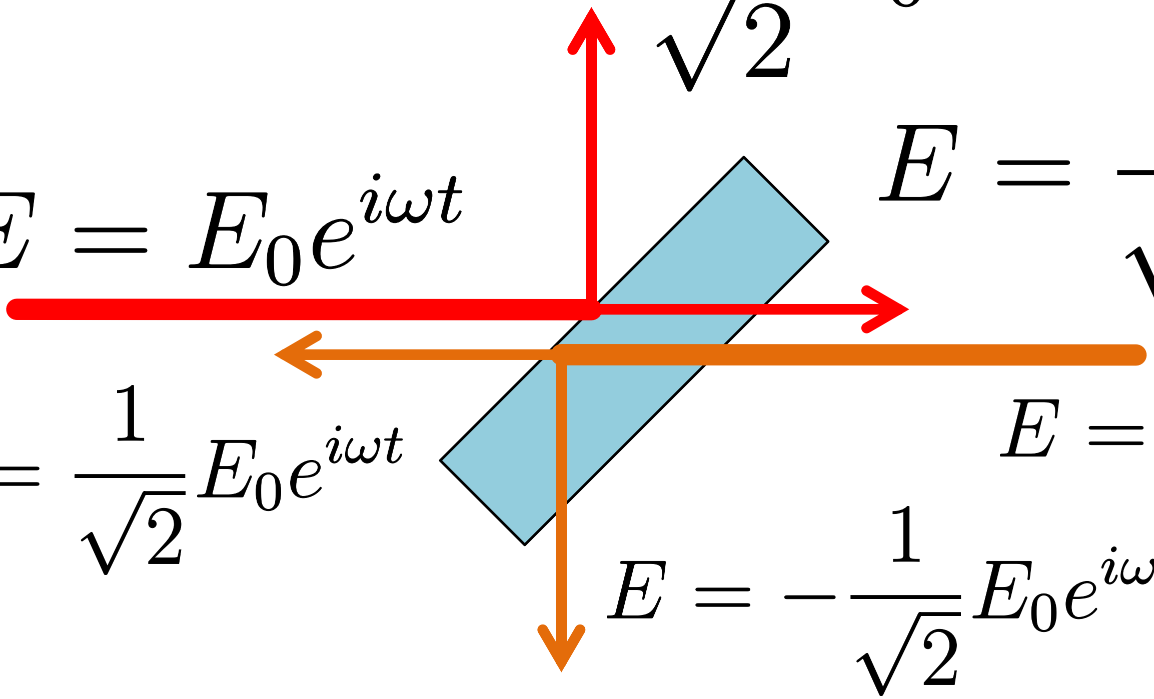
We can only detect power change

Phase change cannot be detected directly



Beam Splitter

- Split beam in two
- Half in power, 1/√2 in amplitude
- Sign flip in back reflection



$$E = \frac{1}{\sqrt{2}} E_0 e^{i\omega t}$$

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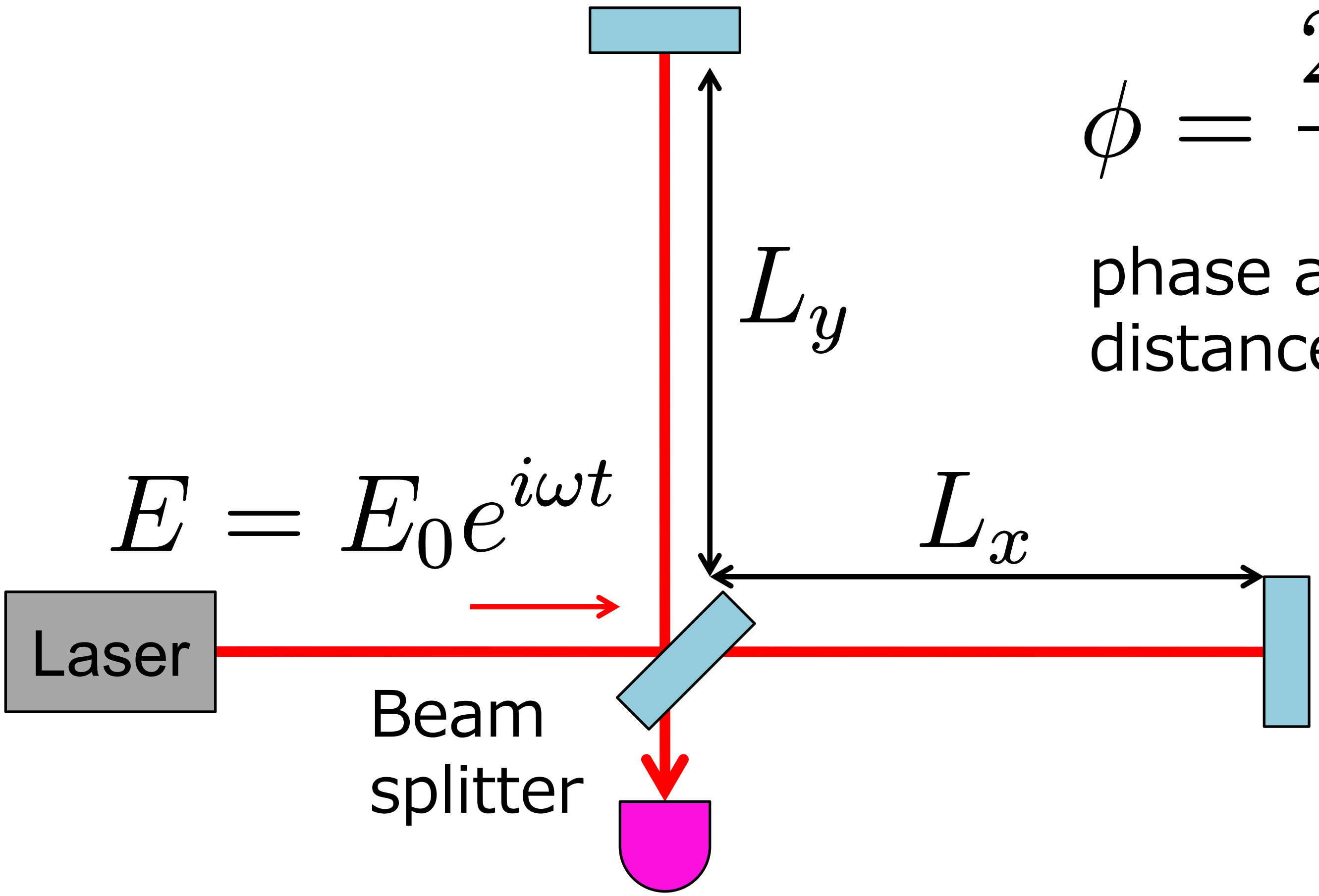
$$E = E_0 e^{i\omega t}$$

$$E = \frac{1}{\sqrt{2}} E_0 e^{i\omega t}$$

$$E = -\frac{1}{\sqrt{2}} E_0 e^{i\omega t}$$

Output of Michelson Interferometer

- What is the power detected at the photodiode?



$$\phi = \frac{2\pi L}{\lambda}$$

phase at distance L

Output of Michelson Interferometer

- What is the power detected at the photodiode?

From Y-am From X-arm

$$P_{\mathrm{PD}} = \left| \frac{1}{2} E_0 e^{i(\omega t - \frac{4\pi L_y}{\lambda})} - \frac{1}{2} E_0 e^{i(\omega t - \frac{4\pi L_x}{\lambda})} \right|^2$$

$$= \frac{1}{4} |E_0|^2 \left| e^{-i\frac{4\pi L_y}{\lambda}} - e^{-i\frac{4\pi L_x}{\lambda}} \right|^2$$

$$= \frac{1}{2} P_0 \left(1 - \cos \frac{4\pi L_-}{\lambda} \right)$$

Input power

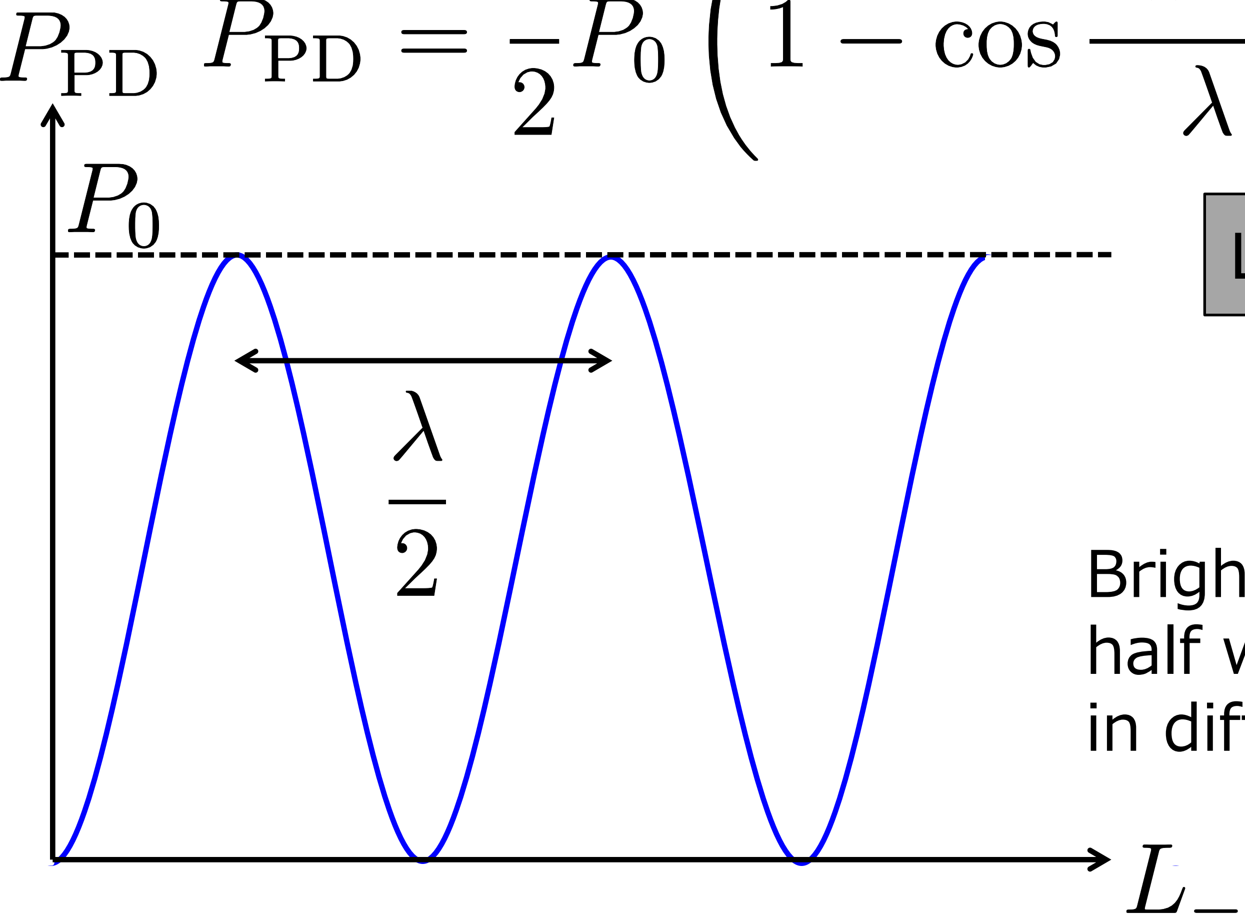
$$L_- = L_y - L_x$$

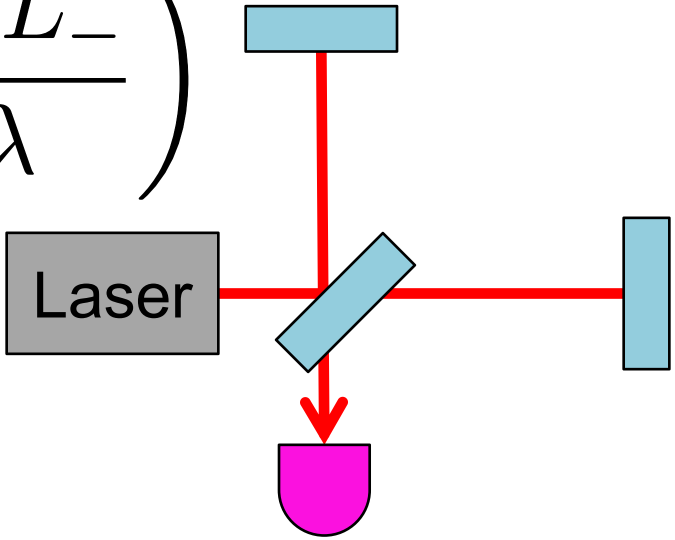
Differential arm length

Output of Michelson Interferometer

- Power changes with differential arm length change (interference)

$$P_{\mathrm{PD}} = \frac{1}{2}P_0\left(1 - \cos\frac{4\pi L_-}{\lambda}\right)$$



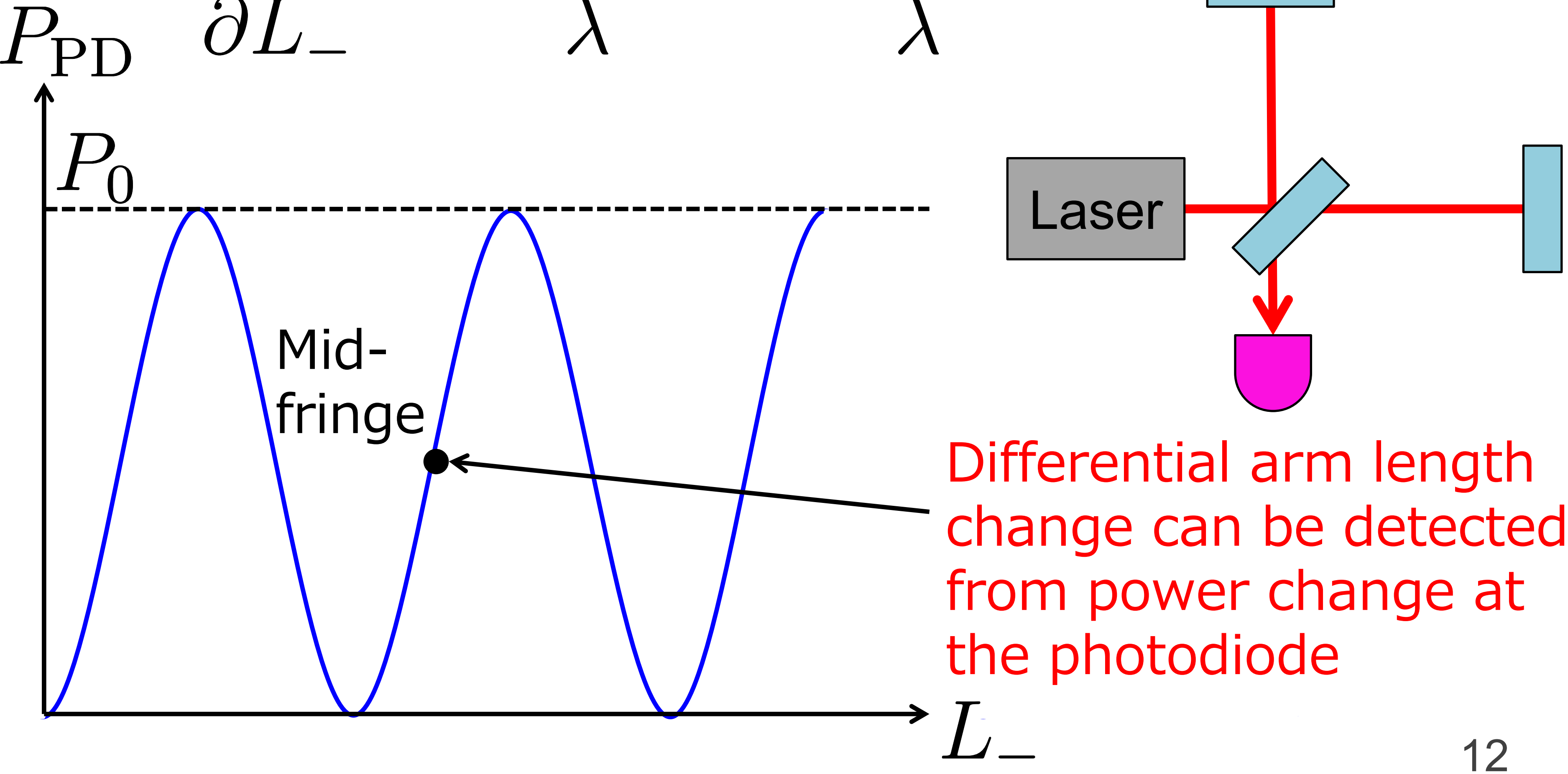


Bright fringe in every half wavelength change in differential arm length

Output of Michelson Interferometer

- Ratio between power change and length change

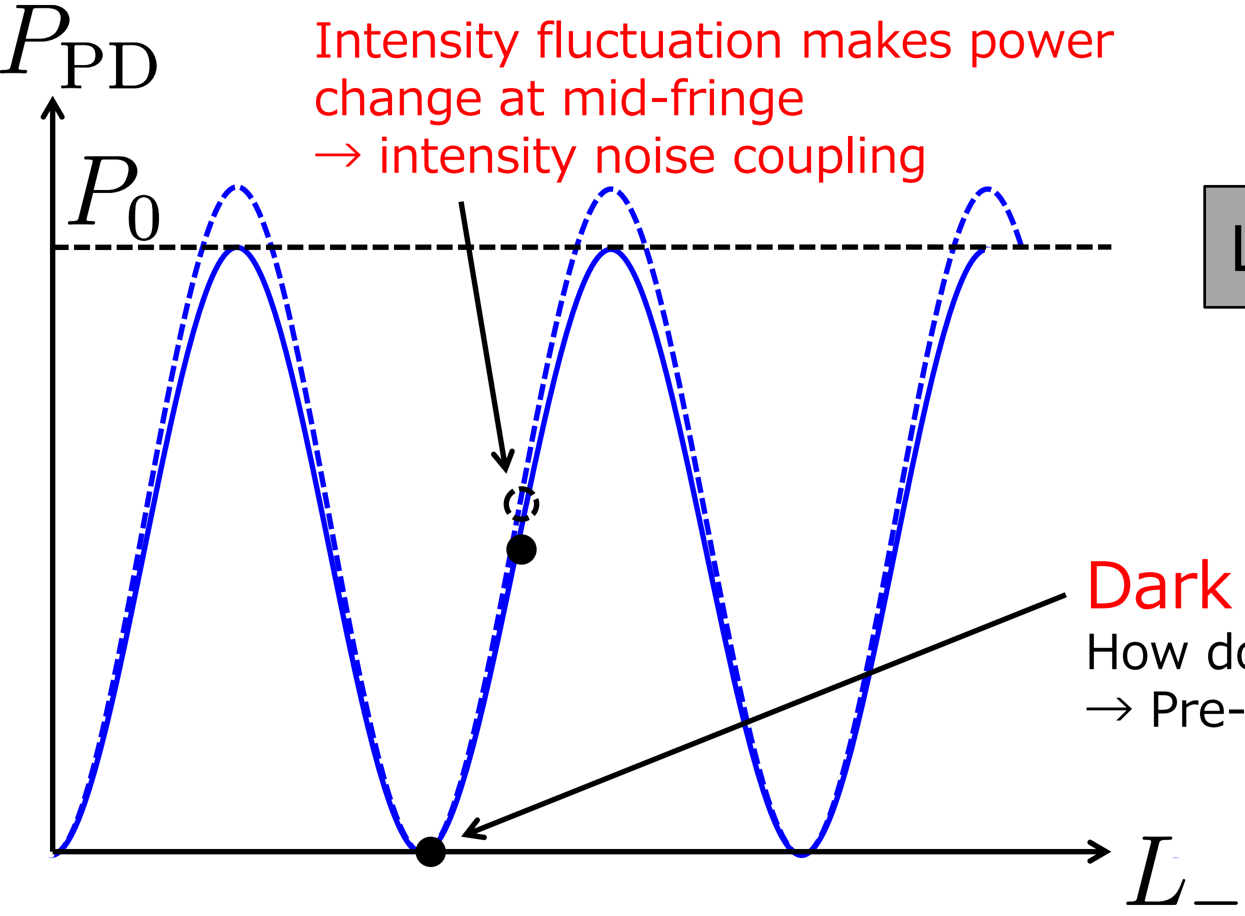
$$\frac{\partial P_{\mathrm{PD}}}{\partial L_{-}} = \frac{2\pi P_0}{\lambda} \sin \frac{4\pi L_{-}}{\lambda}$$



Differential arm length change can be detected from power change at the photodiode

Dark Fringe Operation

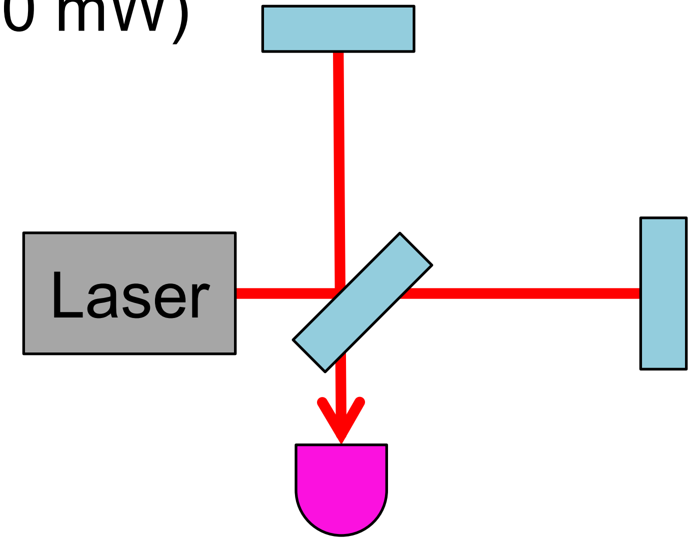
- We often operate at (near) dark fringe to avoid intensity noise coupling and to reduce shot noise (also, PDs can only handle upto ~50 mW)



Intensity fluctuation makes power change at mid-fringe
→ intensity noise coupling

Dark fringe

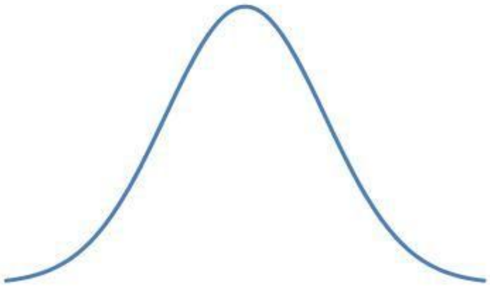
How do we operate at here?
→ Pre-modulation method

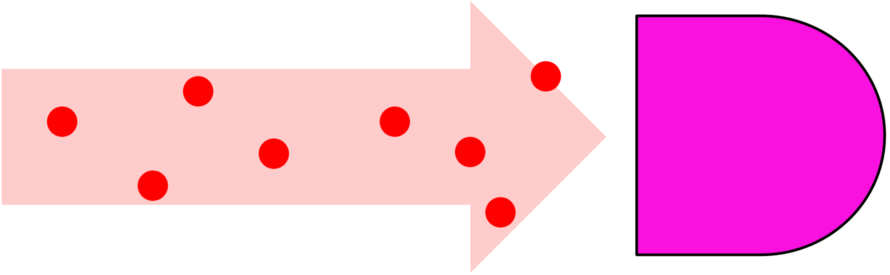


Photon Shot Noise

- Number of photons to photodiodes fluctuates

$$N \pm \sqrt{N}$$





Photon energy

$$p_1 = \frac{hc}{\lambda}$$

Number of photons

$$N = \frac{P_{\mathrm{PD}}}{p_1}$$

- Quantum fluctuation of power

$$\delta P_{\mathrm{shot}} = \sqrt{\frac{2hcP_{\mathrm{PD}}}{\eta\lambda}}$$

Shot noise spectrum

Quantum efficiency

Shot Noise Limit of Michelson

- Power change $$\frac{\partial P_{\mathrm{PD}}}{\partial L_-} = \frac{2\pi P_0}{\lambda} \sin \frac{4\pi L_-}{\lambda}$$
- Shot noise

$$\delta P_{\mathrm{shot}} = \sqrt{\frac{2hcP_{\mathrm{PD}}}{\eta\lambda}} = \sqrt{\frac{hcP_0}{\eta\lambda}\left(1 - \cos\frac{4\pi L_-}{\lambda}\right)}$$

- Shot noise limited sensitivity

$$\delta L_{\mathrm{shot}} = \delta P_{\mathrm{shot}} \left(\frac{\partial P_{\mathrm{PD}}}{\partial L_-}\right)^{-1} \to \frac{1}{2\pi}\sqrt{\frac{hc\lambda}{2\eta P_0}}$$

$$\frac{\sqrt{1-\cos\phi}}{\sin\phi} = \frac{\sqrt{2\sin^2\frac{\phi}{2}}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}} = \frac{1}{\sqrt{2}\cos\frac{\phi}{2}}$$

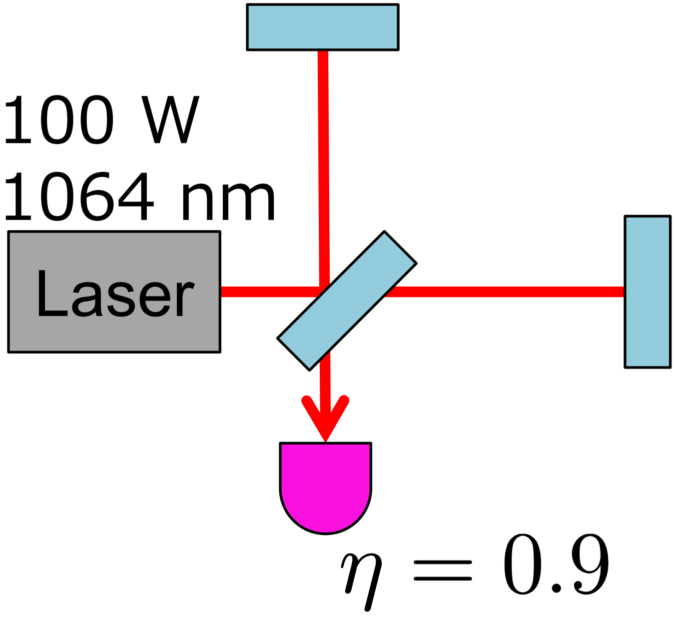
Better shot noise with higher input power
Best at dark fringe (where $P_{\mathrm{PD}}=0$)

Shot Noise Limit of Michelson

- Length detection limit

$$\delta L_{\mathrm{shot}} = \frac{1}{2\pi}\sqrt{\frac{hc\lambda}{\eta P_0}}$$

$$h = 6.626 \times 10^{-34}\ \mathrm{m^2 kg/s}$$



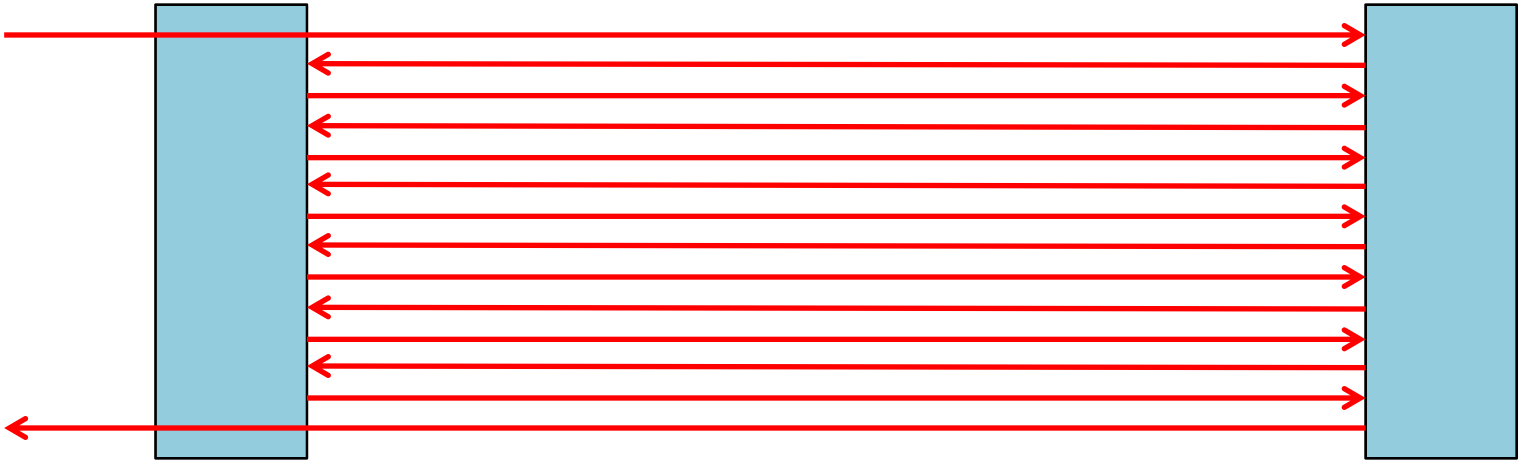
$$\delta L_{\mathrm{shot}} \sim 8 \times 10^{-18}\ \mathrm{m}/\sqrt{\mathrm{Hz}}$$

- This is already incredible, but not enough for reaching h ~ 10^{-21} /√Hz for km detectors

How to Further Enhance the Signal

- Longer arms gives larger length change due to gravitational waves $\delta L = hL$
- But making arm length very long is tough (especially on Earth)
- Use Fabry-Pérot cavity

  laser light go back-and-forth many times to effectively enhance the arm length



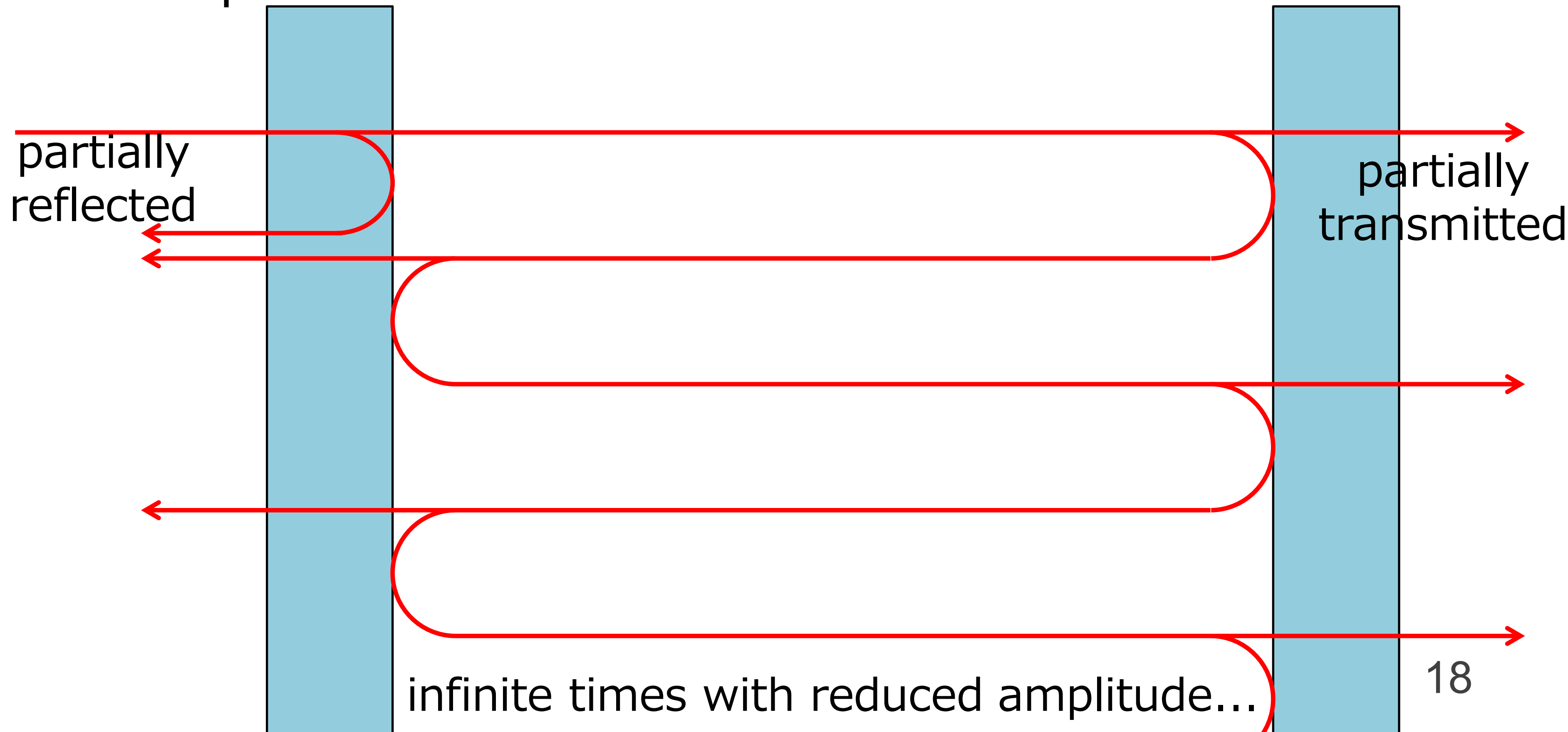
Fabry-Pérot Cavity

- Made from two parallel mirrors

amplitude
reflectivity, transmittance

$r_1, \ t_1$
input mirror

$r_2, \ t_2$
end mirror



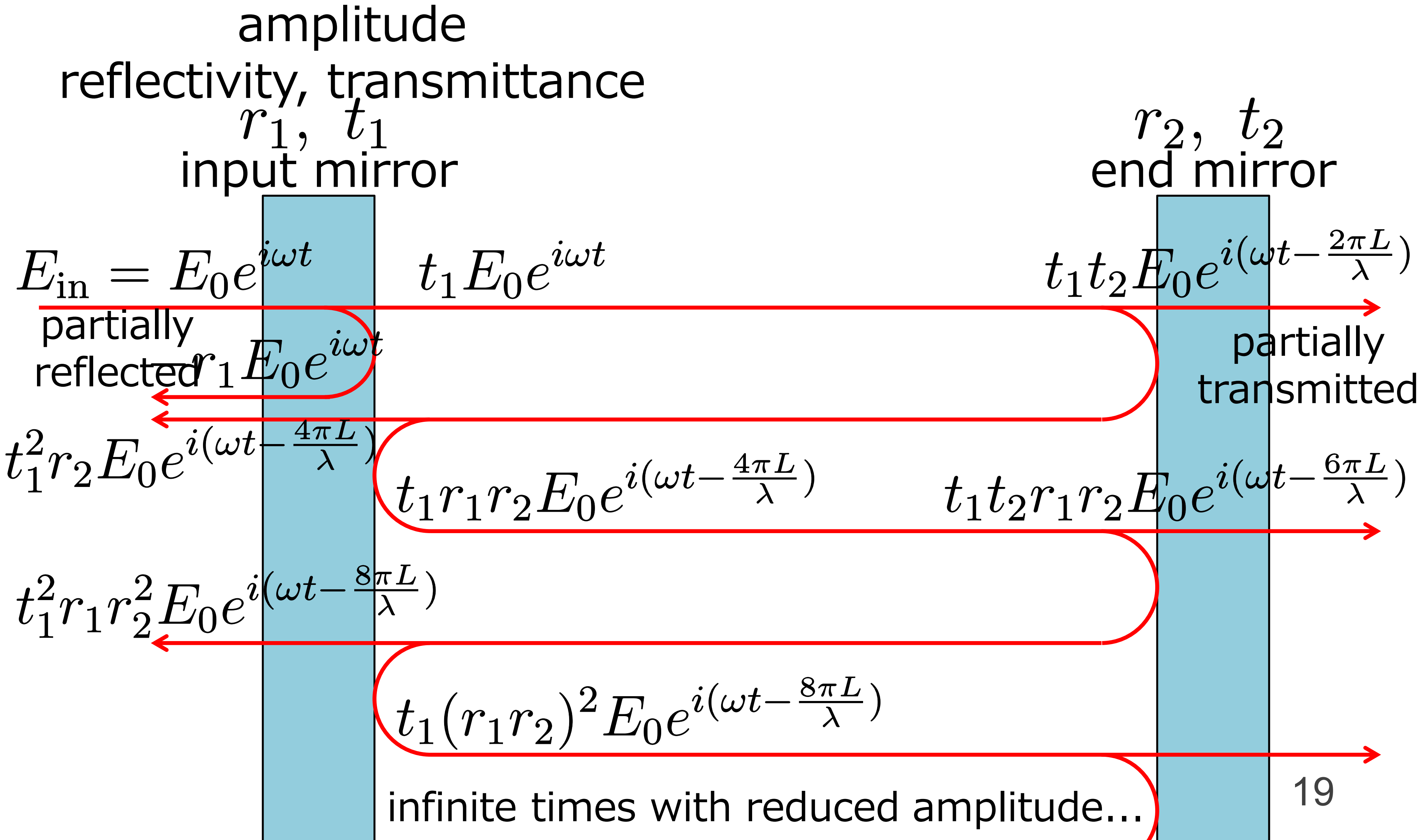
partially
reflected

partially
transmitted

infinite times with reduced amplitude...

Fabry-Pérot Cavity

- Let’s calculate electric field inside the cavity



amplitude

reflectivity, transmittance

$r_1,\ t_1$

input mirror

$r_2,\ t_2$

end mirror

$E_{\mathrm{in}} = E_0 e^{i\omega t}$

$t_1 E_0 e^{i\omega t}$

$t_1 t_2 E_0 e^{i(\omega t - \frac{2\pi L}{\lambda})}$

partially reflected

$-r_1 E_0 e^{i\omega t}$

partially transmitted

$t_1^2 r_2 E_0 e^{i(\omega t - \frac{4\pi L}{\lambda})}$

$t_1 r_1 r_2 E_0 e^{i(\omega t - \frac{4\pi L}{\lambda})}$

$t_1 t_2 r_1 r_2 E_0 e^{i(\omega t - \frac{6\pi L}{\lambda})}$

$t_1^2 r_1 r_2^2 E_0 e^{i(\omega t - \frac{8\pi L}{\lambda})}$

$t_1 (r_1 r_2)^2 E_0 e^{i(\omega t - \frac{8\pi L}{\lambda})}$

infinite times with reduced amplitude...

Intra-Cavity Field

- Intra-cavity field can be expressed as

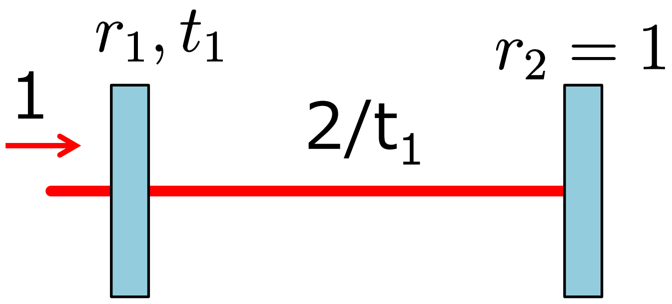
$$E_{\mathrm{cav}} = t_1 E_0 e^{i\omega t} + t_1 r_1 r_2 E_0 e^{i(\omega t - \frac{4\pi L}{\lambda})} + t_1 (r_1 r_2)^2 E_0 e^{i(\omega t - \frac{4\pi L}{\lambda})} + \dots$$

$$= \underbrace{(t_1 + t_1 r_1 r_2 2 e^{i\frac{4\pi L}{\lambda}} + t_1 (r_1 r_2)^2 2 e^{i\frac{8\pi L}{\lambda}} + \dots)}_{\text{infinite geometric series with a common ratio of } r_1 r_2 e^{i\frac{4\pi L}{\lambda}}} \underset{\substack{\uparrow \\ \text{input field}}}{E_0 e^{i\omega t}}$$

$$= \frac{t_1}{1 - r_1 r_2 e^{i\frac{4\pi L}{\lambda}}} E_{\mathrm{in}}$$

For $t_2=0$ and on resonance

$$\simeq \frac{2}{t_1} E_{\mathrm{in}}$$



Reflected Field

- Reflected field can be expressed as

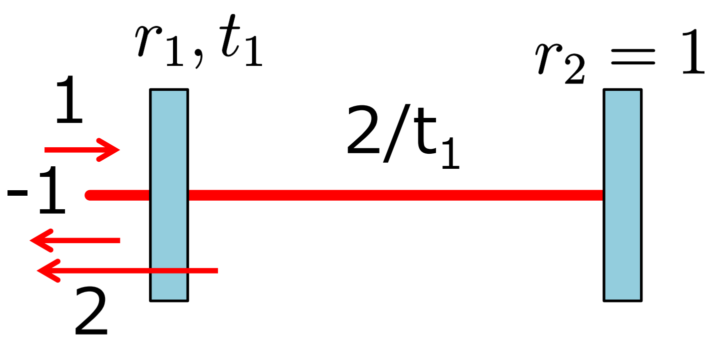
$$E_{\text{refl}} = -r_1 E_0 e^{i\omega t} + t_1^2 r_2 E_0 e^{i(\omega t - \frac{4\pi L}{\lambda})} + t_1^2 r_1 r_2^2 E_0 e^{i(\omega t - \frac{4\pi L}{\lambda})} + \dots$$

$$= (-r_1 + \underbrace{t_1^2 r_2 e^{i\frac{4\pi L}{\lambda}} + t_1^2 r_1 r_2^2 2 e^{i\frac{8\pi L}{\lambda}} + \dots}_{\text{infinite geometric series with a common ratio of } r_1 r_2 e^{i\frac{4\pi L}{\lambda}}})E_0 e^{i\omega t}$$

$$= \left(-r_1 + \frac{t_1^2 r_2 e^{i\frac{4\pi L}{\lambda}}}{1 - r_1 r_2 e^{i\frac{4\pi L}{\lambda}}}\right) E_{\text{in}}$$

For t$_2$=0 and on resonance

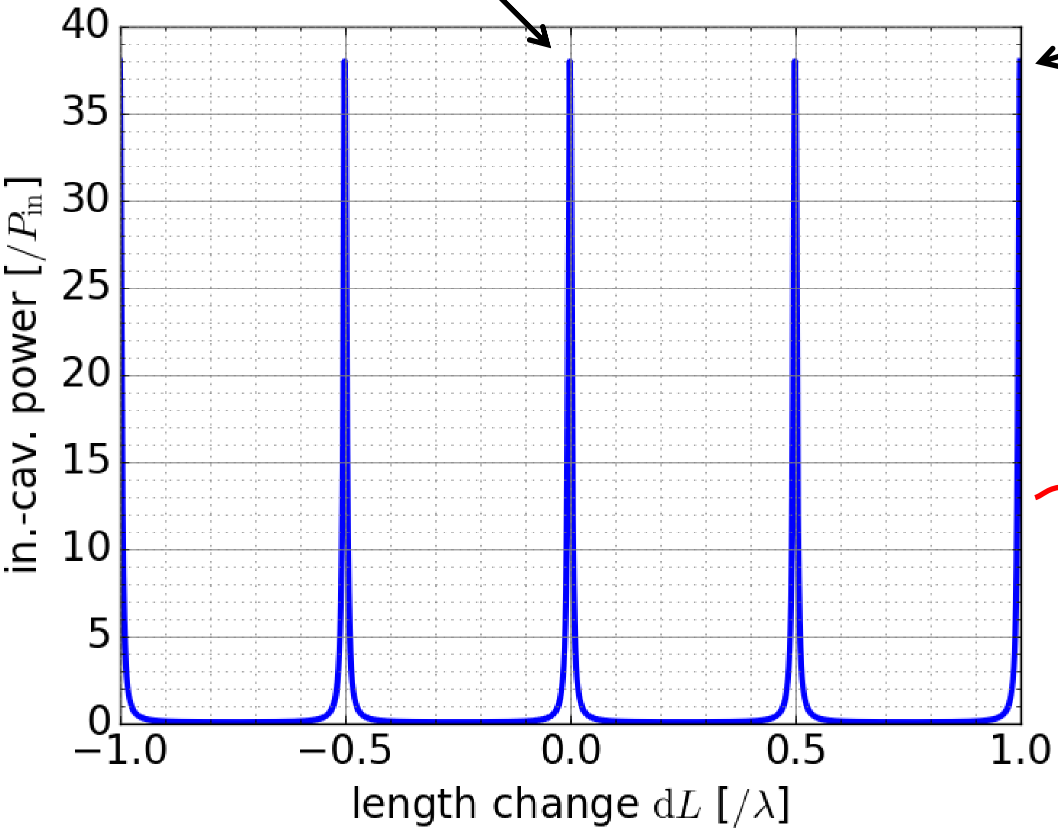
$$\simeq \underbrace{\left(-r_1 + \underbrace{(1 + r_1)}_{\sim 2}\right)}_{=1 \text{ (Energy is conserved)}} E_{\text{in}}$$



Intra-Cavity Power

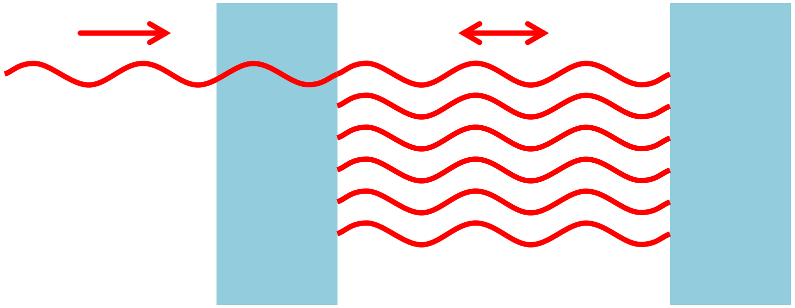
- Power inside the cavity

$$|E_{\mathrm{cav}}|^2 = \left| \frac{t_1}{1 - r_1 r_2 e^{i\frac{4\pi L}{\lambda}}} \right|^2 P_{\mathrm{in}}$$



resonance

Intra-cavity power can be much higher than input power on resonance

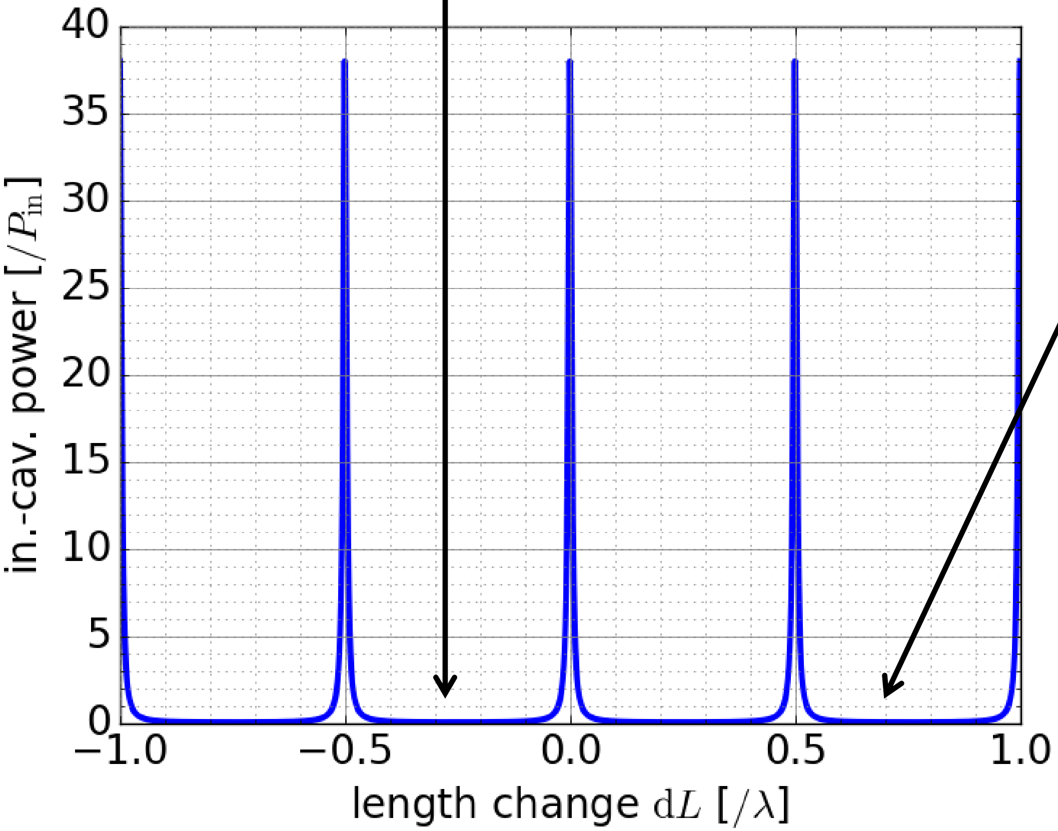


constructive interference

Intra-Cavity Power

- Power inside the cavity

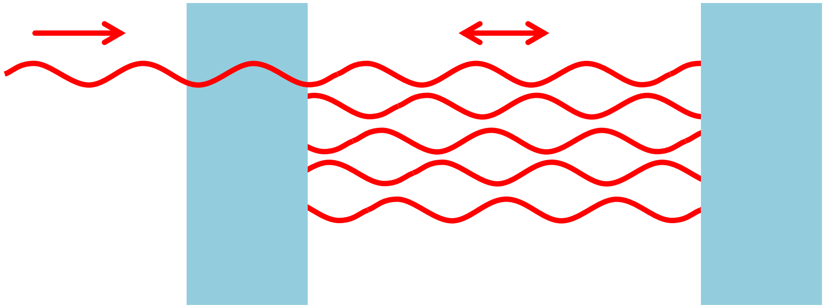
$$|E_{\mathrm{cav}}|^2 = \left|\frac{t_1}{1-r_1r_2e^{i\frac{4\pi L}{\lambda}}}\right|^2 P_{\mathrm{in}}$$



anti-resonance

Almost no intra-cavity power at anti-resonance

destructive interference



Resonant Frequency

- Cavity will be resonant when cavity round-trip length is integer multiples of laser wavelength

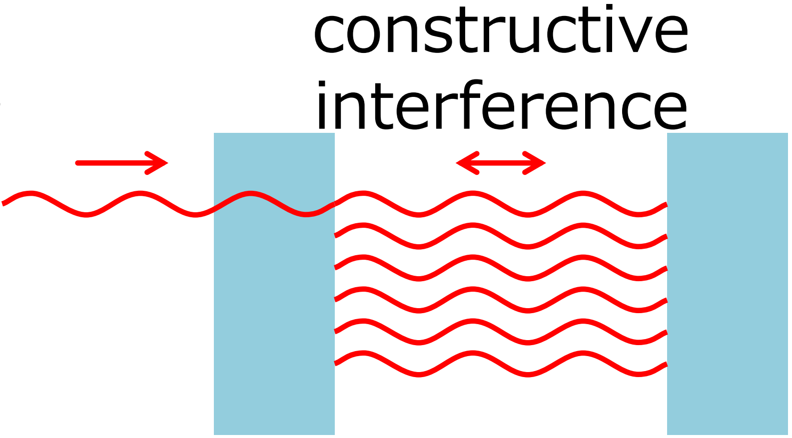
$$2L = N\lambda$$

- In other words, cavity will be resonant when laser frequency is integer multiples of free spectral range

$$\omega_{\mathrm{cav}} = N\omega_{\mathrm{FSR}} = N\frac{\pi c}{L}$$

- Resonant frequency shifts with mirror displacement

$$\delta\omega_{\mathrm{cav}} = \frac{\omega_{\mathrm{cav}}}{L}\delta L$$



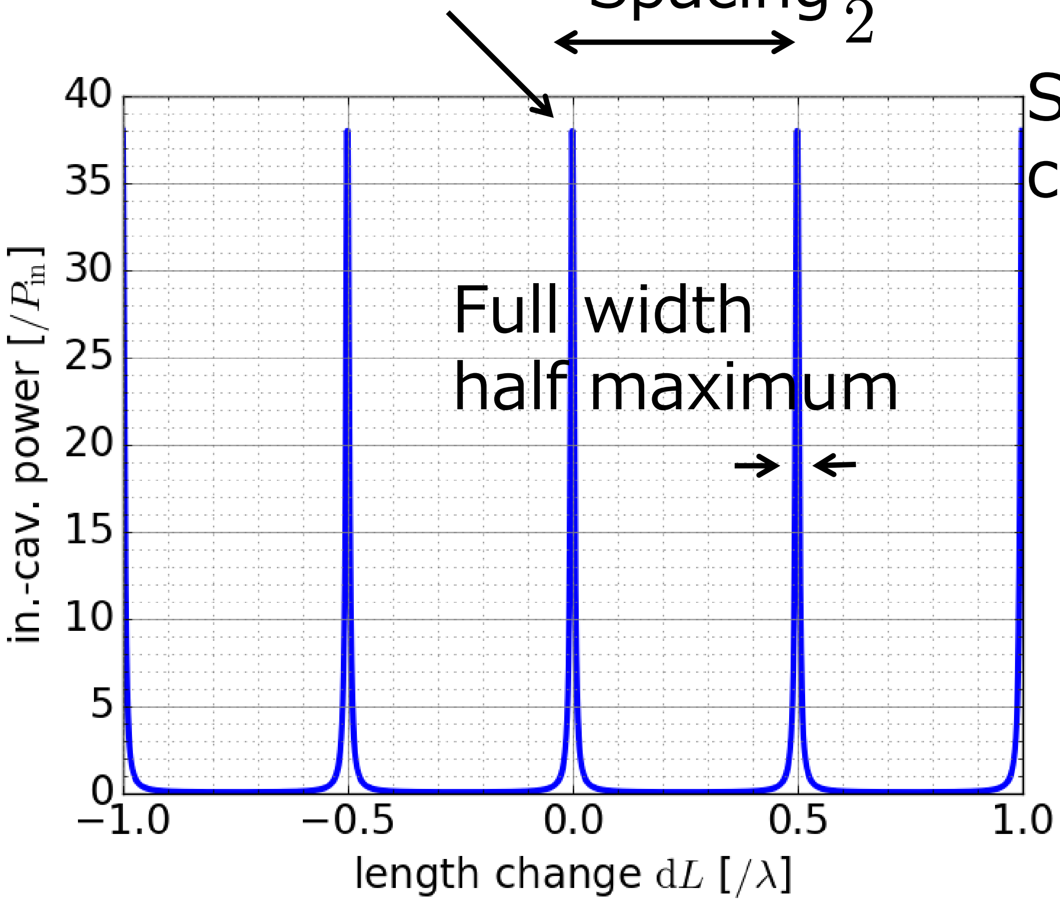
Finesse

- Power inside the cavity

$$|E_{\mathrm{cav}}|^2 = \left| \frac{t_1}{1 - r_1 r_2 e^{i\frac{4\pi L}{\lambda}}} \right|^2 P_{\mathrm{in}}$$

Resonance

Spacing $\frac{\lambda}{2}$



Full width half maximum

Sharpness of the resonance can be evaluated with

$$\frac{\text{Spacing}}{\text{FWHM}} = \frac{\pi\sqrt{r_1 r_2}}{1 - r_1 r_2} \equiv \mathcal{F} \simeq \frac{2\pi}{T_1 + T_2}$$

Finesse

$(T_i = t_i^2)$

Approximation to remember when t_1 and t_2 are small

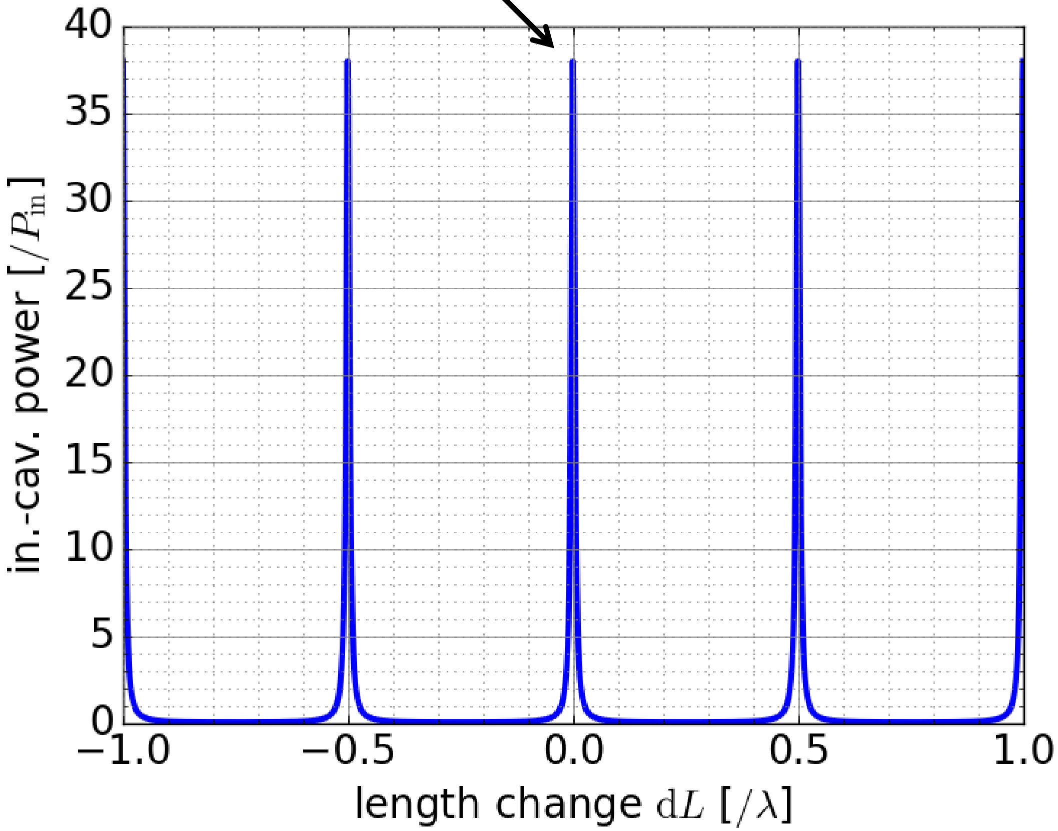
Higher finesse for higher reflectivity

Cavity Build-up

- Power inside the cavity

$$|E_{\mathrm{cav}}|^2 = \left|\frac{t_1}{1-r_1 r_2 e^{i\frac{4\pi L}{\lambda}}}\right|^2 P_{\mathrm{in}}$$

Resonance



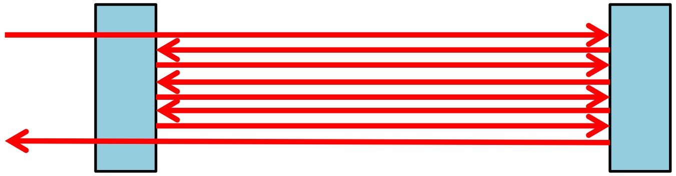
Intra-cavity power at resonance

$$|E_{\mathrm{cav}}|^2_{\mathrm{max}} = \left|\frac{t_1}{1-r_1 r_2}\right|^2 P_{\mathrm{in}}$$

$$\simeq \boxed{\frac{2\mathcal{F}}{\pi}} P_{\mathrm{in}}$$

with
$r_1 \sim 1$, $r_2 = 1$

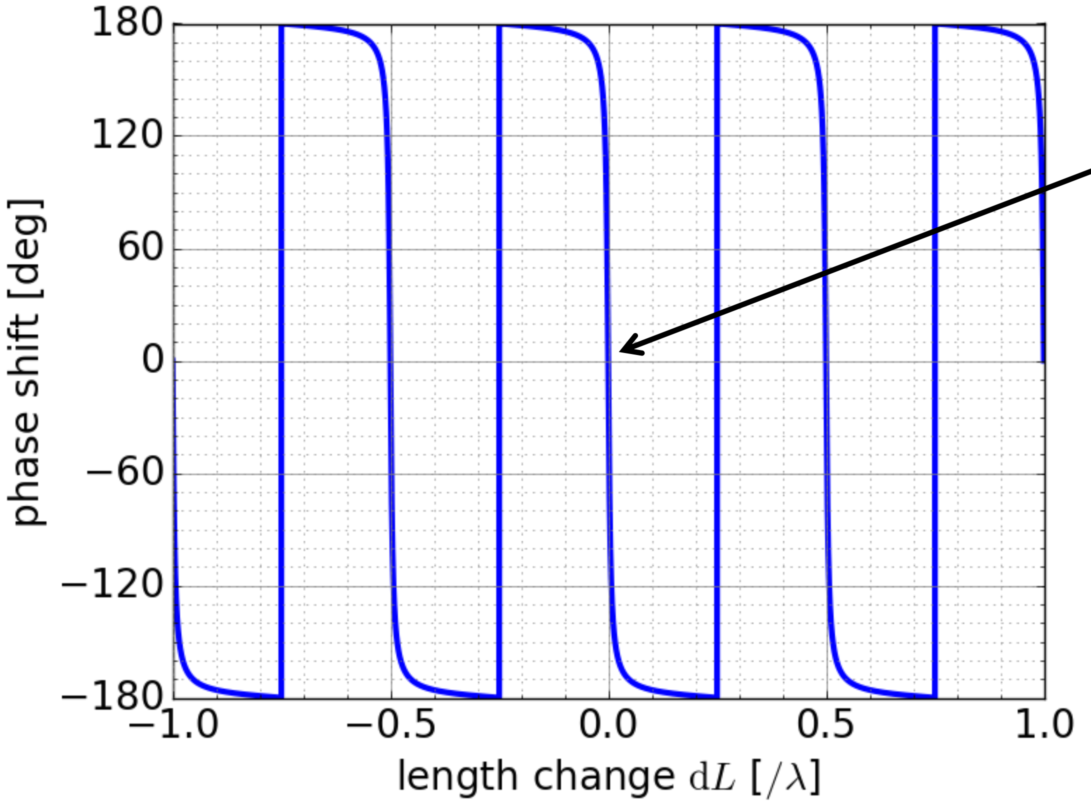
Cavity build-up



Phase of Reflected light

- Reflected field

$$E_{\mathrm{refl}} = \left(-r_1 + \frac{t_1^2 r_2 e^{i\frac{4\pi L}{\lambda}}}{1 - r_1 r_2 e^{i\frac{4\pi L}{\lambda}}} \right) E_{\mathrm{in}}$$



Phase of the reflected beam changes drastically at the resonance

$$\frac{\delta\phi}{\delta L} \simeq \boxed{\frac{2\mathcal{F}}{\pi}}\frac{4\pi}{\lambda}$$

Cavity build-up

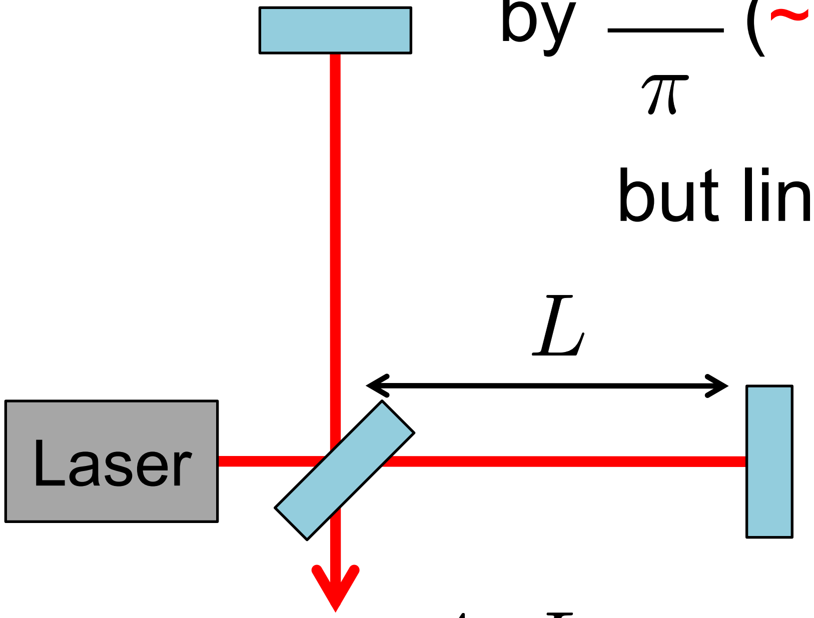
Michelson and Fabry-Pérot

- The phase of the reflected light is different by $\frac{2\mathcal{F}}{\pi}$

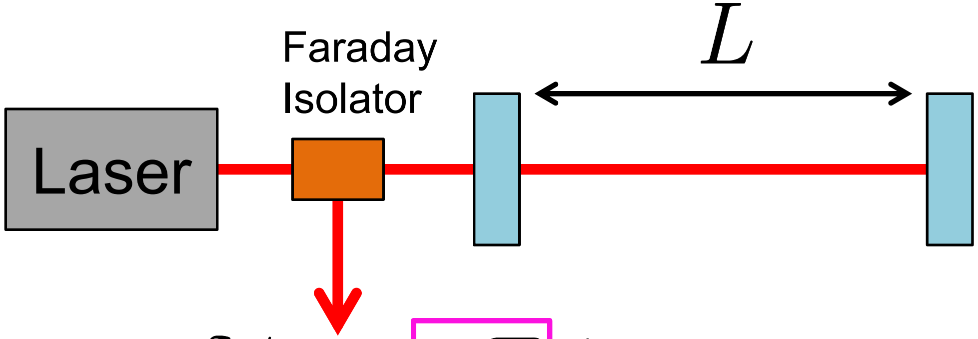
→ FP is more sensitive to mirror displacement

by $\frac{2\mathcal{F}}{\pi}$ (~ finesse)

but linear range is smaller



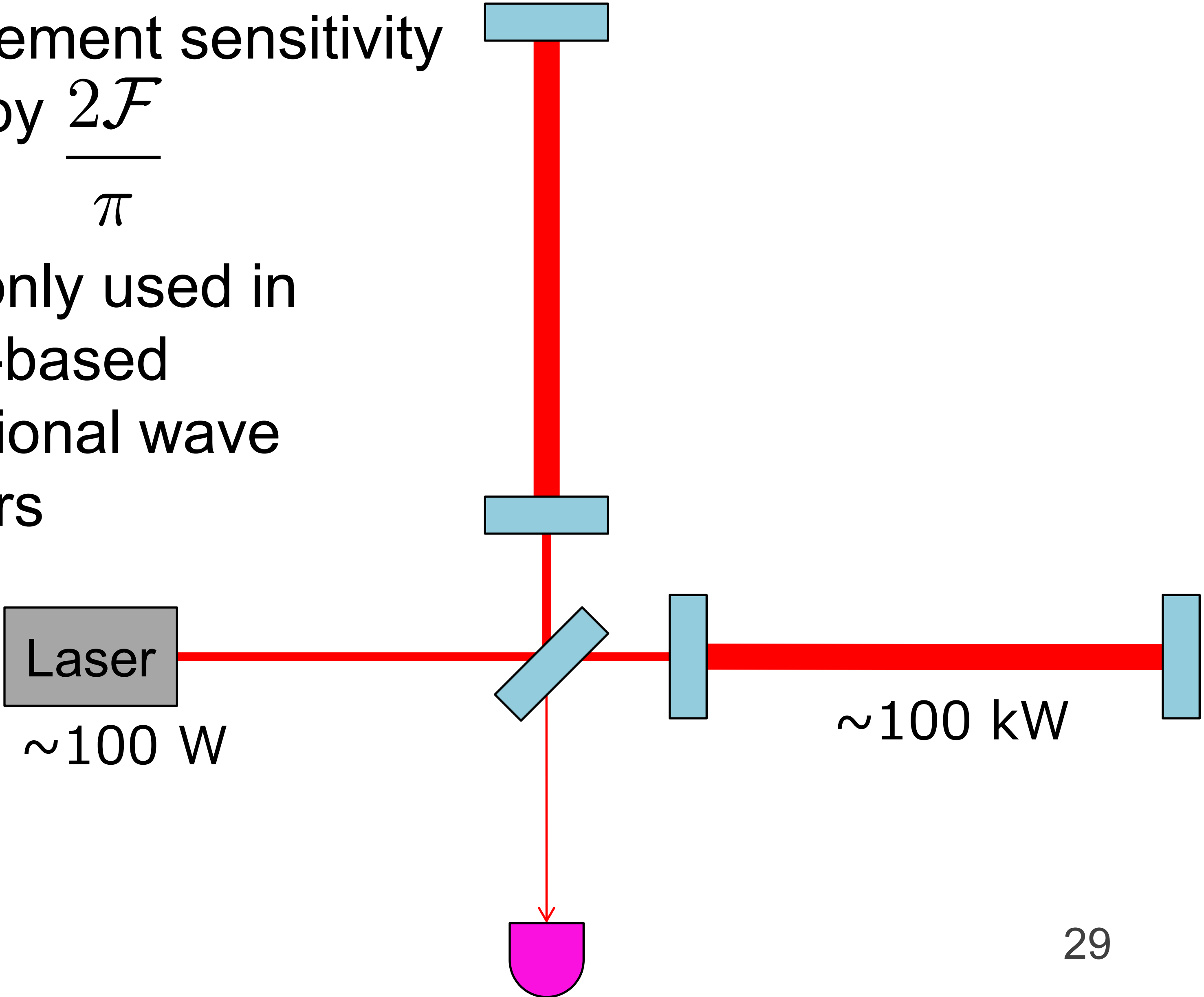
$$\phi = \frac{4\pi L}{\lambda}$$



$$\frac{\delta\phi}{\delta L} \simeq \boxed{\frac{2\mathcal{F}}{\pi}} \frac{4\pi}{\lambda}$$

Fabry-Pérot-Michelson Interferometer

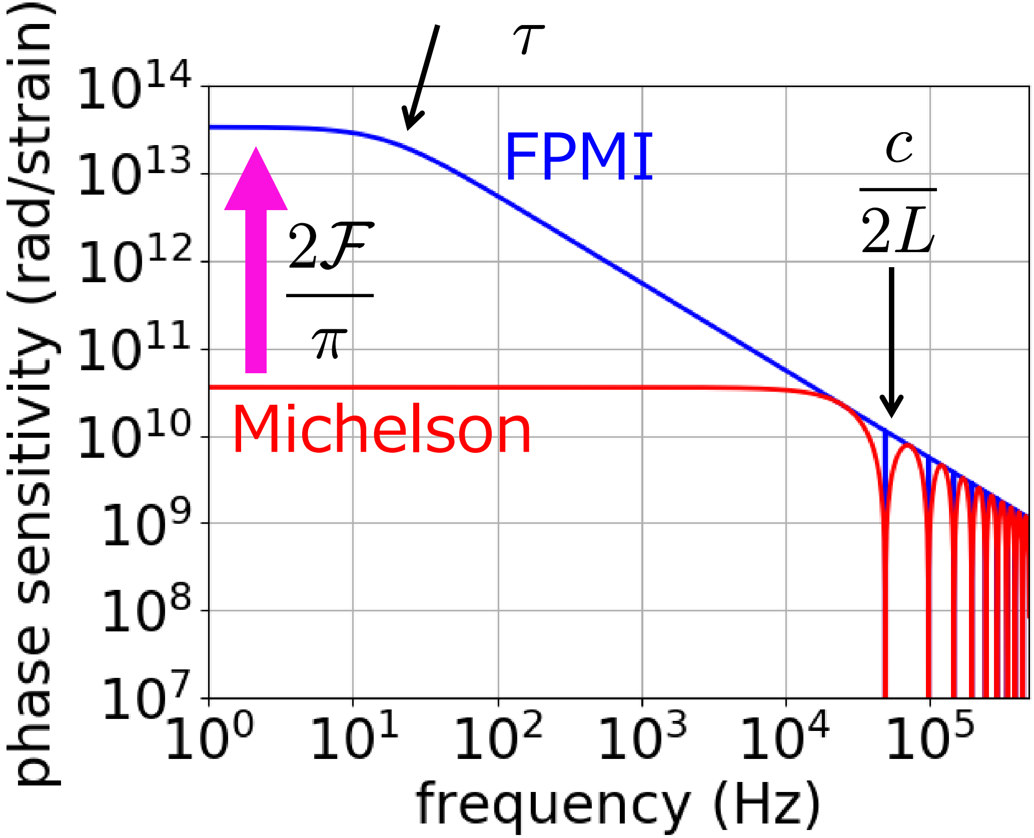
- Displacement sensitivity higher by $\frac{2\mathcal{F}}{\pi}$
- Commonly used in ground-based gravitational wave detectors

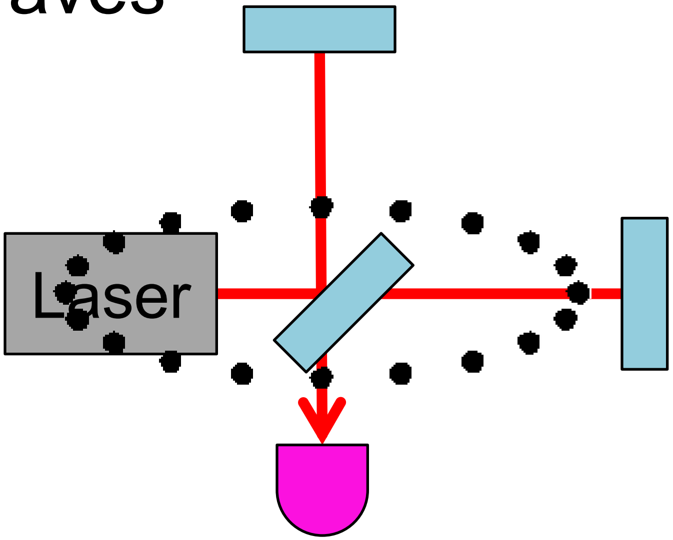


High-Frequency Response

- The effect of gravitational waves cancel at high frequencies

$$f_c = \frac{1}{2\pi}\Bigg(\underbrace{\frac{2\mathcal{F}}{\pi}\frac{L}{c}}_{\tau}\Bigg)^{-1} = \frac{c}{4L\mathcal{F}}$$

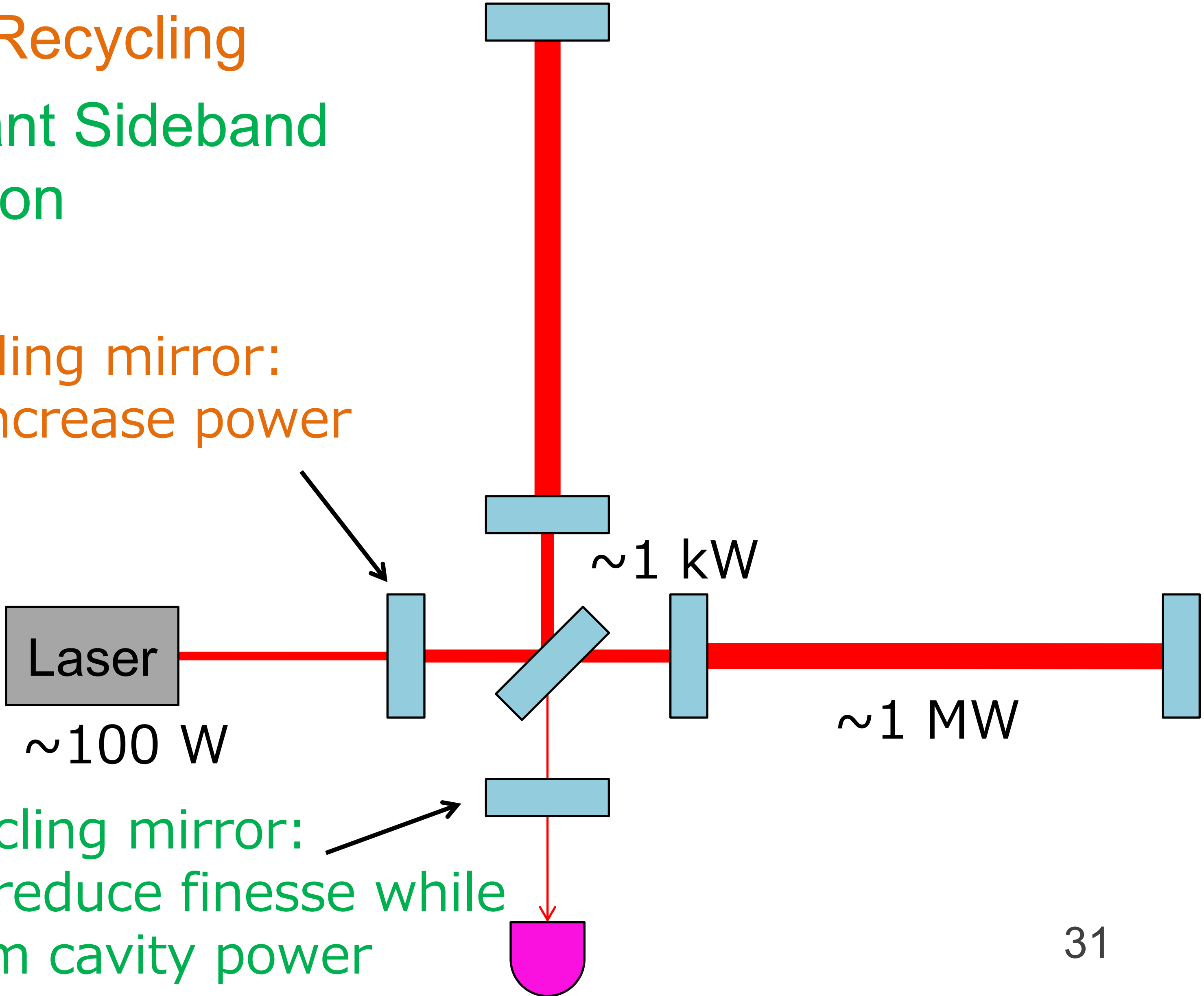




For a given frequency, there is a limit where longer arm length and higher finesse won’t help increasing the sensitivity

Resonant Sideband Extraction

- Power Recycling
- Resonant Sideband Extraction



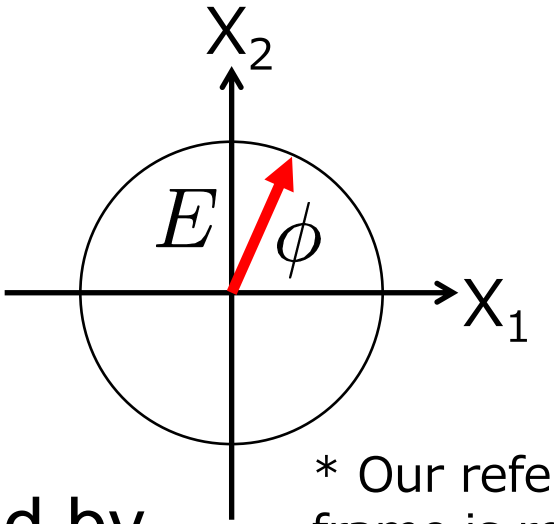
Power recycling mirror:
Effectively increase power

Signal recycling mirror:
Effectively reduce finesse while keeping arm cavity power

Phasor Diagram

- Complex amplitude

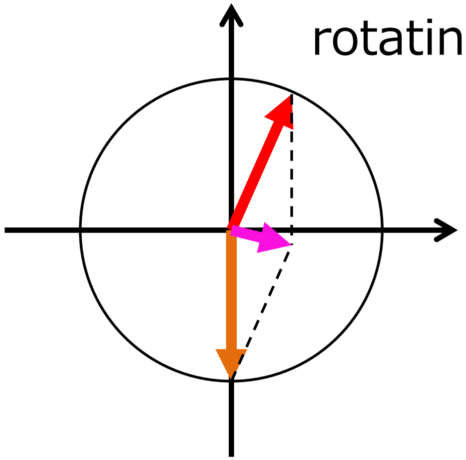
$$Ee^{i(\omega t+\phi)}$$



* Our reference frame is rotating at ωt so that the vector is not rotating

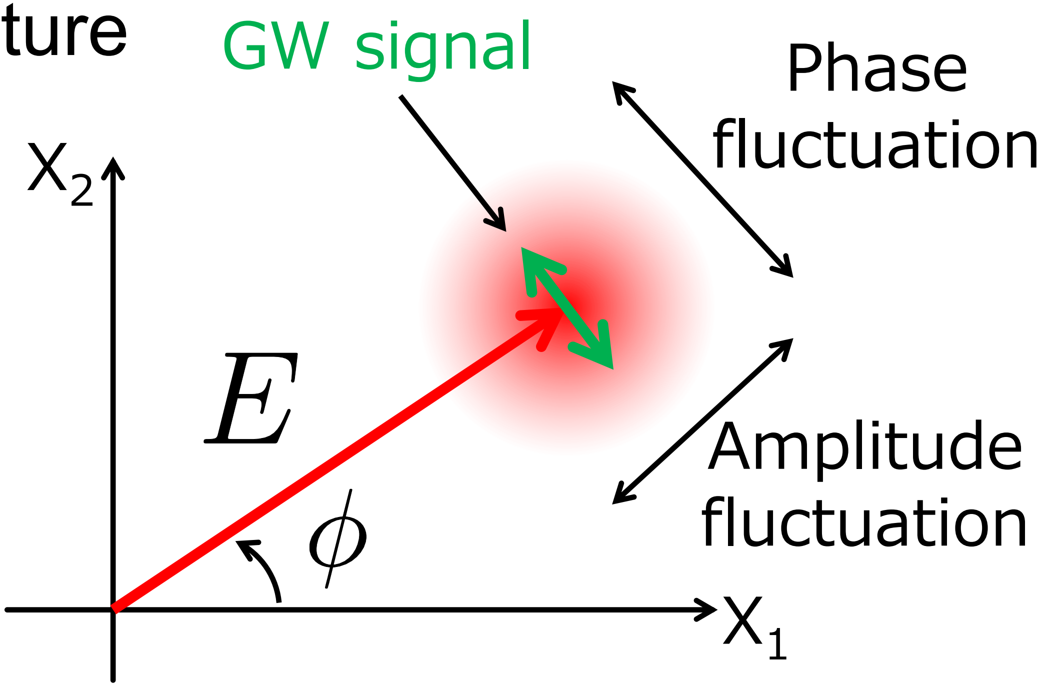
- Interference can be understood by addition of vectors

$$Ee^{i(\omega t+\phi)} + Ee^{i(\omega t+\phi')}$$



Ball-on-Stick Picture

- Field is actually fluctuating (classically and quantum mechanically)
  → Ball-on-stick picture

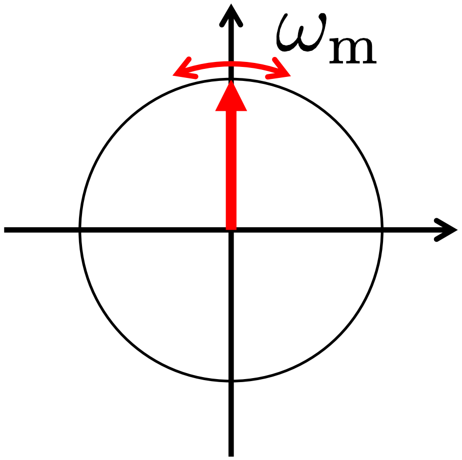


- Gravitational wave creates phase modulation
- We need to convert this into amplitude modulation to detect the signal with a photodiode

Phase Modulation Sidebands

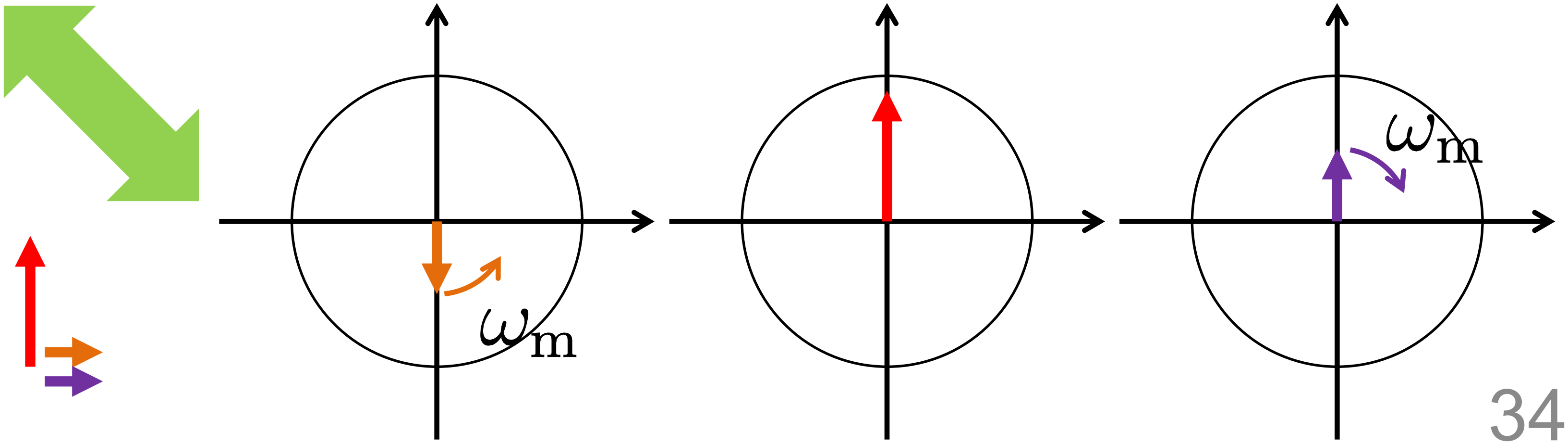
- Phase modulation creates two sidebands (and harmonics)

$$Ee^{i(\omega t+\beta\sin\omega_\mathrm{m}t)}$$



Expansion with Bessel functions of the first kind

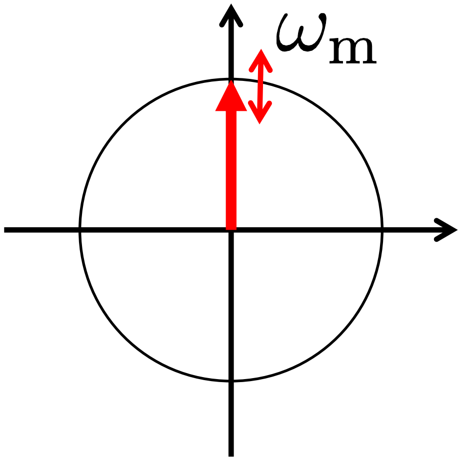
$$E[J_0(\beta)e^{i\omega t}+J_1(\beta)e^{i(\omega+\omega_\mathrm{m})t}-J_1(\beta)e^{i(\omega-\omega_\mathrm{m})t}]$$



Amplitude Modulation Sidebands

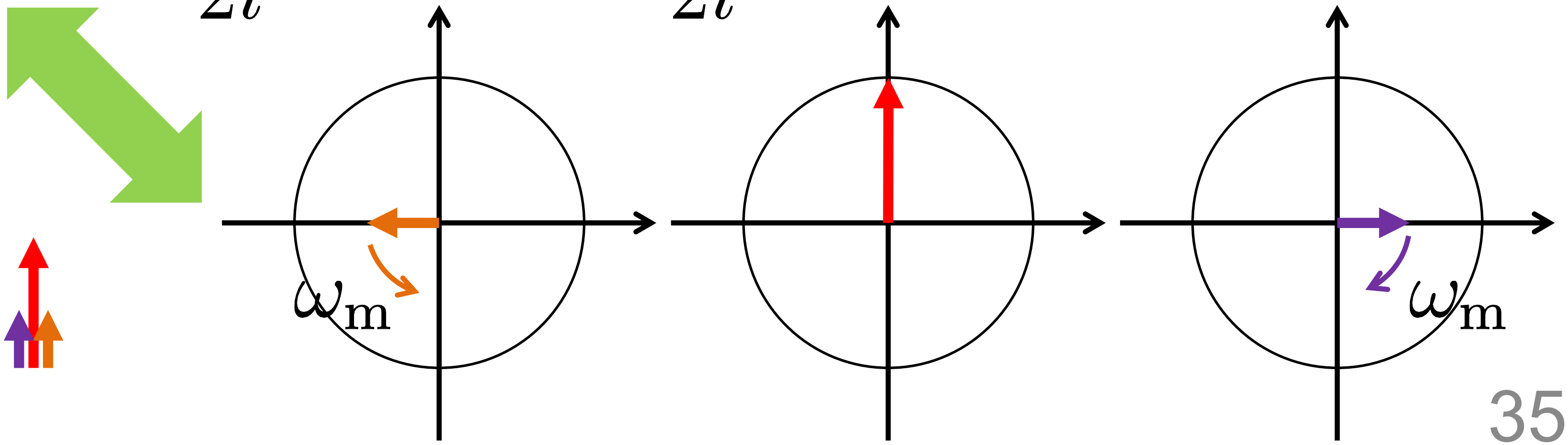
- Amplitude modulation creates 2 sidebands (exact)

$$E(1+\beta\sin\omega_{\mathrm{m}}t)e^{i\omega t}$$



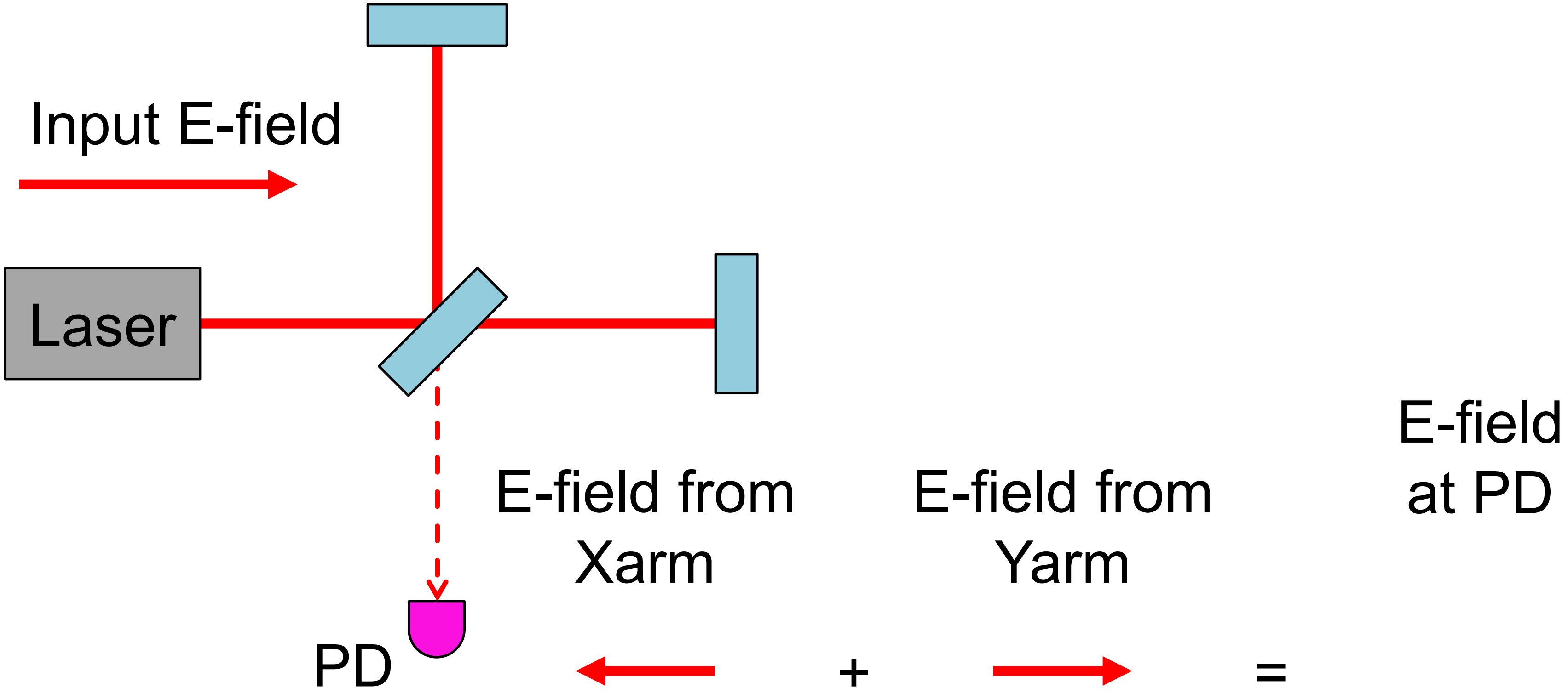
equals to

$$E[e^{i\omega t}+\frac{\beta}{2i}e^{i(\omega+\omega_{\mathrm{m}})t}-\frac{\beta}{2i}e^{i(\omega-\omega_{\mathrm{m}})t}]$$



Michelson Interferometer

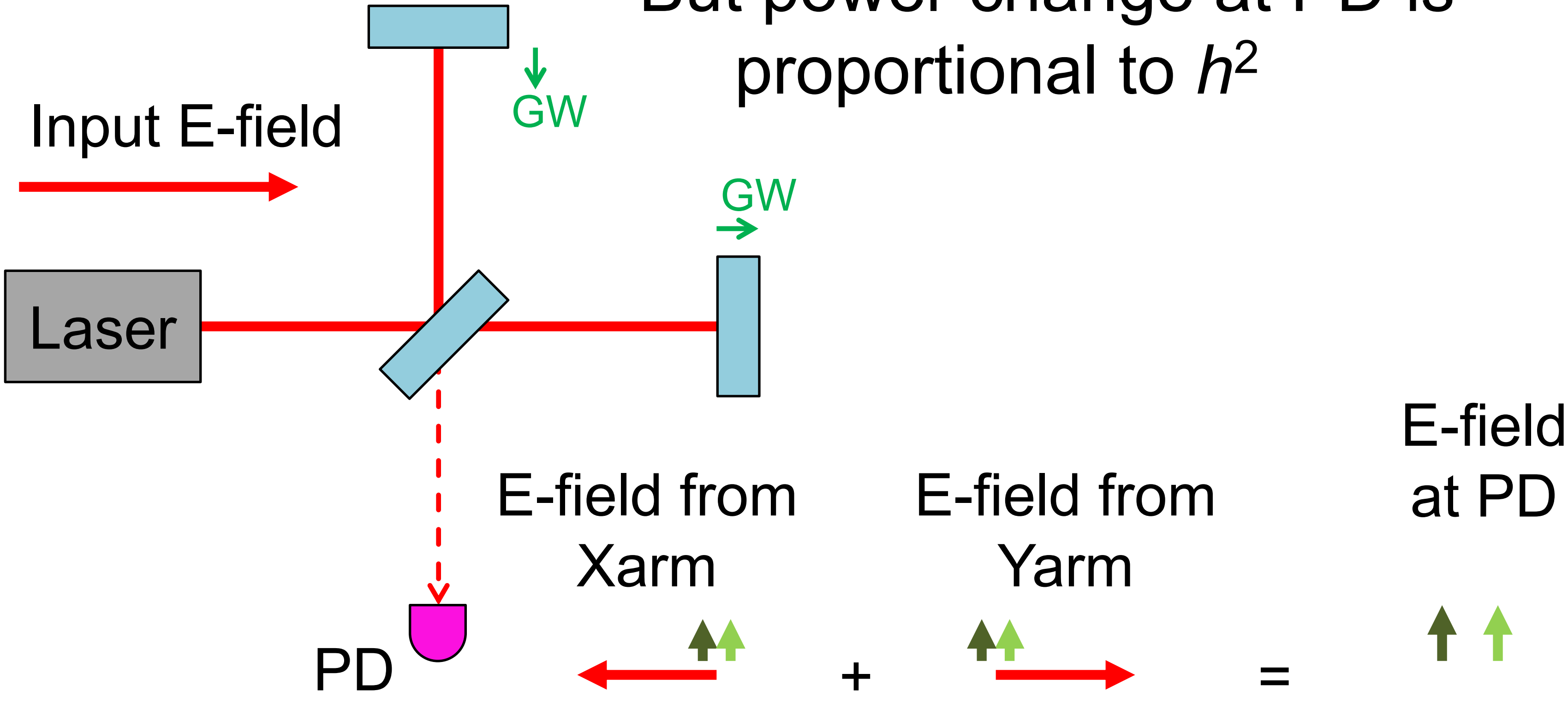
- If length is perfectly the same, no light at PD



Michelson Interferometer

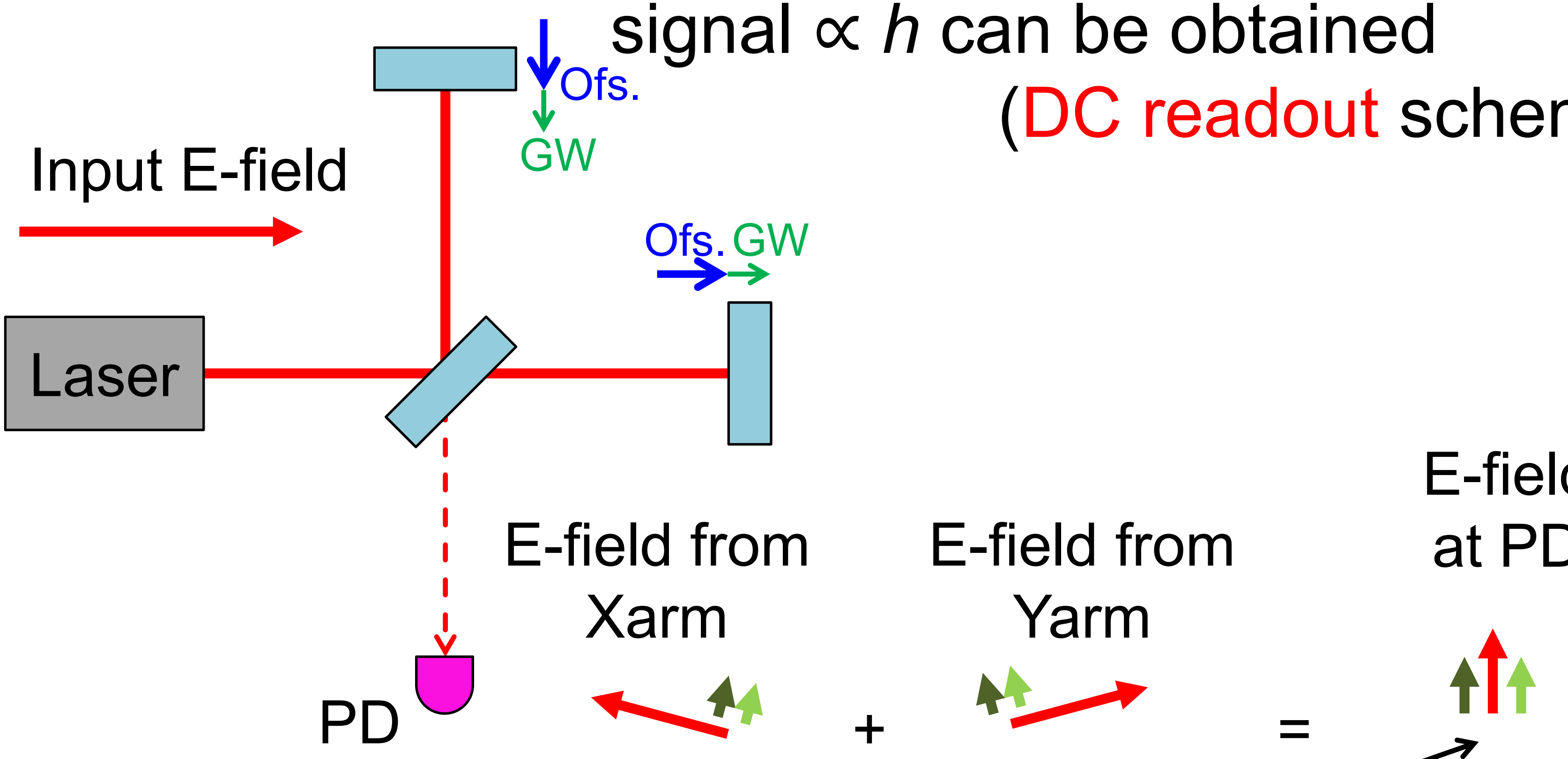
- Gravitational waves makes phase modulation

But power change at PD is proportional to h^2



Michelson Interferometer

- Offset creates intensity modulation at PD, and signal $\propto h$ can be obtained (DC readout scheme)



Intensity modulation

Reference field for signal readout is called local oscillator (LO)

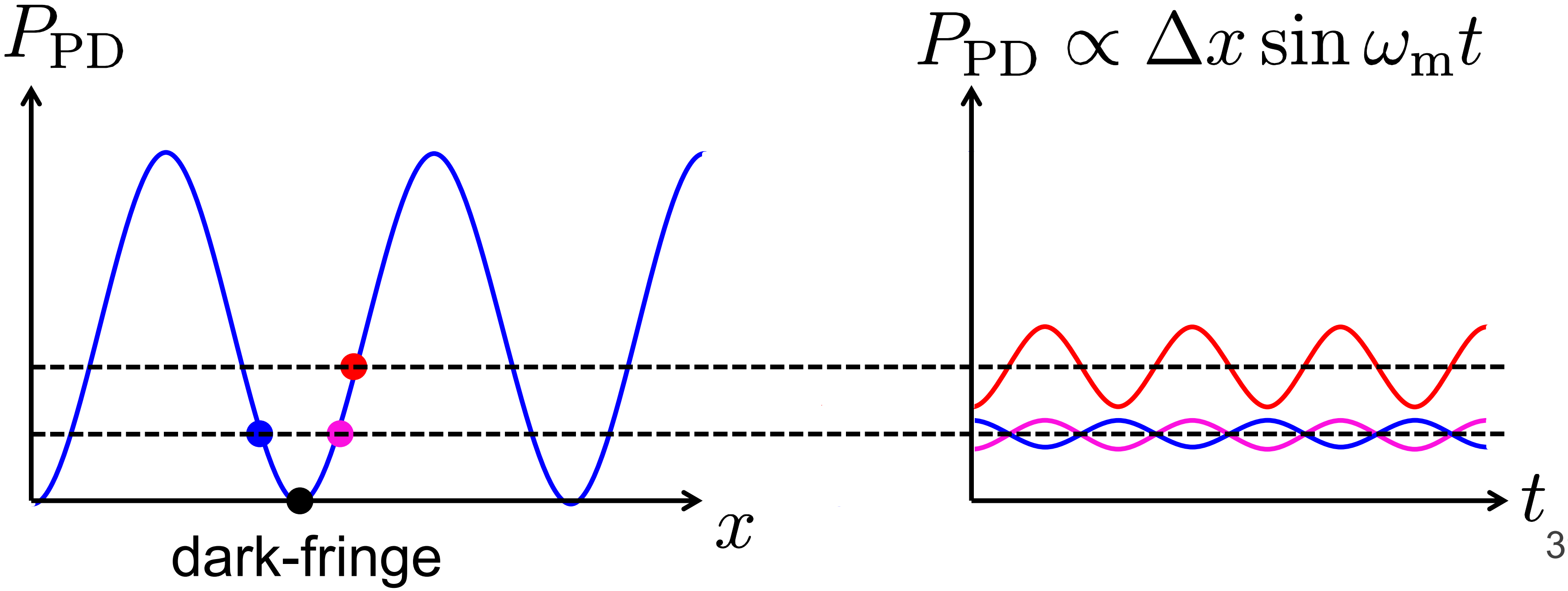
Signal Extraction at Dark Fringe

- Signal proportional to *h* can be obtained at dark fringe if mirror position is constantly modulated, and PD output is demodulated

Remove with a lowpass filter

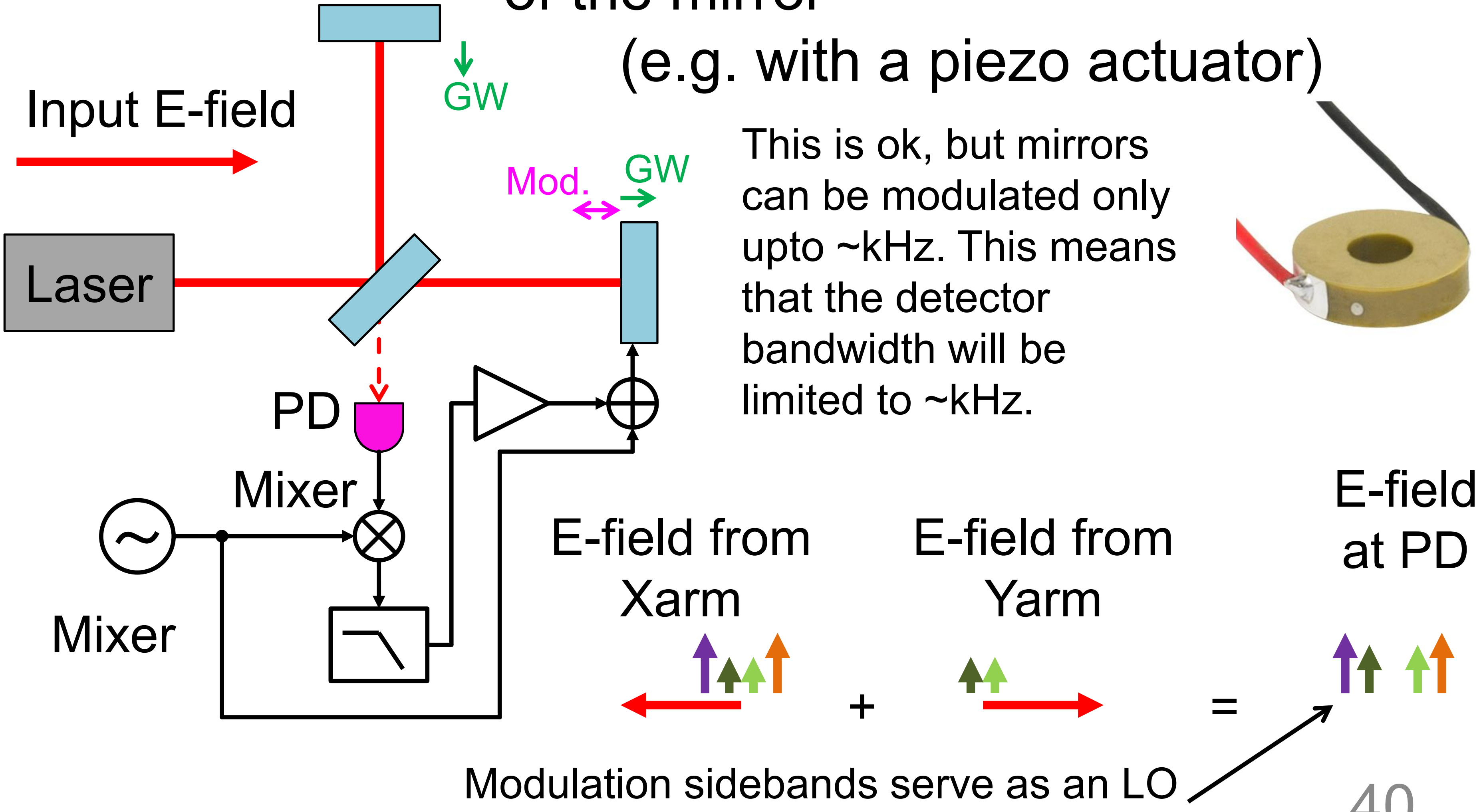
$$\Delta x \sin \omega_{\rm m} t \underset{\uparrow}{\times} \sin \omega_{\rm m} t = \frac{\Delta x}{2}\left(1 - \cancel{\cos 2\omega_{\rm m} t}\right)$$

demodulation



Modulation-Demodulation Method

- This can be simply done by modulating the position of the mirror (e.g. with a piezo actuator)

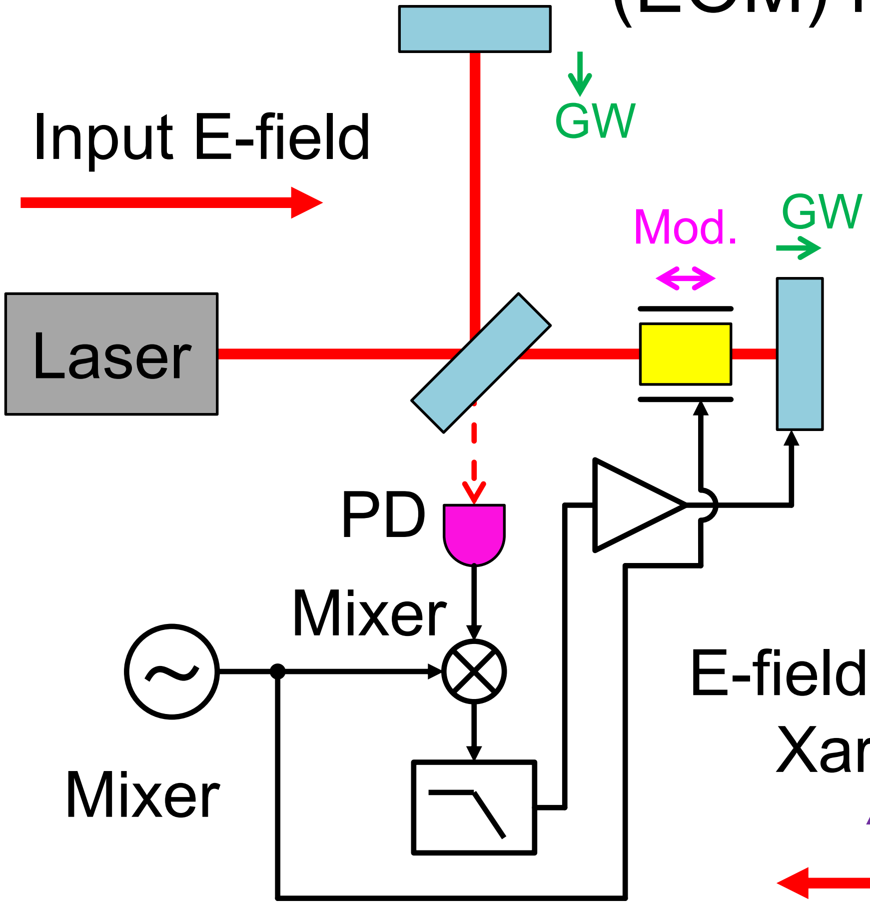


This is ok, but mirrors can be modulated only upto ~kHz. This means that the detector bandwidth will be limited to ~kHz.

Modulation sidebands serve as an LO for GW sidebands

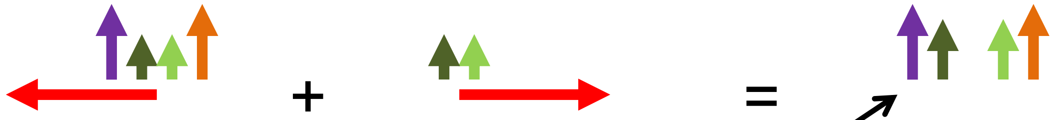
Modulation-Demodulation Method

- Phase modulation with an electro-optic modulator (EOM) increases the bandwidth



Modulation frequency used is upto ~100 MHz

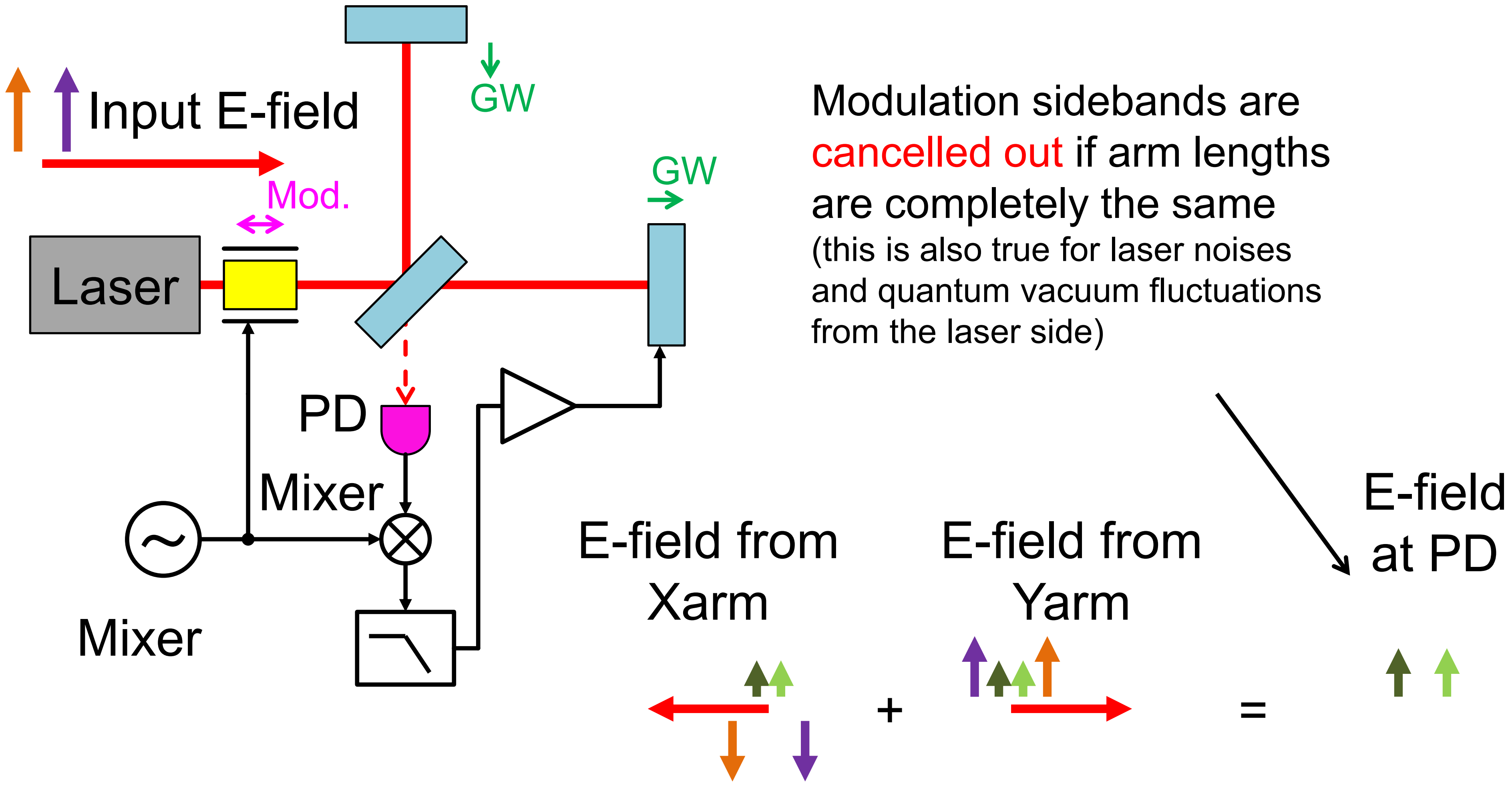
This is ok, but adding something in the arm creates optical losses.



Modulation sidebands serve as an LO for GW sidebands

Pre-Modulation Method

- Put an EOM before the beam splitter instead

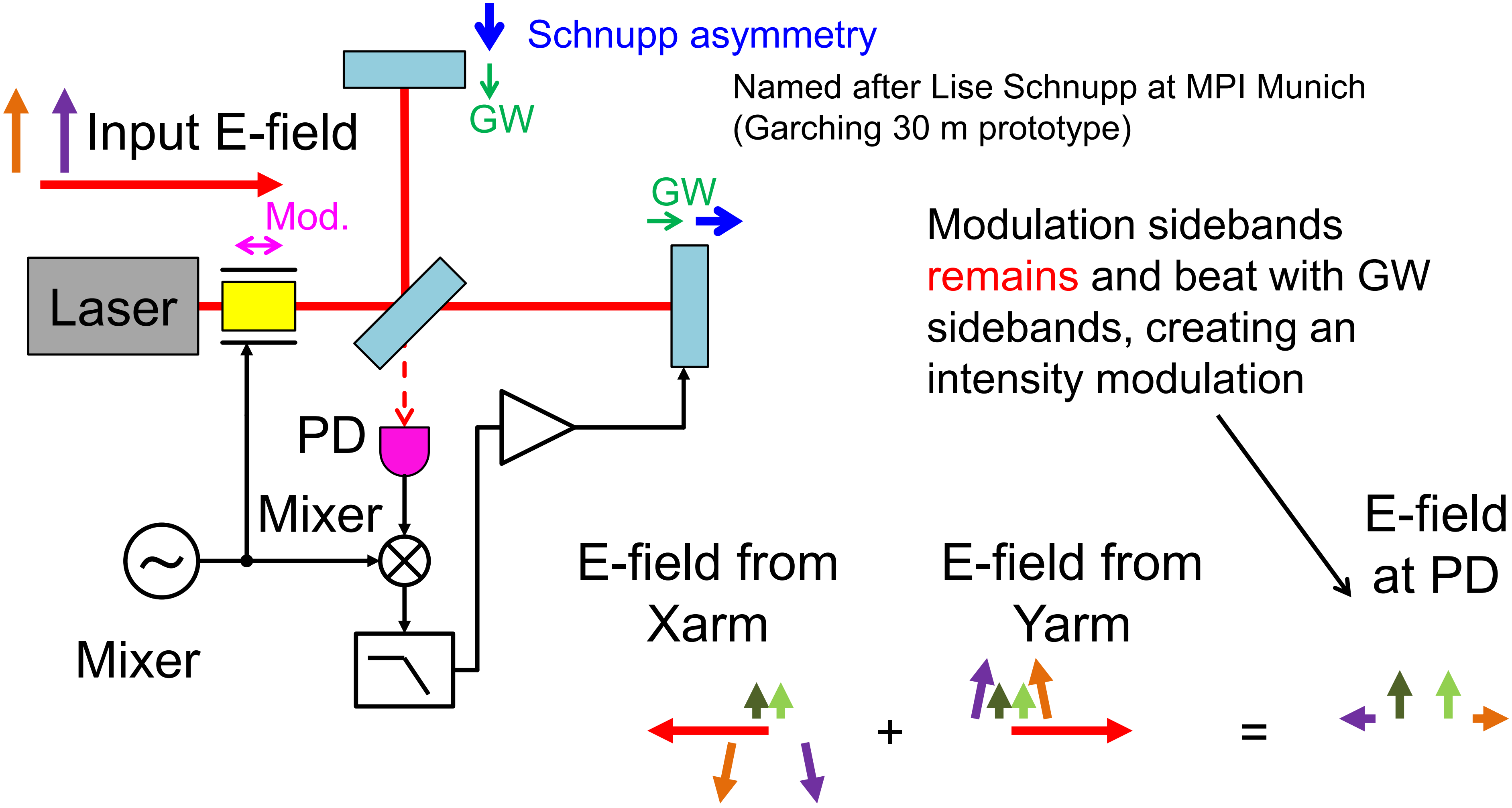


Modulation sidebands are cancelled out if arm lengths are completely the same

(this is also true for laser noises and quantum vacuum fluctuations from the laser side)

Schnupp Asymmetry

- Make arm length macroscopically different



Named after Lise Schnupp at MPI Munich
(Garching 30 m prototype)

Modulation sidebands remains and beat with GW sidebands, creating an intensity modulation

Schnupp Asymmetry: How Much?

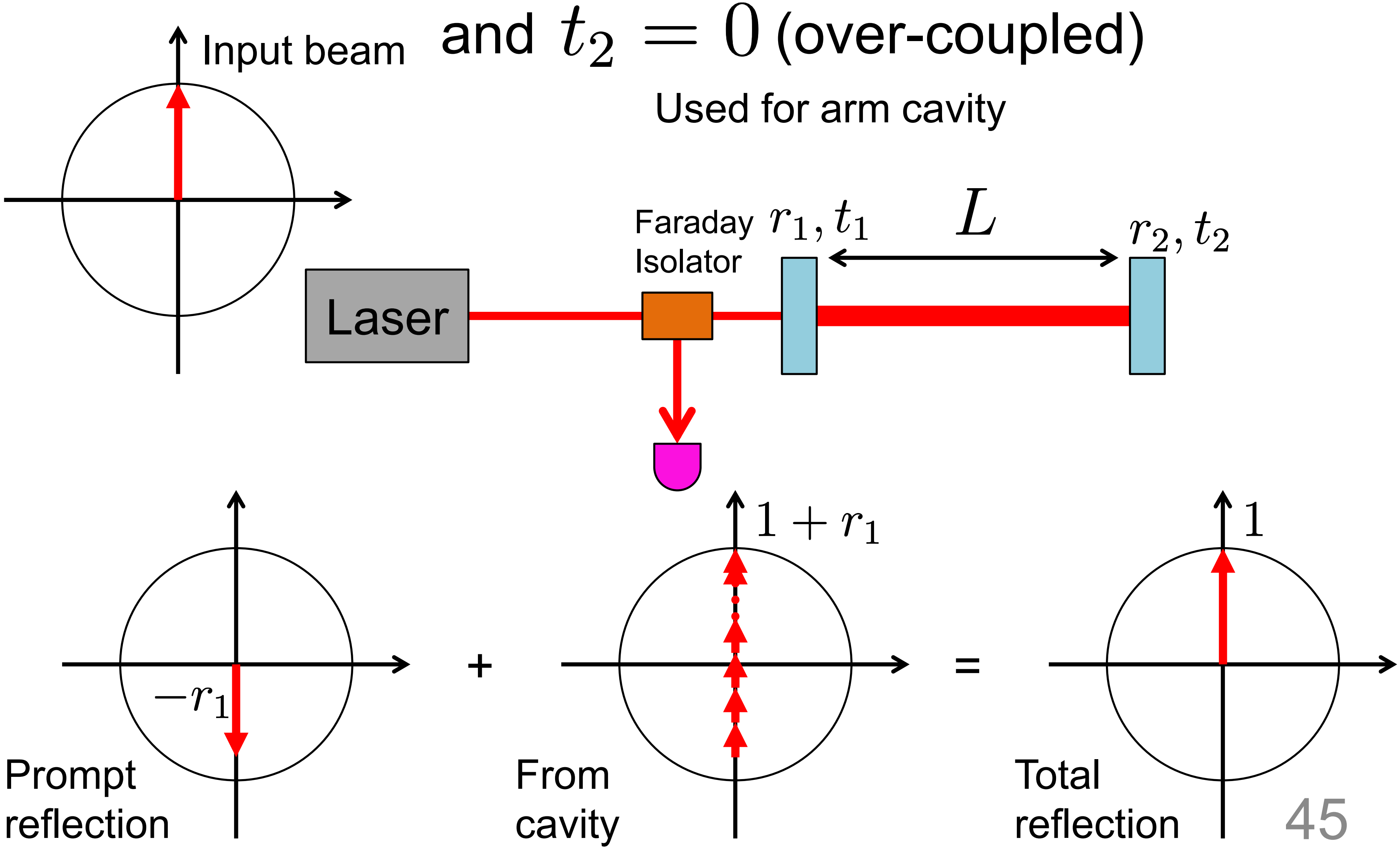
- Schnupp asymmetry of $l_{-} \equiv l_{\mathrm{x}} - l_{\mathrm{y}}$ gives sideband amplitude transmission of

$$t_{\mathrm{sb}} = \sin\left(\frac{l_{-}\omega_{\mathrm{m}}}{c}\right) \equiv \sin\left(\alpha\right)$$

- For modulation frequency $\omega_m/(2\pi)$ = 300 MHz, l_- = 1 m creates α = 2π
- For KAGRA, modulation frequencies are f1 = 16.88 MHz, f2 = 45.02 MHz and l_- = 3.33 m
- For aLIGO, modulation frequencies are f1 = 9.1 MHz, f2 = 45.5 MHz and l_- = 0.08 m

Fabry-Pérot Cavity and Phasor

- First, assume a resonant condition $2L = N\lambda$ and $t_2 = 0$ (over-coupled)

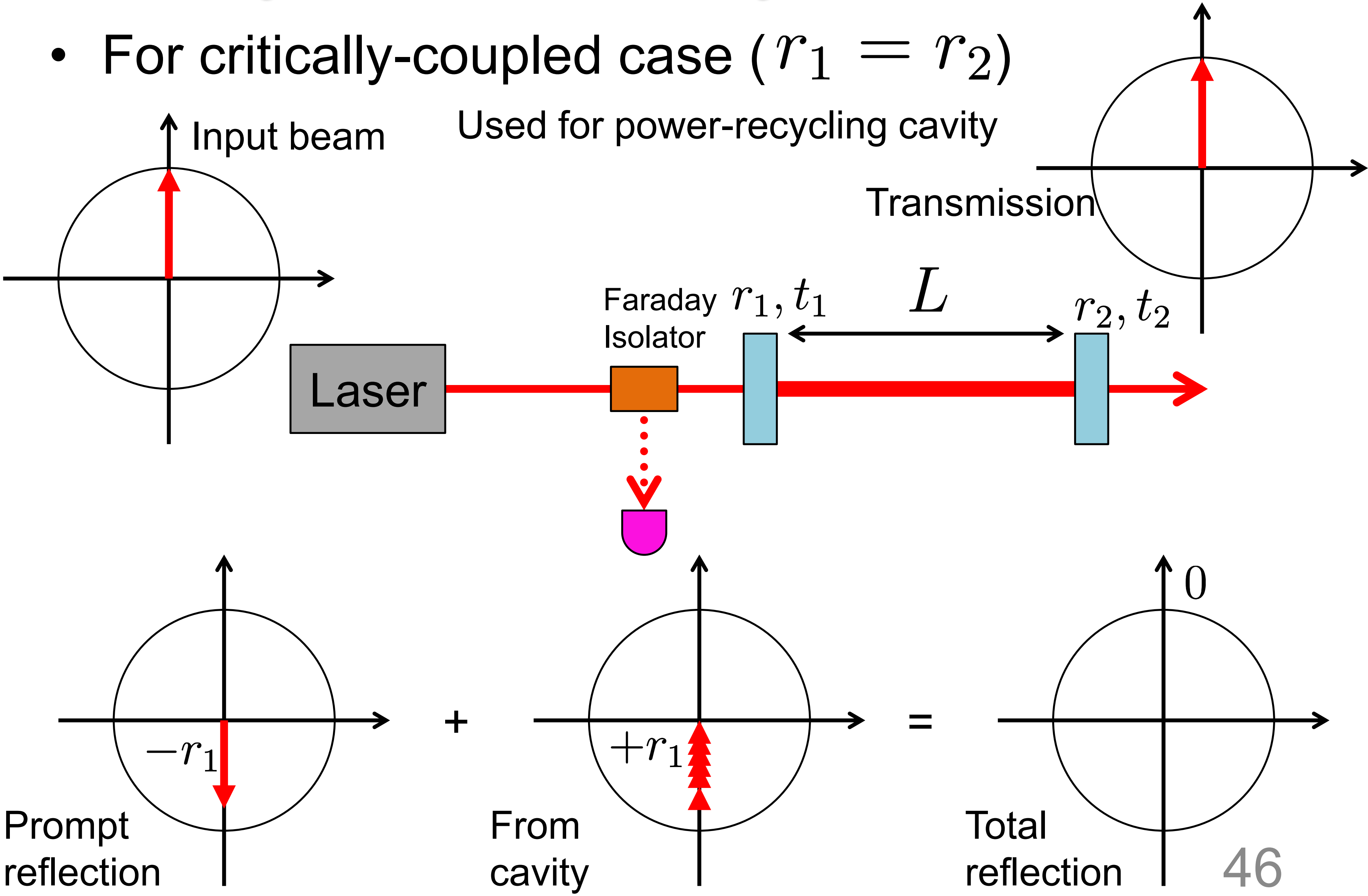


Used for arm cavity

Fabry-Pérot Cavity and Phasor

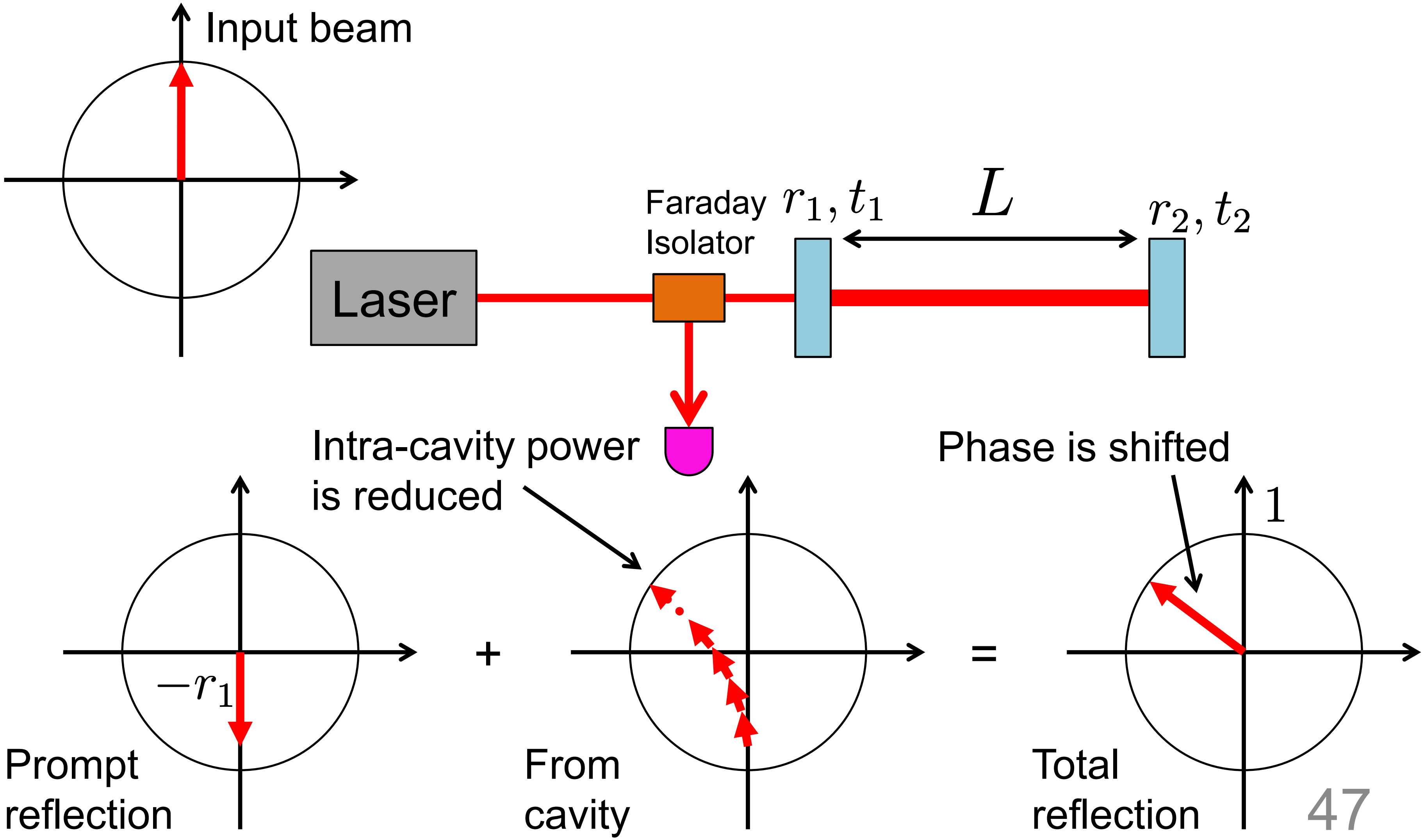
- For critically-coupled case ($r_1 = r_2$)

Used for power-recycling cavity



Fabry-Pérot Cavity and Phasor

- If $t_2 = 0$ and not on resonance



Input beam

Laser

Faraday Isolator

r_1, t_1

L

r_2, t_2

Intra-cavity power is reduced

Phase is shifted

$-r_1$

1

Prompt reflection

+

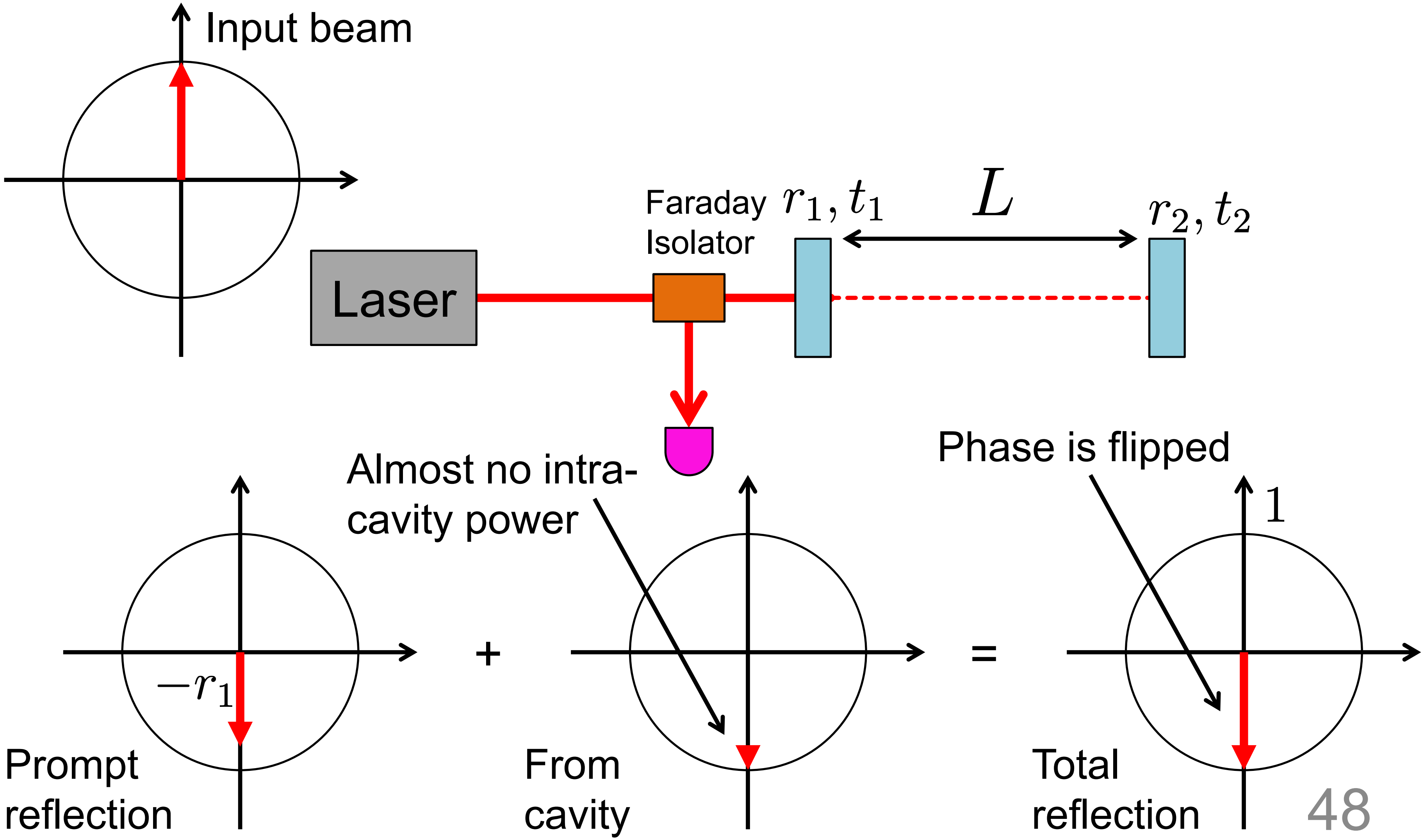
From cavity

=

Total reflection

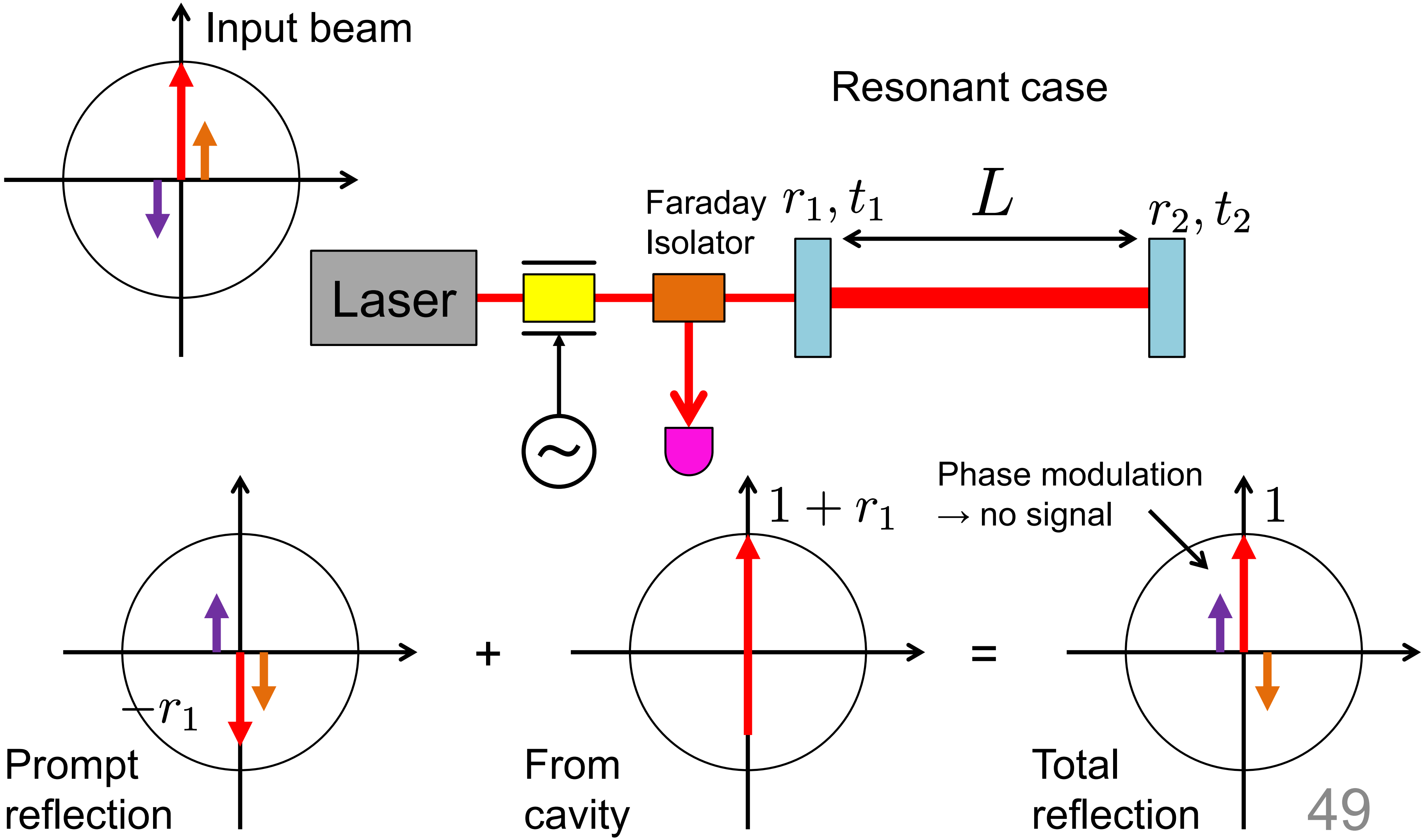
Fabry-Pérot Cavity and Phasor

- If $t_2 = 0$ and anti-resonance



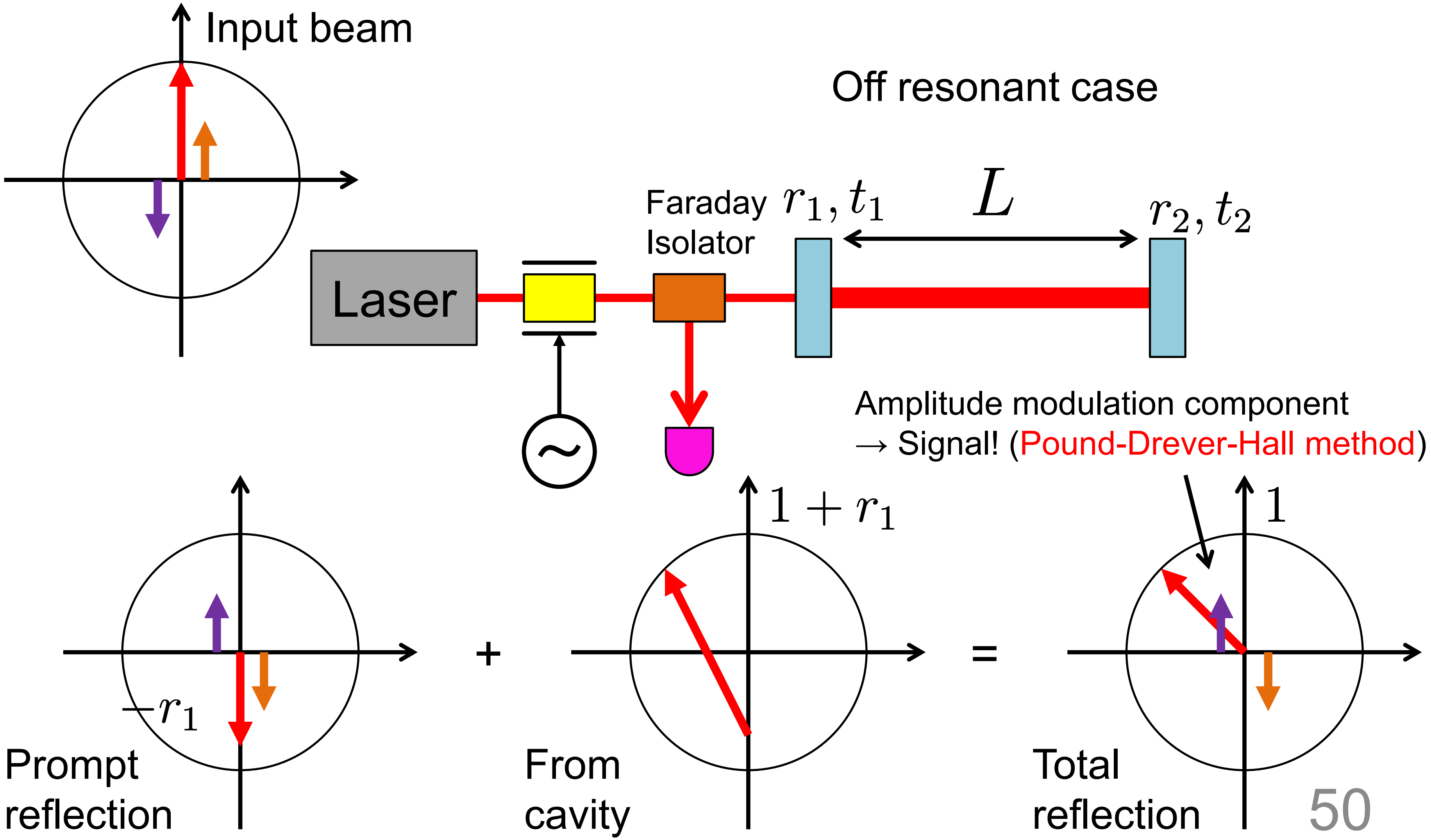
Fabry-Pérot Cavity with Sidebands

- Anti-resonant sidebands can be used as LOs



Fabry-Pérot Cavity with Sidebands

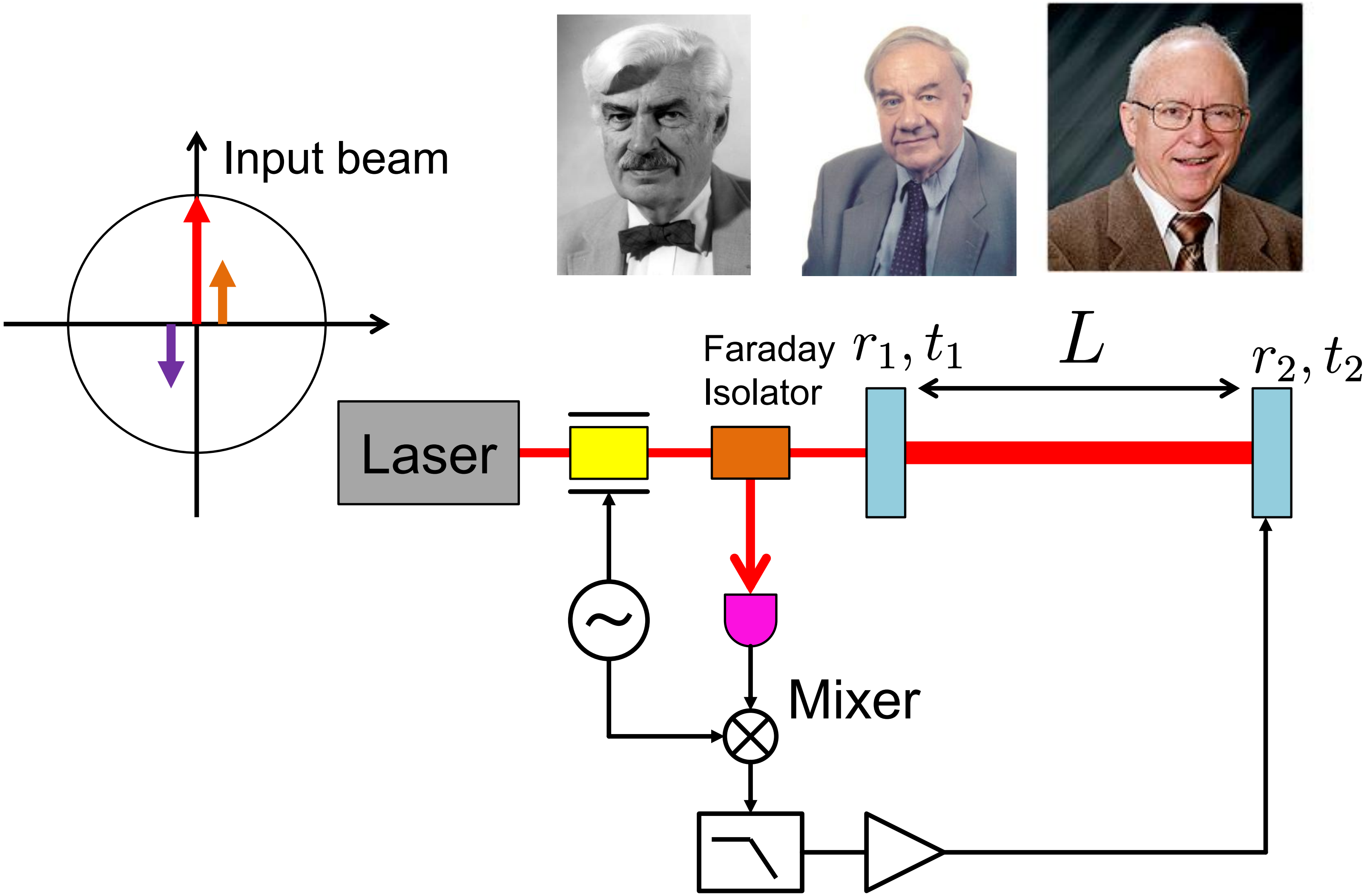
- Anti-resonant sidebands can be used as LOs



Off resonant case

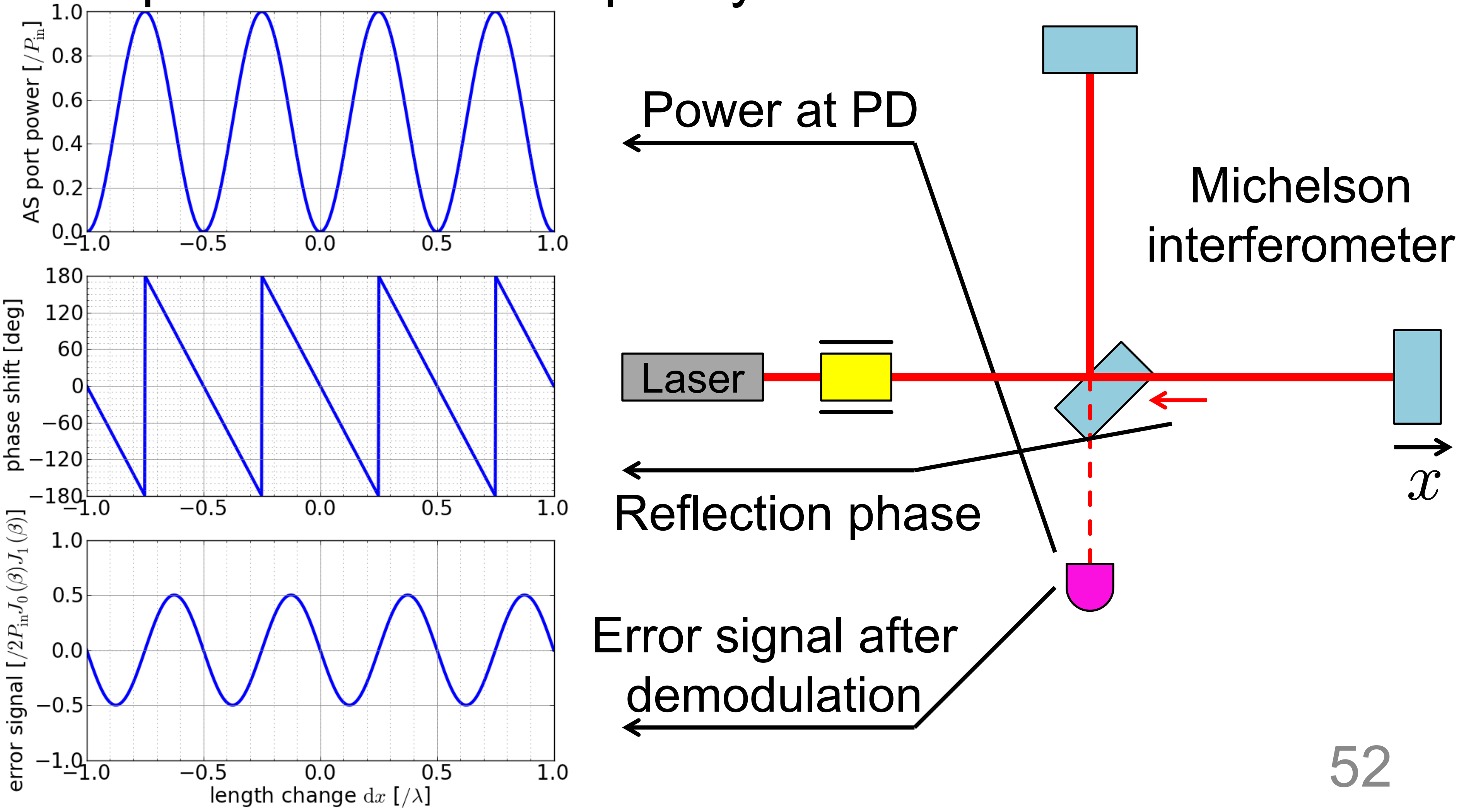
Amplitude modulation component
→ Signal! (Pound-Drever-Hall method)

Pound-Drever-Hall Method



Comparing Michelson and FP

- Both detect phase shift, but in Fabry-Pérot cavity, phase shift is steep only near the resonance



Power at PD

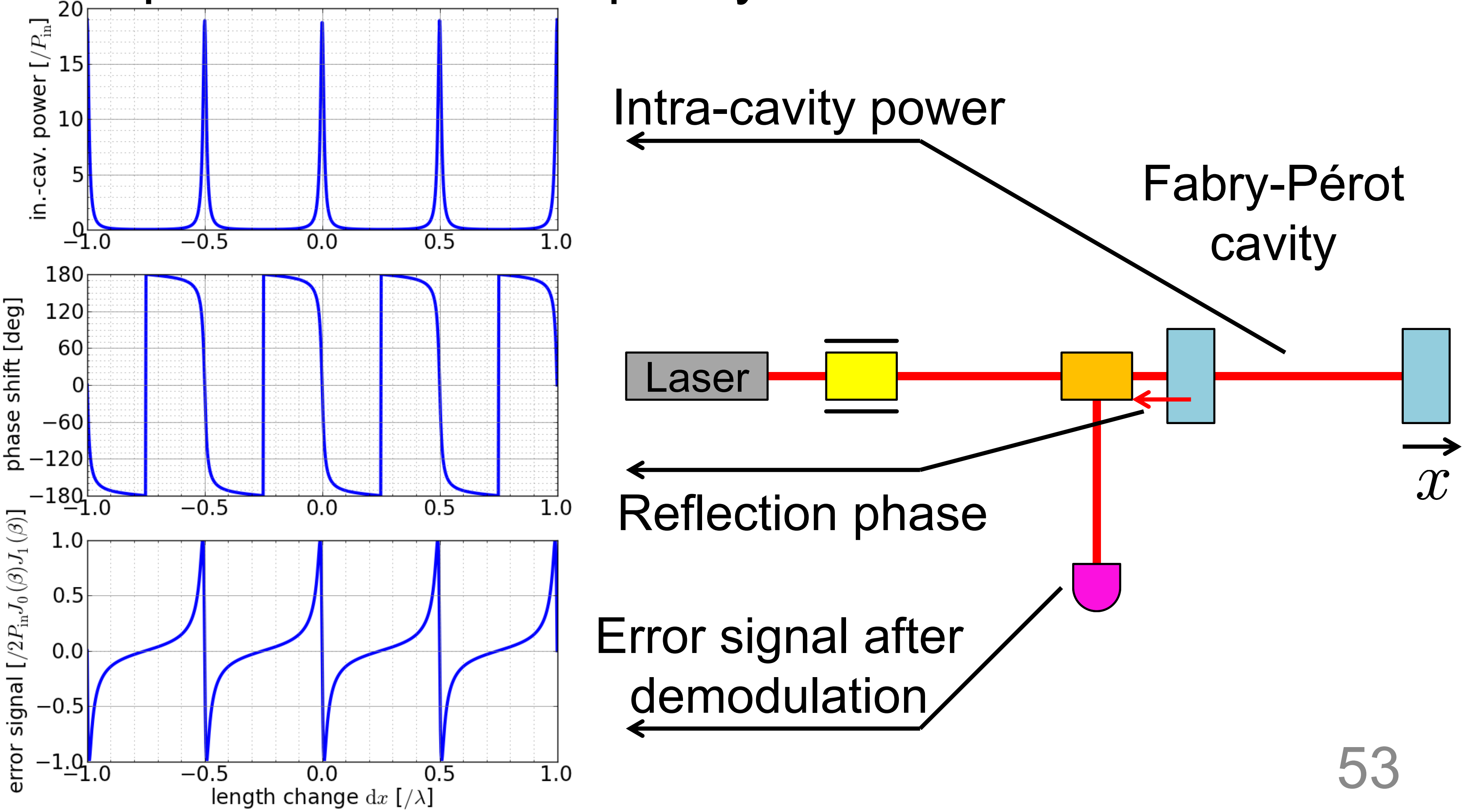
Michelson interferometer

Reflection phase

Error signal after demodulation

Comparing Michelson and FP

- Both detect phase shift, but in Fabry-Pérot cavity, phase shift is steep only near the resonance



Intra-cavity power

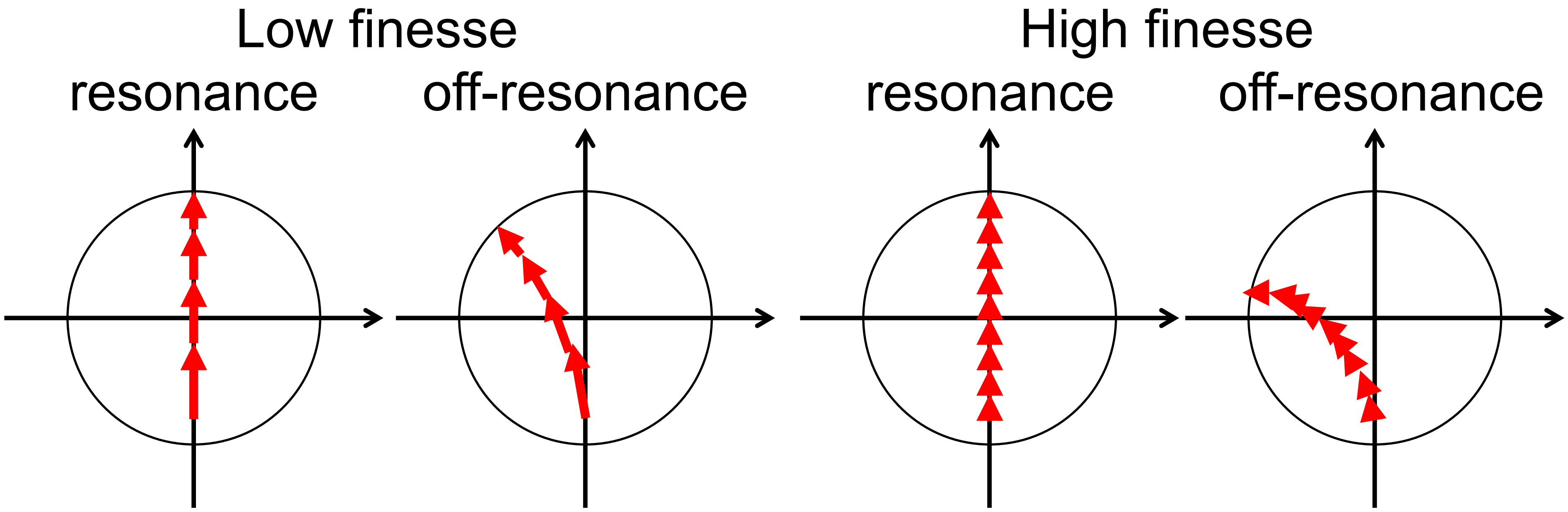
Fabry-Pérot cavity

Reflection phase

Error signal after demodulation

Finesse and Phase Shift

- Finesse is higher with lower transmission
- Higher the finesse, narrower the resonance and steeper the phase shift around the resonance



Low finesse

resonance off-resonance

High finesse

resonance off-resonance

Summary

- Gravitational waves are phase modulation and interferometers convert this into amplitude modulation signal
- Shot noise of Michelson interferometer is better at dark fringe
- Fabry-Pérot cavities are used to enhance the signal, at the cost of bandwidth
- Modulation-demodulation methods are used to extract the signal at Michelson dark fringe and on cavity resonance

Assignment for Oct 24

- What are the possible noise sources for laser interferometers? List up the most ridiculous noise sources you can imagine.
- You may also answer from the Google Form below https://forms.gle/6AwJ48XcpWQXqMon9

Don’t forget to put your name and student#

You may answer in any language