Hilbert Transform and its Application to Laser Interferometry

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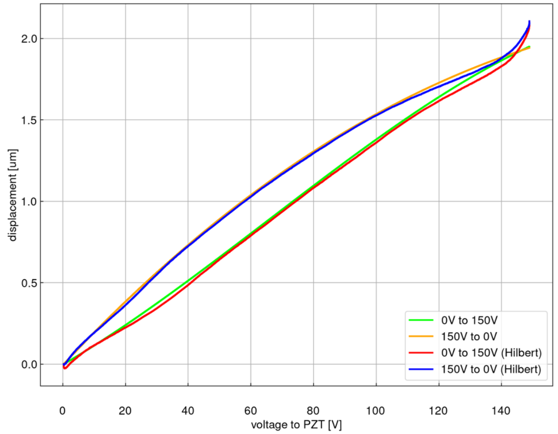
Overview

◆Hilbert Transform

- An analytical method to obtain the amplitude $A(t)$ and the phase $\theta(t)$ from a signal $x(t)$
- Usually used in offline analysis based on FFT
- Easy to implement (scipy.signal.hilbert)

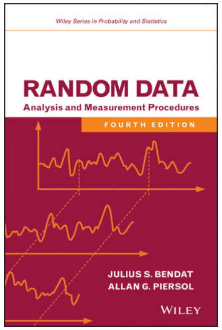


◆Application to laser interferometry

➢Analysis of the nonlinearity and the hysteresis of piezo actuators

☐ Reference

- "Random Data: Analysis and Measurement Procedures, 4th Edition"



Contents

- Introduction of Hilbert Transform
- Application to simulated chirp signal
- Application to analysis of nonlinearity and hysteresis of piezo actuators
- Comparison with quadrature phase interferometer
- Introduction of papers
- Summary

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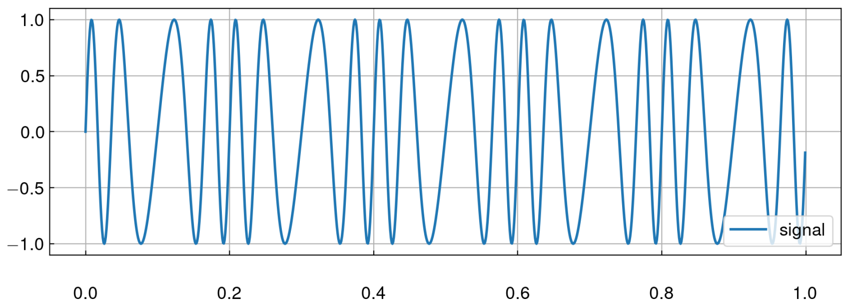
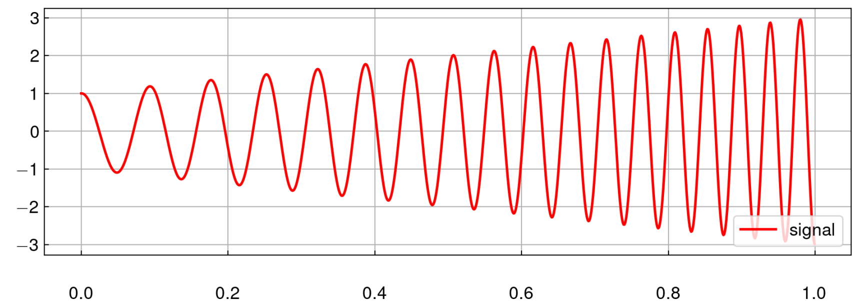
Motivation

- Consider a signal $x(t)$ with time-varying amplitude $A(t)$ and phase $\theta(t)$:

Phase-modulated signal



Chirp signal



- We want to know $A(t)$ and $\theta(t)$ from a given signal $x(t)$:

$$x(t) = A(t)\cos\theta(t)$$

$A(t)$: instantaneous amplitude

$\theta(t)$: instantaneous phase

$\frac{1}{2\pi}\frac{d\theta(t)}{dt}$: instantaneous frequency

How can we obtain $A(t)$ and $\theta(t)$ individually?

Analytic signal

- If we have the analytic signal $z(t)$ of $x(t)$, $A(t)$ and $\theta(t)$ can be obtained:

original signal (real function) → $x(t)$

Imaginary part (real function) → $\tilde{x}(t)$

$$z(t) = x(t) + i\tilde{x}(t) = A(t)\cos\theta(t) + iA(t)\sin\theta(t)$$

$$= A(t)e^{i\theta(t)}$$

Analytic signal of $x(t)$ (complex function)

$$\Rightarrow \begin{cases} A(t) = \sqrt{x^2(t) + \tilde{x}^2(t)} \\ \theta(t) = \tan^{-1}\left(\frac{\tilde{x}(t)}{x(t)}\right) \end{cases}$$

At this moment, analytic signal $z(t)$ has not been defined yet

How should we define $z(t)$?

(How can we obtain analytic signal $z(t)$ from $x(t)$?)

Construction of $z(t)$ from $x(t)$

- There are infinite ways to construct a complex function $z(t)$ from $x(t)$ satisfying $\mathrm{Re}[z(t)] = x(t)$:

Example) $x(t) = \cos\omega_0 t$

- $z(t) = e^{i\omega_0 t} \Rightarrow \begin{cases} A(t) = 1 \\ \theta(t) = \omega_0 t \end{cases}$
- $z(t) = \cos\omega_0 t + iB\sin\omega_0 t \Rightarrow \begin{cases} A(t) = \sqrt{\cos^2\omega_0 t + B^2\sin^2\omega_0 t} \\ \theta(t) = \tan^{-1}(B\tan\omega_0 t) \end{cases}$

Instantaneous amplitude $A(t)$ and phase $\theta(t)$ can change with how to construct $z(t)$

➡ What is the best definition of analytic signal $z(t)$?

Requirement for the definition of $z(t)$

- With the natural definition of the analytic signal $z(t)$, we want

$$x(t) = \cos \omega_0 t \;\mapsto\; z(t) = e^{i\omega_0 t}$$

$$\Rightarrow \begin{cases} A(t) = 1 \\ \theta(t) = \omega_0 t \end{cases}$$

➡ The natural definition of analytic signal $z(t)$ lies in Fourier transform

Definition of analytic signal $z(t)$

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} X(\omega) e^{i\omega t}$$: Inverse Fourier transform

$$x(t) = \int_0^{\infty} \frac{d\omega}{2\pi} \left[X(\omega)e^{i\omega t} + X(-\omega)e^{-i\omega t}\right]$$

$X(\omega) = X^*(-\omega)$

$$= \int_0^{\infty} \frac{d\omega}{2\pi} 2\mathrm{Re}\left[X(\omega)e^{i\omega t}\right]$$

$$= \mathrm{Re}\left[\int_0^{\infty} \frac{d\omega}{2\pi} 2X(\omega)e^{i\omega t}\right]$$

Considering $\mathrm{Re}[z(t)] = x(t)$, the definition of the analytic signal $z(t)$ of $x(t)$ should be

$$z(t) \equiv \int_0^{\infty} \frac{d\omega}{2\pi} 2X(\omega)e^{i\omega t}$$

Analytic signal of $x(t)$

Hilbert transform

- Let's calculate $z(t) = x(t) + i\tilde{x}(t)$ from its definition:

$$z(t) \equiv \int_0^{\infty} \frac{d\omega}{2\pi} 2X(\omega) e^{i\omega t}$$

$$= \int_0^{\infty} \frac{d\omega}{\pi} \int_{-\infty}^{\infty} dt'\, x(t') e^{-i\omega t'} e^{i\omega t}$$

$$\int_0^{\infty} d\omega\, e^{i\omega(t-t')} = \pi\delta(t-t') + \frac{i}{t-t'}$$

$$= \cdots = x(t) + i \int_{-\infty}^{\infty} dt' \frac{x(t')}{t-t'}$$

Convolution: $x(t) * 1/t$

$x(t)$: Original signal

$i \int_{-\infty}^{\infty} dt' \frac{x(t')}{t-t'} \equiv \tilde{x}(t)$: Hilbert transform of $x(t)$

➡ We can obtain analytic signal $z(t)$ by calculating Hilbert transform $\tilde{x}(t)$

($x(t) \mapsto z(t)$ is also called Hilbert transform in practice)

Practical derivation of analytic signal

- Calculating $\tilde{x}(t)$ is sometimes troublesome

$$z(t) = x(t) + i \underbrace{\int_{-\infty}^{\infty} dt' \frac{x(t')}{t-t'}}_{}$$

$\tilde{x}(t)$

- More practical method to obtain the analytic signal $z(t)$:
  Use inverse Fourier transform

$$z(t) \equiv \int_0^{\infty} \frac{d\omega}{2\pi} 2X(\omega) e^{i\omega t}$$

$$= \mathcal{F}^{-1}[X'(\omega)]$$

$$X'(\omega) \equiv \begin{cases} 2X(\omega) & (\omega > 0) \\ 0 & (\omega < 0) \end{cases}$$

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$X(\omega) \downarrow \begin{matrix} \times 2 \ (\omega > 0) \\ \times 0 \ (\omega < 0) \end{matrix} X'(\omega)$$

$$z(t) = x(t) + i\tilde{x}(t) \xleftarrow{\mathcal{F}^{-1}} X'(\omega)$$

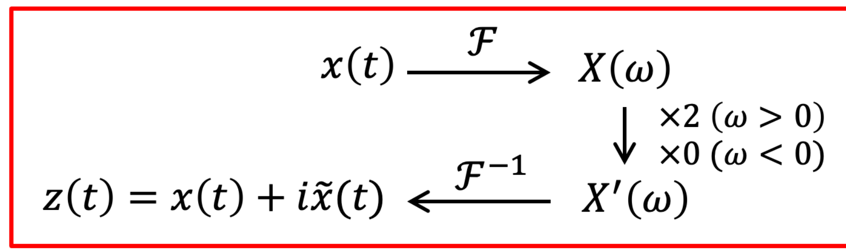
In practice, FFT are applied to finite time series data of $x(t)$

Summary of Hilbert transform

- The amplitude $A(t)$ and the phase $\theta(t)$ of $x(t)$ can be obtained by calculating the analytic signal $z(t)$ (Hilbert transform)

$$z(t) = x(t) + i\tilde{x}(t)$$
$$= A(t)e^{i\theta(t)}$$

Calculation process



$A(t) = \sqrt{x^2(t) + \tilde{x}^2(t)}$: instantaneous amplitude

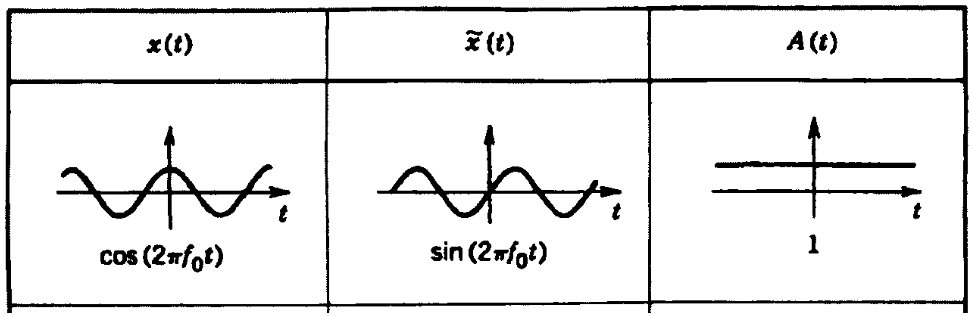
$\theta(t) = \tan^{-1}\left(\frac{\tilde{x}(t)}{x(t)}\right)$: instantaneous phase

$\frac{1}{2\pi}\frac{d\theta(t)}{dt}$: instantaneous frequency

Examples of Hilbert transform

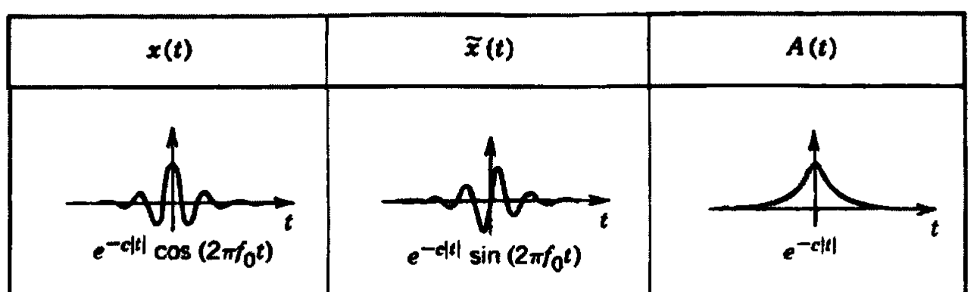
- Examples)
  - $x(t) = \cos\omega_0 t \;\Rightarrow\; z(t) = \cos\omega_0 t + i\sin\omega_0 t$

    $A(t) = 1,\; \theta(t) = \omega_0 t$



  - $x(t) = e^{-c|t|}\cos\omega_0 t \;\Rightarrow\; z(t) = e^{-c|t|}\cos\omega_0 t + i\, e^{-c|t|}\sin\omega_0 t$

    $A(t) = e^{-c|t|},\; \theta(t) = \omega_0 t$



(Quoted from “Random Data: Analysis and Measurement Procedures, 4th Edition”)

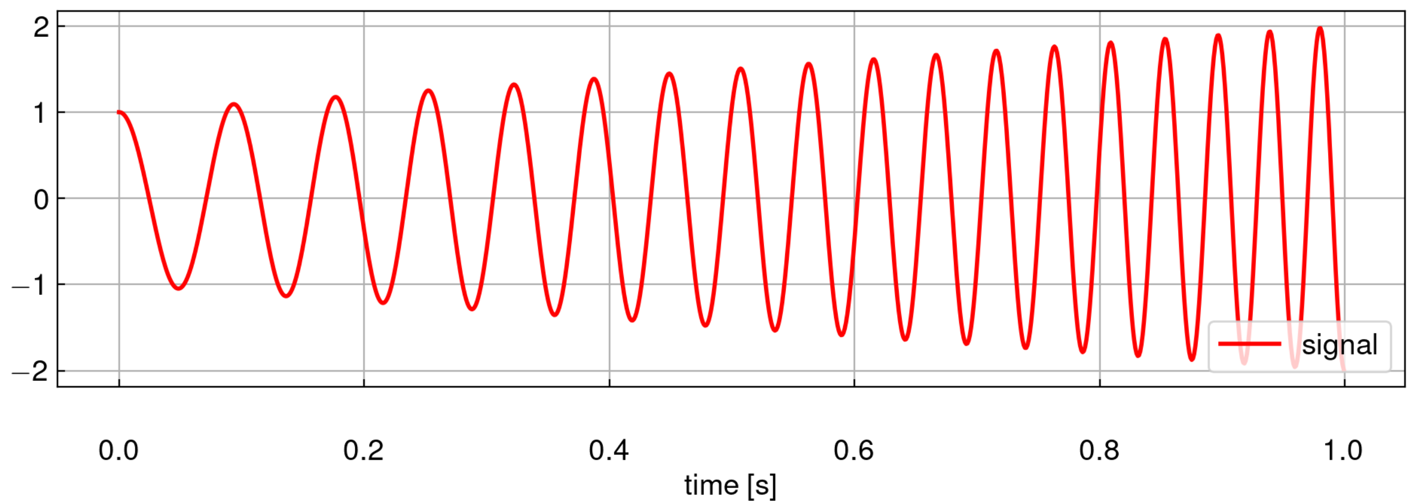
Contents

Application to simulated chirp signal

- Apply Hilbert transform to a simulated chirp signal

  - Chirp signal: Time duration: 1 s
    Frequency: 10 Hz ⇒ 25Hz in 1 s
    Amplitude: 1 ⇒ 2 in 1s

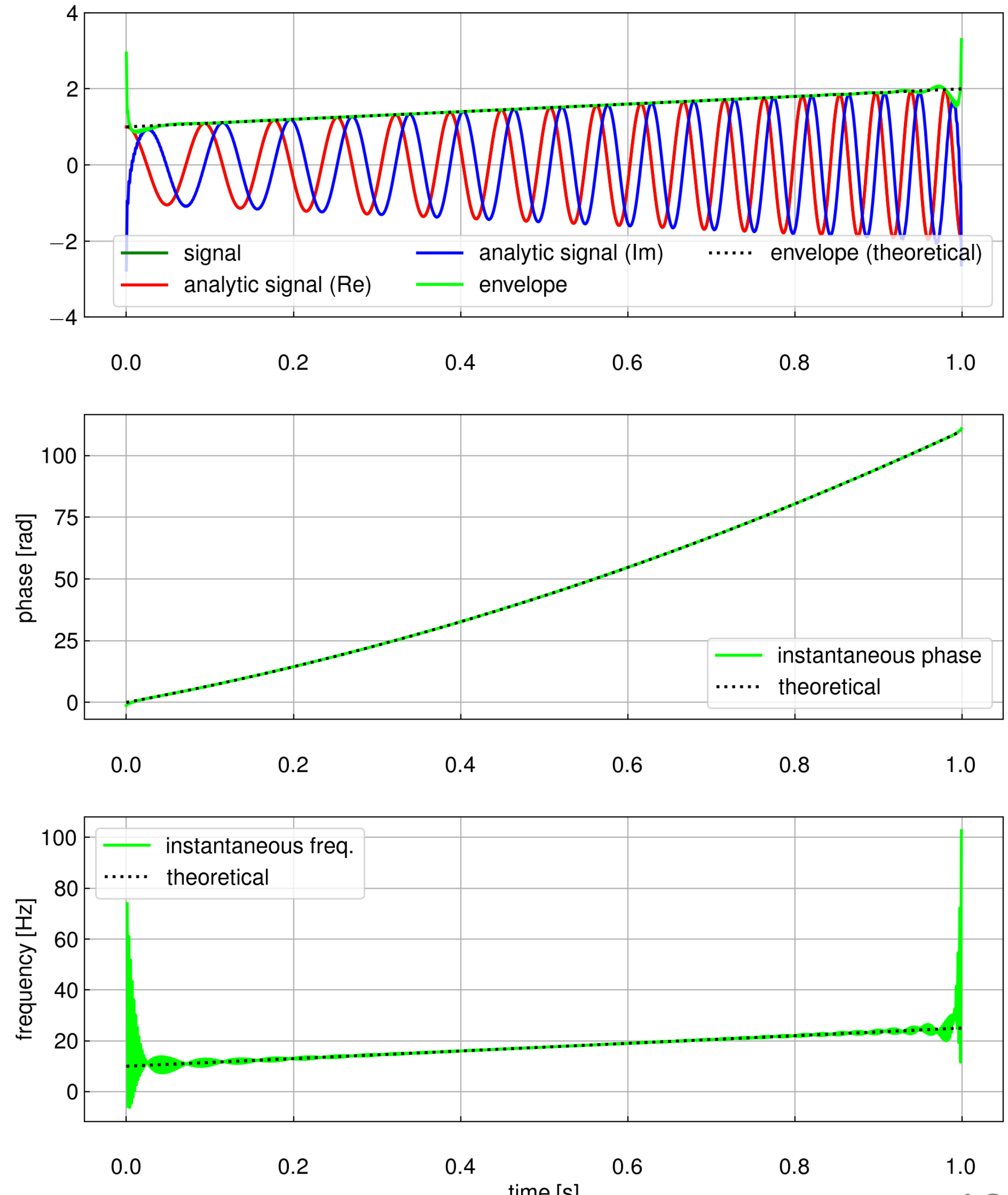
  - Function for Hilbert transform: [scipy.signal.Hilbert]()
    *There is no option for window functions



Result

- Obtained amplitude and phase are consistent with the theoretical curve without the edge of the data

- At the edge, Hilbert transform is not going well (maybe due to the edge effect in the FFT process)



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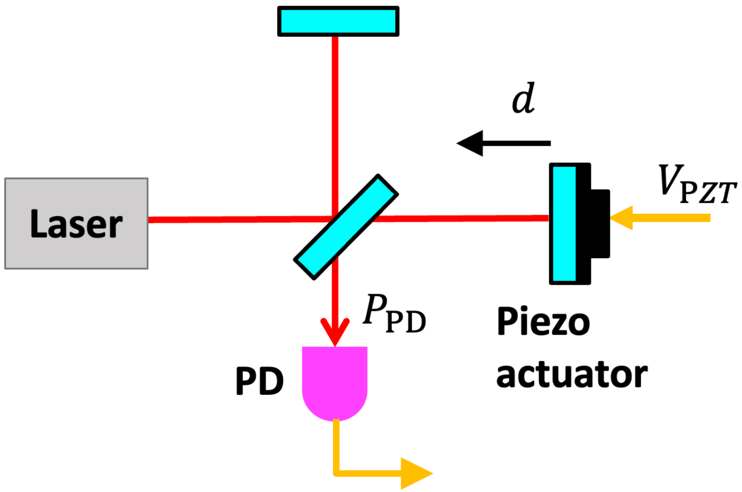
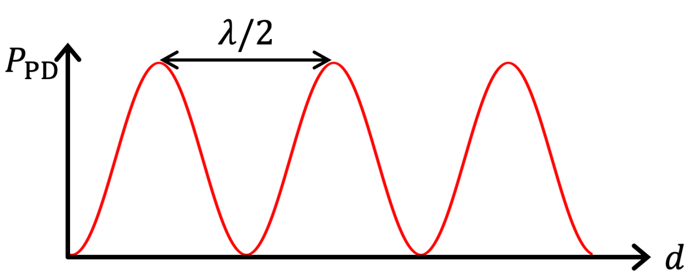
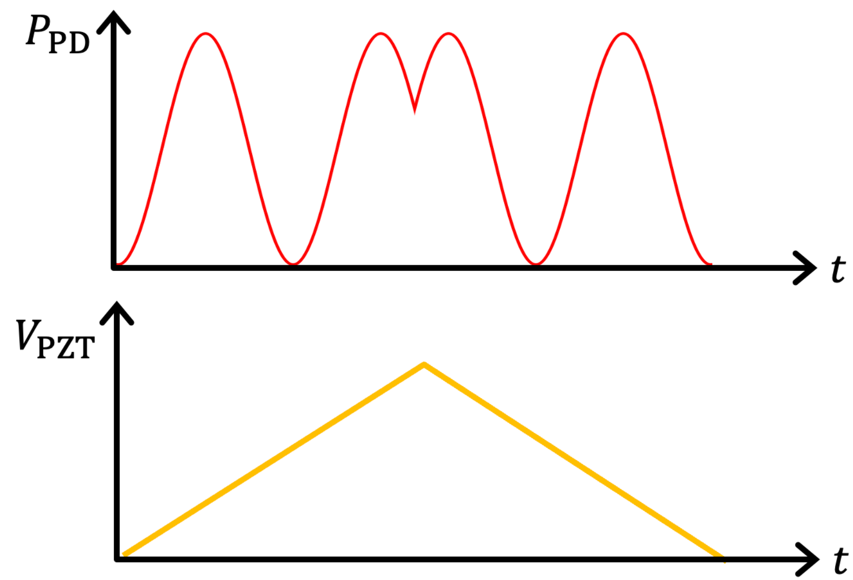
Michelson interferometer with piezo actuator





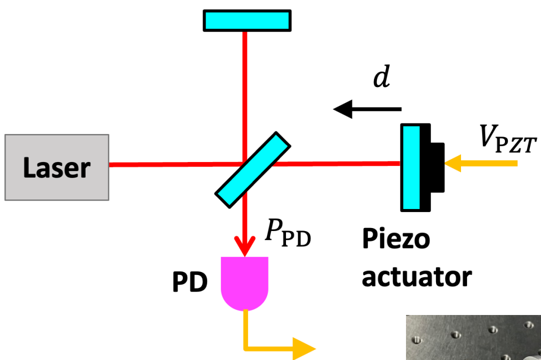
If $d = kV_{\mathrm{PZT}}$, V_{PZT} of triangular wave makes pure sine wave output:

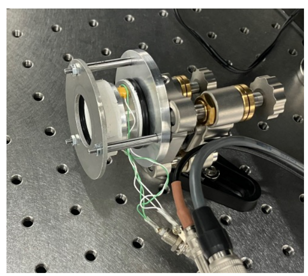


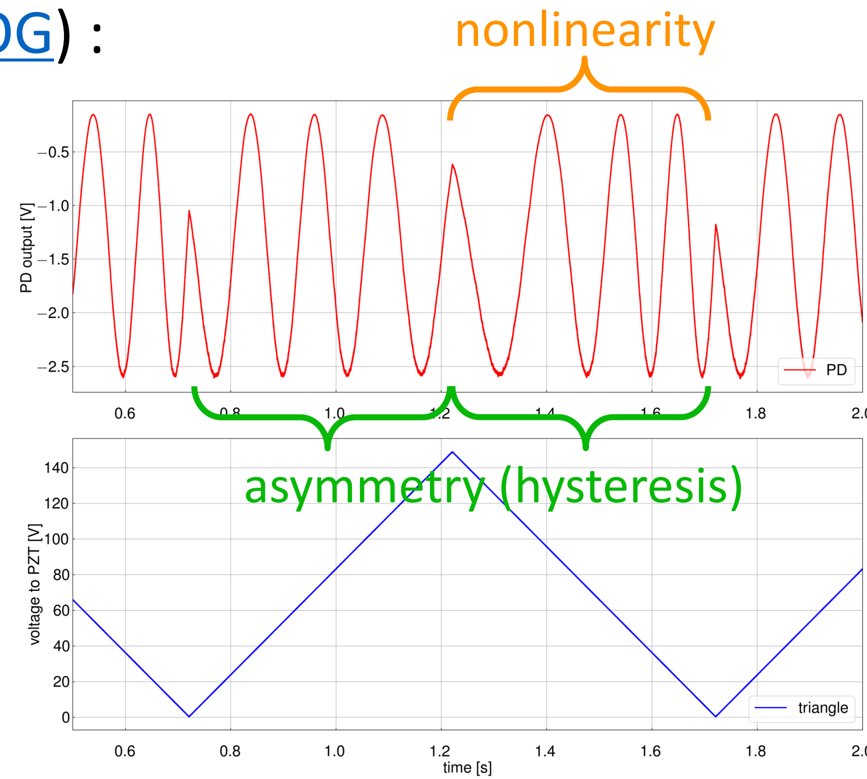
This is not true in reality

Nonlinearity and hysteresis of piezo actuator

- Actual fringes in reality (ELOG) :





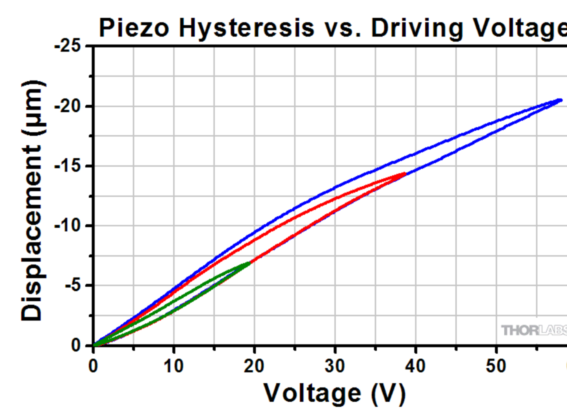


- Generally, piezo actuators have nonlinearity and hysteresis

  ➢Hysteresis model of piezo actuator

  -M. Riepold *et al.*: *Vibroengineering PROCEDIA*, Vol. 22, pp. 47–52, Mar. 2019

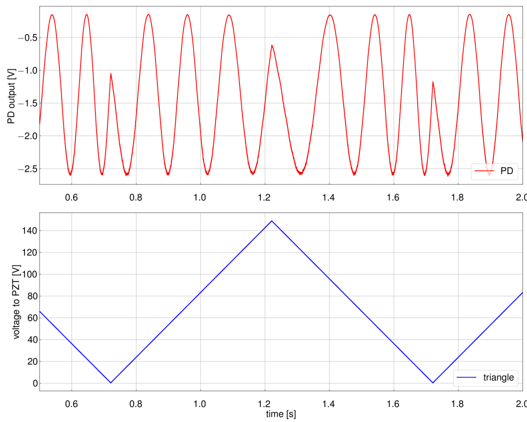
  -M. Altaher and S. S. Aphale: *Computers* **2018**, *7*, 10



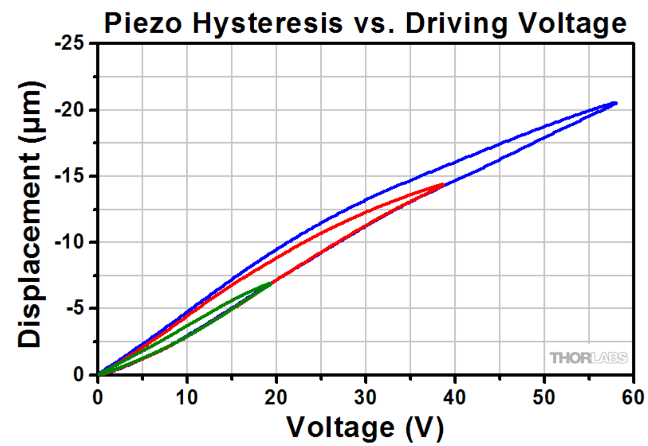
Quoted from Thorlabs documents on PZT

Measurement of hysteresis curve

- If we measure the phase $\theta(t)$ of the PD output, we can obtain its hysteresis curve:



Measure $\theta(t)$ →



- Two methods for the measurement of $\theta(t)$:

  1. Apply curve-fitting to the measured PD output
  2. Use Hilbert transform to the measured PD output

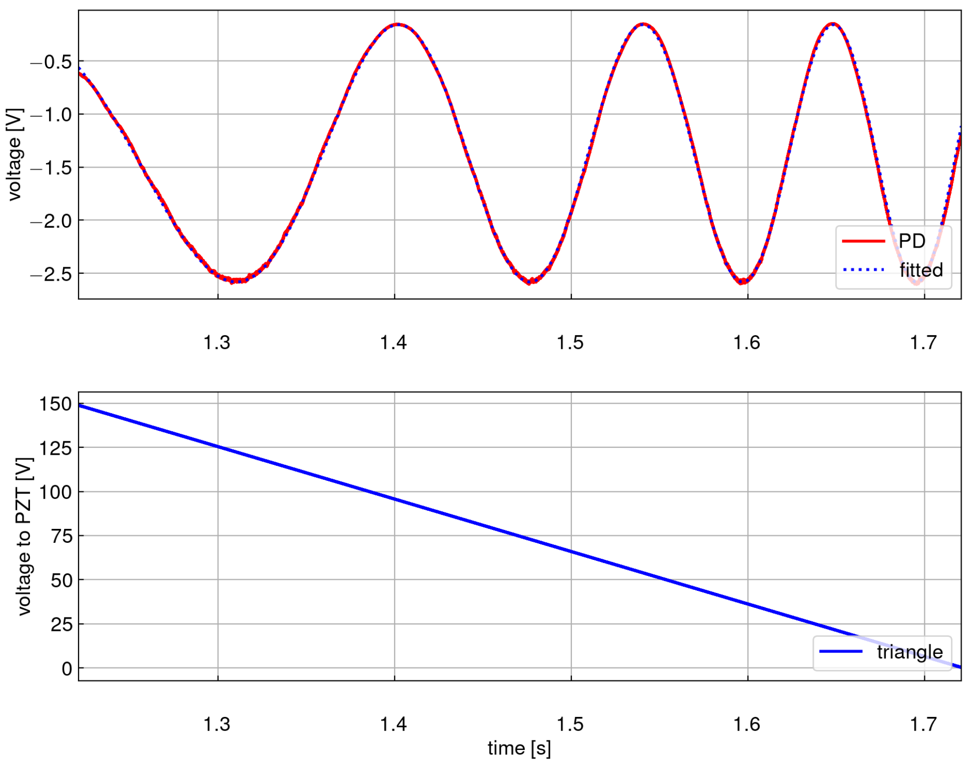
I tested the both methods and compared the results

1. Curve-fitting method

- Fitting model: sine wave with cubic polynomial phase

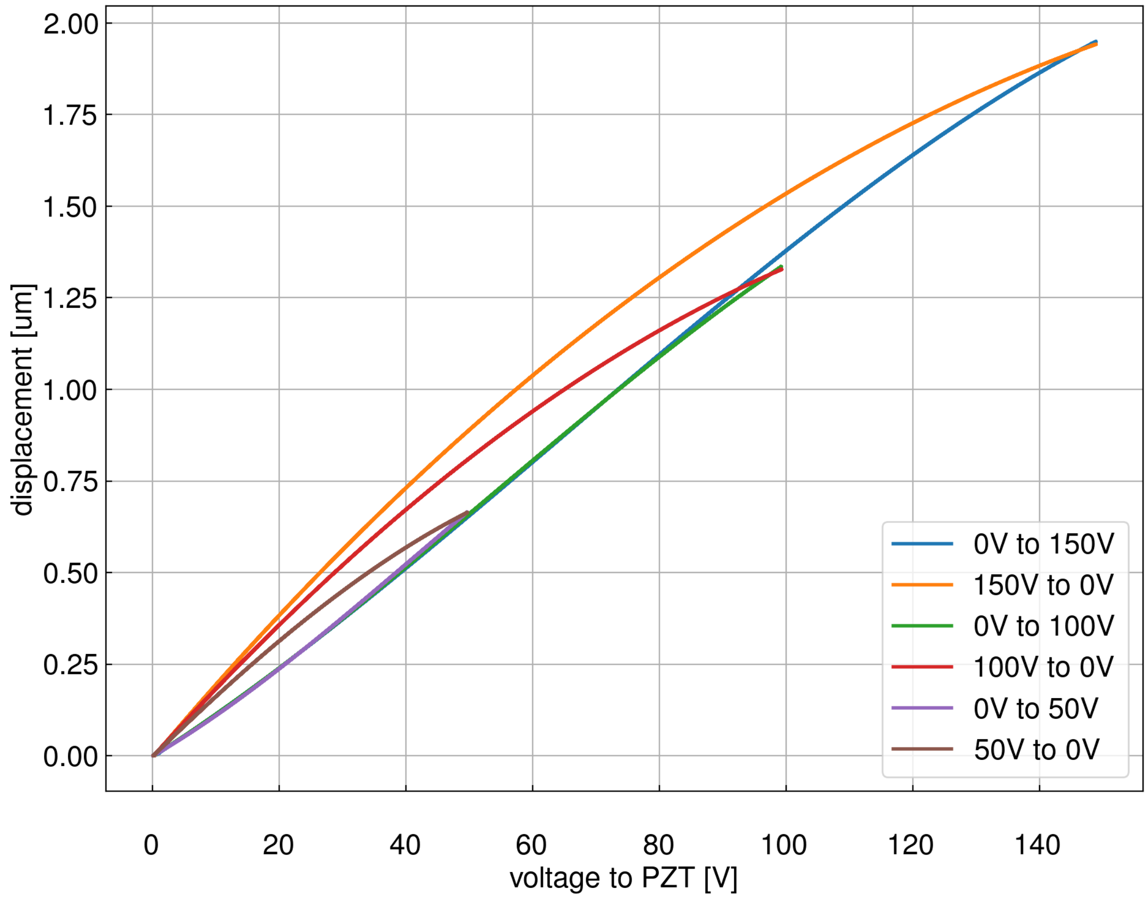
$$V(t) = V_{\mathrm{DC}} + V_0 \sin\theta(t)$$
$$\theta(t) = at^3 + bt^2 + ct + d$$

- Applied voltage: $0\ \mathrm{V} \leftrightarrow 50\ \mathrm{V}$, $0\ \mathrm{V} \leftrightarrow 100\ \mathrm{V}$, $0\ \mathrm{V} \leftrightarrow 150\ \mathrm{V}$,



1. Curve-fitting method: Result

- Hysteresis curves were obtained for three applied voltage ranges
- The edge of $(0\ \mathrm{V} \leftrightarrow 100\ \mathrm{V})$ curve is not consistent with other curves for some reason…



2. Hilbert transform method

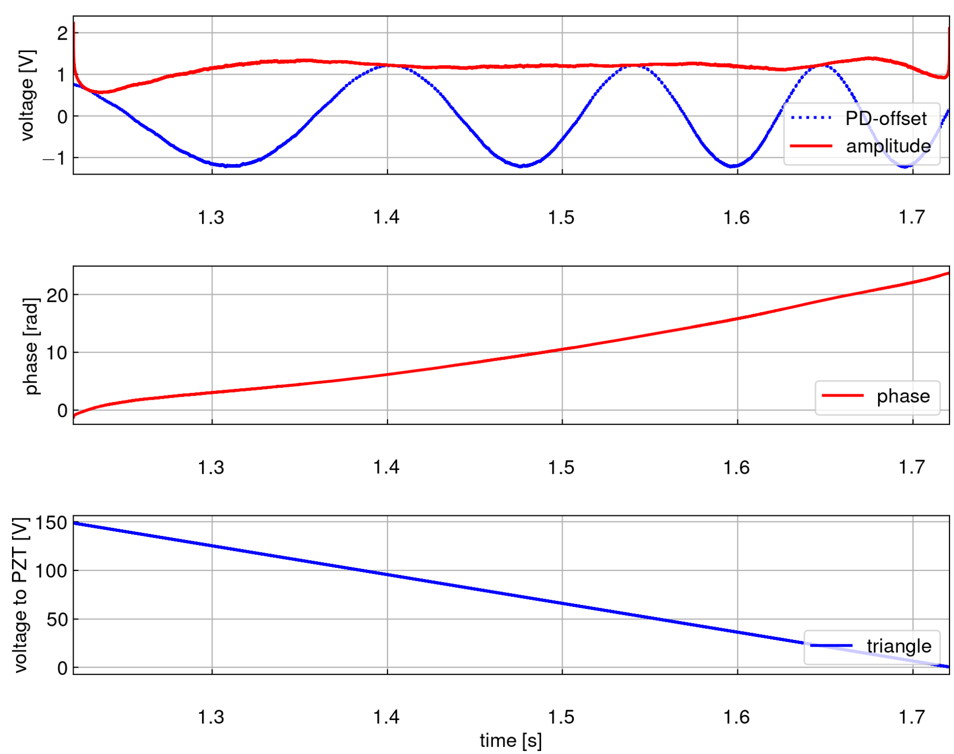
1. Subtract dc offset: $V'_{\mathrm{PD}} = V_{\mathrm{PD}} - \frac{V_{\mathrm{PD,max}} + V_{\mathrm{PD,min}}}{2}$

2. Apply Hilbert transform with scipy.signal.hilbert:

$$z(t) = \mathcal{H}[V'_{\mathrm{PD}}] = V'_{\mathrm{PD}} + i\widetilde{V}'_{\mathrm{PD}}$$

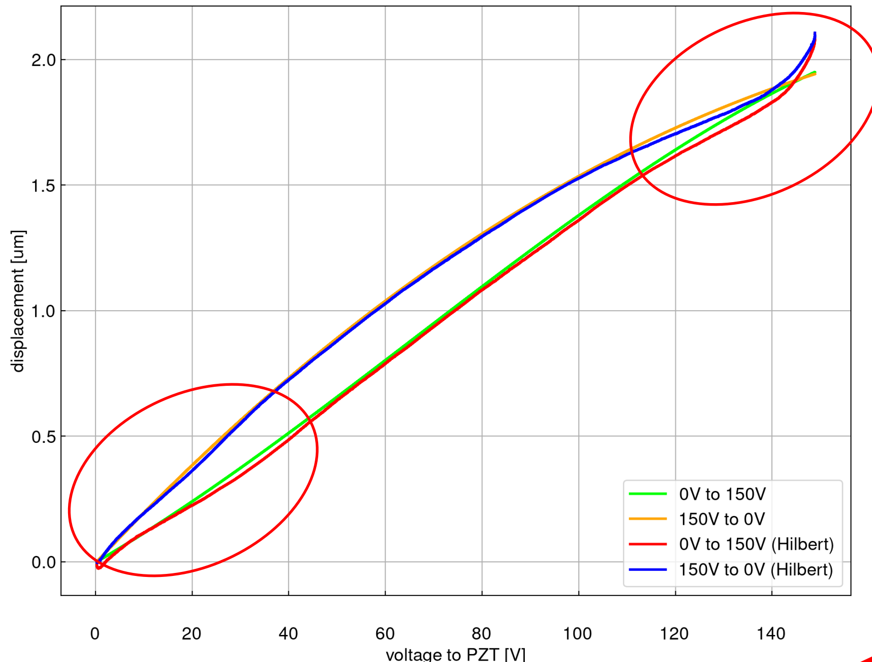
3. Obtain phase and amplitude: $\theta(t) = \tan^{-1}\left(\frac{\widetilde{V'}_{\mathrm{PD}}}{V'_{\mathrm{PD}}}\right), A(t) = \sqrt{V'^{2}_{\mathrm{PD}} + \tilde{V}'^{2}_{\mathrm{PD}}}$

4. Unwrap $\theta(t)$



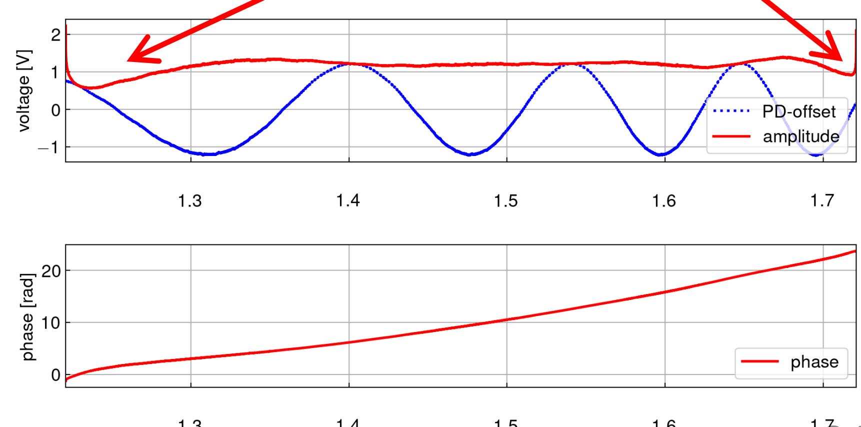
2. Hilbert transform method: Result

- Not so good especially around the edges compared to the curve-fitting method



We know that the amplitude is constant but Hilbert transform does not

- Reason for the inaccuracy (my guess):
  - ➢Hilbert transform can't distinguish whether the amplitude is decreasing or only the phase is changing around the edges



Summary: Hilbert transform vs. curve-fitting

- My feeling through this experiment:

	Hilbert transform	Curve-fitting
Accuracy	Bad around the edges of the data	Good
Difficulty in implementation	Very easy with a few lines in code	A little difficult to choose the initial value of fitting parameters

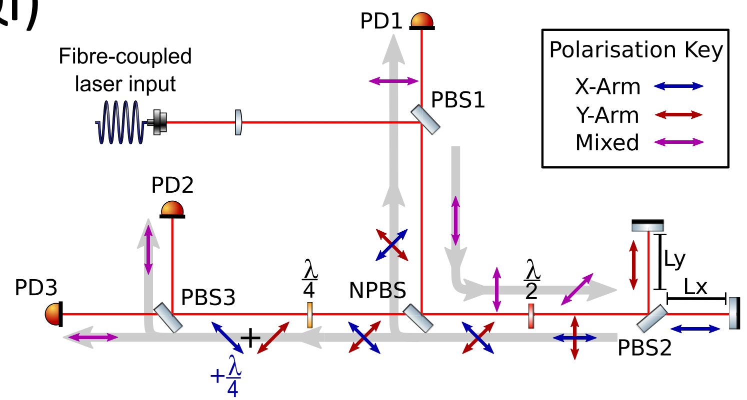
➡ Hilbert transform can be used for rough analysis or finding the initial values of fitting parameters

If you come up with good uses of Hilbert transform, please let me know!

Contents

Comparison with quadrature phase interferometer

- In laser interferometry, $A\cos\theta(t)$ and $A\sin\theta(t)$ can be measured simultaneously with quadrature phase interferometer (e.g., HoQI)



Optical layout of HoQI quoted from S. J. Cooper+, *Class. Quantum Grav.* **35** 095007 (2018)

	Hilbert transform	Quadrature phase interferometer
Difficulty in implementation	Very easy with a simple experimental setup and a few lines in code	A little complicated setup
Process	Usually offline	online
SNR	Bad around near-zero region. Because only $A\cos\theta(t)$ is measured.	Good

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Introduction of papers

- Papers on the applications of Hilbert transform to laser interferometry

◆Nonlinear frequency sweep in high-speed tunable laser

T.J. Ahn and D. Y. Kim: Appl. Opt. **46**, 2394-2400 (2007)

◆Vibration measurement for self-mixing interferometer

Z. Zhang *et al*.: Optics Communications **436**, 192-196 (2019)

Z. Zhang *et al*.: *Opt Rev* **27**, 90–97 (2020)

◆High-speed absolute distance measurement with frequency scanning interferometry

X. Li *et al*,: Appl. Opt. **61**, 3150-3155 (2022)

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Summary

◆Hilbert Transform

- A method to obtain the analytic signal $z(t) = x(t) + i\tilde{x}(t)$ $= A(t)e^{i\theta(t)}$ from a signal $x(t)$
- From $z(t)$, the amplitude $A(t)$ and the phase $\theta(t)$ can be obtained
- Usually used in offline analysis based on FFT
- Easy to implement (scipy.signal.hilbert)

◆Application to analyzing hysteresis curve of piezo actuators

➢It worked. But has inaccuracy around the edges of the data
➢Might be useful for instant rough analysis