Binary black holes (BBHs)

$1M_\odot \sim 1.5 \text{ km} \sim 5 \times 10^{-6} \text{ s}$
Gravitational waves from BBHs

Gravitational waveform

Inspiral: Post-Newtonian
Merger: Numerical Relativity
Ringdown: Black hole perturbation
Gravitational wave observations of BBHs

Various gravitational wave detectors and $f_{\text{merge}} \sim 200 \left( \frac{20 M_{\odot}}{M} \right) \text{Hz}$

(from http://www.ast.cam.ac.uk/~rhc26/sources/)
By Christopher Moore, Robert Cole and Christopher Berry from the Gravitational Wave Group at the Institute of Astronomy, University of Cambridge
Numerical INJection Analysis (NINJA): 1

NINJA-1: Numerical relativity (NR) and Data analysis (DA) communities

- 23 numerical waveforms (10 NR groups)
- Injected into Gaussian noise colored with the frequency sensitivity of first generation detectors
- Search and parameter-estimation (9 DA groups)

Aylott et al., Class. Quantum Grav. 26 (2009) 114008.
Aylott et al., Class. Quantum Grav. 26 (2009) 165008.

Major limitations
- No length or accuracy requirements for the NR waveforms
- No non-Gaussian noise transients
NINJA-2: for real science

- Use real noise through LSC/Virgo MOU
- More NR waveforms (13 NR groups)
- PN-NR stitching to cover mass down to $\sim 10M_\odot$.

Accuracy requirements

- 5 usable orbits before merger
- NR (2, 2) amplitude accuracy below 5%
- NR (2, 2) accumulated phase error $\leq 0.05$ radian
- GW stitching at $M\omega_{2,2} \leq 0.075$
- Hybrid GWs start at $M\omega_{2,2} \leq 0.006 \leftarrow 10M_\odot$ at 20Hz
- Highest PN order available for phase and amplitude

Only $\ell = |m| = 2$ required, but all harmonic modes welcome.
Numerical INJection Analysis (NINJA): 3

Aligned (anti-aligned) spinning (non-precessing) cases

$q$: mass ratio, $\chi_i$: dimensionless spins
Numerical INJection Analysis (NINJA): 4

Equal-mass, equal-spin waveforms \( (\chi = (\chi_1 + \chi_2)/2) \)

\( 10M_\odot, 100\text{Mpc} \)

Ajith et al., Class. Quantum Grav. 29 (2012) 124001.
Ajith et al., Class. Quantum Grav. 30 (2013) 199401.
In progress

- Real data from GW detectors
- Recolored data for the sensitivities expected from aLIGO/aVirgo in 2015-16.
- “blind injections”

https://www.ninja-project.org/doku.php?id=ninja2:home
A catalog of 171 high-quality binary black-hole simulations for gravitational-wave astronomy, Mroue et al., [arXiv:1304.6077]

\( \chi \): dimensionless spin, \((\theta, \phi)\): initial spin direction, 
\( q = m_1/m_2 \): mass ratio, \( N_{\text{orbits}} \): number of orbits before merger, 
\( e \): initial eccentricity
Recent progress: 2

A catalog of 171 high-quality binary black-hole simulations for gravitational-wave astronomy, Mroue et al., [arXiv:1304.6077]

(See also Pekowsky et al., [arXiv:1304.3176], over 220 waveforms)
Collaboration between Numerical and Analytical Relativity

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Group name</th>
</tr>
</thead>
<tbody>
<tr>
<td>JCP</td>
<td>Jena-Cardiff-Palma</td>
</tr>
<tr>
<td>FAU</td>
<td>Florida Atlantic University</td>
</tr>
<tr>
<td>GATech</td>
<td>Georgia Tech</td>
</tr>
<tr>
<td>RIT</td>
<td>Rochester Institute of Technology</td>
</tr>
<tr>
<td>Lean</td>
<td>Ulrich Sperhake</td>
</tr>
<tr>
<td>AEI</td>
<td>Albert Einstein Institute</td>
</tr>
<tr>
<td>PC</td>
<td>Palma-Caltech</td>
</tr>
<tr>
<td>SXS</td>
<td>Simulating eXtreme Spacetimes (Caltech, Cornell, CITA, CSU Fullerton)</td>
</tr>
<tr>
<td>UIUC</td>
<td>University of Illinois, Urbana-Champaign</td>
</tr>
</tbody>
</table>

Hinder et al., [arXiv:1307.5307]
Plan:

- **Stage 1:** Basic coverage \((q = 1, 2, 3, \text{mild spins})\)
- **Stage 2:** Additional precessing configurations
- **Stage 3:** More challenging configurations
  - Higher mass ratios \((q \gg 10)\)
  - Large spins \((|\chi| > 0.95)\)
  - Long \((> 40\text{ orbits})\)

Targets:

- \(\sim 20\) usable GW cycles
- Eccentricity: Aim for \(e < 0.002\) as conservative target
- Phase error \(\Delta \phi(t) < 0.25\text{radians up to } M\omega_{GW} = 0.2\) \((< 1\text{ orbit before merger})\)
- Relative amplitude error \(\Delta A/A < 0.01\)

First stage of the NRAR collaboration
(60 simulations, for spinning non-precessing configurations)
Numerical Relativity/Analytical Relativity (NRAR): 4

Analyzed configurations in Hinder et al., [arXiv:1307.5307]

| # | Group | Label | \( q \) | \( S_1/m_1^2 \) | \( S_2/m_2^2 \) | \( M_f/M \) | \( |\chi_f| \) |
|---|---|---|---|---|---|---|---|
| 1 | JCP | J1p+49+11 | 1 | \((-0.128, 0.171, 0.494)\) | \((0.129, -0.149, 0.106)\) | 0.941 | 0.774 |
| 2 | | J2−15+60 | 2 | \(-0.150\) | | 0.600 | 0.611 |
| 3 | FAU | F1p+30−30 | 1 | \((0.000, -0.520, 0.300)\) | \((0.520, 0.000, -0.300)\) | 0.952 | 0.704 |
| 4 | | F3+60+40 | 3 | 0.600 | | 0.400 | 0.958 |
| 5 | GATech | G1+60+60 | 1 | 0.603 | | 0.603 | 0.927 |
| 6 | | G2+15−60 | 2 | 0.150 | | -0.607 | 0.962 |
| 7 | | G2+30+00 | 2 | 0.301 | | 0.000 | 0.955 |
| 8 | | G2+60+60 | 2 | 0.601 | | 0.607 | 0.940 |
| 9 | RIT | R10 | 10 | 0.000 | 0.000 | 0.992 | 0.263 |
| 10 | Lean | L4* | 4 | 0.000 | 0.000 | 0.978 | 0.472 |
| 11 | | L3+60+00 | 3 | 0.600 | 0.000 | 0.957 | 0.792 |
| 12 | AEI | A1+30+00 | 1 | 0.300 | 0.000 | 0.947 | 0.732 |
| 13 | | A1+60+00 | 1 | 0.602 | 0.000 | 0.942 | 0.775 |
| 14 | PC | P1+80−40 | 1 | 0.802 | -0.400 | 0.945 | 0.744 |
| 15 | | P1+80+40 | 1 | 0.801 | 0.400 | 0.927 | 0.856 |
| 16 | SXS | S1+44+44* | 1 | 0.437 | 0.437 | 0.936 | 0.814 |
| 17 | | S1−44−44* | 1 | -0.438 | -0.438 | 0.961 | 0.548 |
| 18 | | S1+30+30 | 1 | 0.300 | 0.300 | 0.942 | 0.775 |
| 19 | | S2+30+30 | 2 | 0.300 | 0.300 | 0.953 | 0.734 |
| 20 | | S3+30+30 | 3 | 0.300 | 0.300 | 0.965 | 0.680 |
| 21 | | S3p+00−15 | 3 | 0.000 | (0.260, 0.005, -0.150) | 0.972 | 0.536 |
| 22 | | S1p+30+30 | 1 | (0.054, -0.514, 0.305) | (0.054, -0.514, 0.305) | 0.937 | 0.804 |
| 23 | | S1p−30−30 | 1 | (0.000, 0.520, -0.300) | (0.000, 0.520, -0.300) | 0.958 | 0.638 |
| 24 | | S3−60+00 | 3 | -0.599 | 0.000 | 0.978 | 0.271 |
| 25 | UIUC | U1+30+00 | 1 | 0.300 | 0.000 | 0.947 | 0.732 |

\[ q = \frac{m_1}{m_2}, \quad S_i/m_i^2: \text{ dimensionless spin}, \quad M = m_1 + m_2, \]
\[ M_f \text{ and } |\chi_f|: \text{ mass and dimensionless spin of the final BH} \]
Error estimates for the NR simulations

- Finite numerical resolution
- Waveform measurement at finite distance from the source
- Computation of $h$ from $\psi_4$
Spinning non-precessing template models

Comparison with time-domain SEOBNRv1 model

Spin-aligned EOBNR model.
Spinning non-precessing template models

Comparison with Frequency-domain phenomenological IMRPhenomB

Frequency domain (non-precessing spins) inspiral-merger-ringdown templates of Ajith et al. (2011).
Spinning non-precessing template models

Comparison with Frequency-domain phenomenological IMRPhenomC

Frequency domain (non-precessing spins) inspiral-merger-ringdown templates of with phenomenological coefficients defined in the Table II of Santamaria et al. (2010).
Non-spinning $q = 10$ waveform

Unfaithfulness of analytical non-spinning $q = 10$ waveforms generated by IMRPhenomB, IMRPhenomC and three versions of EOB models with the numerical non-spinning $q = 10$ waveform (R10).
Intermediate-mass-ratio BBHs

**Mass ratio:** \(1/10 \geq q \geq 1/100\)

"Full numerical simulations" and "Analytic treatments"

**NR:** challenge in the exploration of BBH parameter space

**AR:** Post-Newtonian approach \((v \ll 1)\)
  Effective one body approach
  Gravitational self-force \((q \ll 1)\) and so on.

**Simple as possible**

- Regge-Wheeler-Zerilli formalism (BHP) + remnant BH’s spin
- TaylorT4 orbital phase evolution (PN) + fitting parameters

We want to introduce the spin effect into the black hole perturbation approach.

BH ($M = m_1 + m_2$)
+ particle ($\mu = m_1 m_2 / M$)
Analytic, perturbative approach

Spin-Regge-Wheeler-Zerilli (SRWZ) formalism

- Extension of the RWZ for Schwarzschild perturbations.
- Include, perturbatively, a term linear in the remnant BH's spin.
- 2nd order perturbations.

\[
\Psi_{\ell m}(t, r) = \Psi_{\ell m}^{(1)}(t, r) + \Psi_{\ell m}^{(2)}(t, r) ,
\]

\[
\Psi_{\ell m}^{(o)} = \Psi_{\ell m}^{(o,1)}(t, r) + 2 \int dt \Psi_{\ell m}^{(o,Z,2)}(t, r) ,
\]

- \(\Psi_{\ell m}^{(1)}\), \(\Psi_{\ell m}^{(2)}\): Even parity Zerilli function
- \(\Psi_{\ell m}^{(o,1)}\): Odd parity Regge-Wheeler function
- \(\Psi_{\ell m}^{(o,Z,2)}\): Odd parity Zerilli function
Example (even parity wave equation)

\[ -\frac{\partial^2}{\partial t^2} \psi_{\ell m}(t, r) + \frac{\partial^2}{\partial r^*} \psi_{\ell m}(t, r) - V_{\ell}^{(even)}(r)\psi_{\ell m}(t, r) + im\alpha \hat{P}_{\ell}^{(even)}\psi_{\ell m}(t, r) \]

\[ = S_{\ell m}^{(even)}(t, r; r_p(t), \phi_p(t)), \]

\[ r^* = r + 2M \ln[r/(2M) - 1] \]

\[ \alpha: \text{Nondimensional spin parameter} \]

\[ V_{\ell}^{(even/odd)}: \text{Potentials} \]

\[ \hat{P}_{\ell}^{(even)}: \text{Differential operator} \]

\[ S_{\ell m}: \text{Source term (2nd order)} \]

- We need the particle’s trajectory \((r_p(t), \phi_p(t))\).
Based on TaylorT4 evolution [Boyle et al. (2007)]

\[
\frac{d\Omega}{dt} = \frac{96}{5} \Omega^{11/3} M^{5/3} \eta \left(1 + B \left(\frac{\Omega}{\Omega_0}\right)^{\beta/3}\right)^{-1} \left[1 + \left(-\frac{743}{336} - \frac{11}{4} \eta\right) \left(M\Omega\right)^{2/3} + 4 \pi M\Omega \right.
\]
\[
+ \left(\frac{34103}{18144} + \frac{13661}{2016} \eta + \frac{59}{18} \eta^2\right) \left(M\Omega\right)^{4/3} + \left(-\frac{4159}{672} \pi - \frac{189}{8} \eta \pi\right) \left(M\Omega\right)^{5/3}
\]
\[
+ \left(\frac{16447322263}{139708800} + \frac{16}{3} \pi^2 - \frac{1712}{105} \gamma - \frac{1712}{315} \ln(64 M\Omega) - \frac{56198689}{217728} \eta + \frac{451}{48} \eta \pi^2 + \frac{541}{896} \eta^2
\]
\[
- \frac{5605}{2592} \eta^3\right) \left(M\Omega\right)^2 + \left(-\frac{4415}{4032} \pi + \frac{358675}{6048} \eta \pi + \frac{91495}{1512} \eta^2 \pi\right) \left(M\Omega\right)^{7/3} + A \left(\frac{\Omega}{\Omega_0}\right)^{\alpha/3} \right]
,\]

\[
\Omega = \frac{d\phi}{dt} , \quad M = m_1 + m_2 , \quad \eta = \frac{m_1 m_2}{M^2} ,
\]

- **A, α, B and β:** **Fitting parameters**
- \(M\Omega_0 = (1/3)^{3/2} \sim 0.19\) at \(R_{\text{Sch}} = 3M\) for circular orbit.
- \(\alpha > 7\) and \(\beta > 7\) to be consistent with the 3.5PN formula.
Based on the ADM (Arnowitt, Deser and Misner)-TT PN (NR \sim ADM-TT/“trumpet” stationary 1 + log slice of Schwarzschild)

\[ R = \frac{M}{(M\Omega)^2/3} \left[ 1 + \left( -1 + \frac{1}{3} \eta \right) (M\Omega)^{2/3} + \left( -\frac{1}{4} + \frac{9}{8} \eta + \frac{1}{9} \eta^2 \right)(M\Omega)^{4/3} \right. \\
\left. + \left( -\frac{1}{4} - \frac{1625}{144} \eta + \frac{167}{192} \eta \pi^2 - \frac{3}{2} \eta^2 + \frac{2}{81} \eta^3 \right)(M\Omega)^2 \right] / (1+a_0(\Omega/\Omega_0)^{a_1}) + C , \]

- \( R \) and \( \Omega \): in the NR coordinates
- \( a_0, a_1, C \): **Fitting parameters**
- \( M\Omega_0 = (1/3)^{3/2} \sim 0.19 \)
- \( a_1 > 2 \) to be consistent with the 3PN calculation.

\( \textbf{C} \)

Inconsistent with the ADM-TT PN formula. → But, we need!
Wave calculation in the SRWZ formalism

Radial transformation to remove the offset $C$ between the NR and the “trumpet” coordinates by assuming $T_{NR} = T_{Log}$,

$$R_{NR} \rightarrow R_{Log} = R_{NR} - C.$$  

To the standard Schwarzschild coordinates,

$$(T_{Log}, R_{Log}) \rightarrow (T_{Sch}, R_{Sch}),$$

**Final plunge trajectory**: Plunging (Schwarzschild) orbit from a matching radius $R_M \sim 3M$ to the horizon $R = 2M$: Geodesic without the radiation reaction.

**The SRWZ waveforms** at a sufficiently distant location $R_{Obs}$:

$$\frac{R_{Obs}}{M} (h_+ - i h_\times) = \sum_{\ell m} \frac{\sqrt{(\ell - 1)\ell(\ell + 1)(\ell + 2)}}{2M} \left( \psi_{\ell m}^{(even)} - i \psi_{\ell m}^{(odd)} \right) - 2 Y_{\ell m}.$$
Extended TaylorT4 (Trajectory) + SRWZ formalism (Wave generation)
Wave extrapolation for NR waveforms

Perturbative formula for NR waveforms

We extrapolate waveforms to $r \to \infty$,

$$
\lim_{r \to \infty} [r \psi_4^m(r, t)] = \left[ r \psi_4^m(r, t) - \frac{(\ell - 1)(\ell + 2)}{2} \int_0^t dt \psi_4^m(r, t) \right]_{r=r_{\text{Obs}}} + O(R_{\text{Obs}}^{-2}),
$$

$r_{\text{Obs}}$: Approximate areal radius of the sphere $R_{\text{Obs}}=\text{const.}$

- This formula gives reliable extrapolations for $R_{\text{Obs}} \gtrsim 100M$.
  (Numerical study by [M. C. Babiuc et al. (2011)])
- $\psi_4 \to h$: PYGWANALYSIS code
  [Reisswig and Pollney (2011)] in EINSTEINTOOLKIT
Results: Gravitational wave phase \((q = 1/10)\)

**NR vs. SRWZ waveforms**

\[ q = 1/10, \phi = 0 \text{ at } t = -830M. \]

\((\ell = 2, m = 2)\)

\((\ell = 2, m = 1)\)

\((\ell = 3, m = 3)\)
Results: Gravitational wave phase \((q = 1/15)\)

**NR vs. SRWZ waveforms**

\(q = 1/15, \phi = 0\) at \(t = -600M\).

\((\ell = 2, m = 2)\)

\((\ell = 2, m = 1)\)

\((\ell = 3, m = 3)\)
Results: Gravitational wave phase \( (q = 1/100) \)

**NR vs. SRWZ waveforms**

\( q = 1/100, \phi = 0 \) at \( t = -85M \).

\((\ell = 2, \, m = 2)\)

\((\ell = 3, \, m = 3)\)
Match between the NR and SRWZ ($\ell = 2$, $m = 2$) GWs in aLIGO (Zero Det, High Power). (Integration from $f_{\text{low}} \sim 10\text{Hz}$.)

<table>
<thead>
<tr>
<th></th>
<th>$q = 1/10$</th>
<th>$q = 1/15$</th>
<th>$q = 1/100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range ($M\Omega_{22}$)</td>
<td>$\geq 0.075$</td>
<td>$\geq 0.09$</td>
<td>$\geq 0.15$</td>
</tr>
<tr>
<td>Total mass ($M_{\odot}$)</td>
<td>242</td>
<td>290</td>
<td>484</td>
</tr>
<tr>
<td>$\mathcal{M}_{22}$</td>
<td>0.994669</td>
<td>0.996039</td>
<td>0.995477</td>
</tr>
</tbody>
</table>

Next plan: Longer NR simulations, Various $q$ cases

→ Fitting parameters in fitting functions for the trajectory

$$(A, \alpha, B, \beta, a_0, a_1, C) = c_0 \eta^{c_1}.$$
Gravitational waves from binary black holes

- Total mass $\rightarrow$ Frequency $\rightarrow$ Detectors
- Analytical Relativity – Numerical Relativity – Data Analysis
- $d\Omega/ dt(\Omega)$ for $q = 1/10$
- Up to $M\Omega = 0.175$:
  (NR orbital frequency around $R_{\text{Sch}} = 3M$)
- End point of the NR curve:
  $R_{\text{Sch}} = 2M$

<table>
<thead>
<tr>
<th>Mass-ratio</th>
<th>$A$</th>
<th>$\alpha$</th>
<th>$B$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = 1/10$</td>
<td>17.0500</td>
<td>7.21975 ($&gt;7$)</td>
<td>8.18920</td>
<td>12.5197 ($&gt;7$)</td>
</tr>
<tr>
<td>$q = 1/15$</td>
<td>26.0150</td>
<td>7.54047 ($&gt;7$)</td>
<td>8.65525</td>
<td>13.6168 ($&gt;7$)</td>
</tr>
<tr>
<td>$q = 1/100$</td>
<td>93.0650</td>
<td>4.32071 ($&lt;7$)</td>
<td>5.42457</td>
<td>14.9711 ($&gt;7$)</td>
</tr>
</tbody>
</table>
suppl.: Fitting for orbital radius

- $R(\Omega)$ for $q = 1/10$.
- Up to $M\Omega = 0.175$ ($R_{\text{Sch}} = 3M$)
- End point of the NR curve: $R_{\text{Sch}} = 2M$

<table>
<thead>
<tr>
<th>Mass-ratio</th>
<th>$C$</th>
<th>$a_0$</th>
<th>$a_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = 1/10$</td>
<td>$0.216953 \neq 0$</td>
<td>$0.513214$</td>
<td>$4.68472 (&gt; 2)$</td>
</tr>
<tr>
<td>$q = 1/15$</td>
<td>$0.237427 \neq 0$</td>
<td>$0.600321$</td>
<td>$4.57899 (&gt; 2)$</td>
</tr>
<tr>
<td>$q = 1/100$</td>
<td>$0.198137 \neq 0$</td>
<td>$0.923360$</td>
<td>$5.29681 (&gt; 2)$</td>
</tr>
</tbody>
</table>

Black: NR, Red: Fitting, Blue: PN