Gravitational-Wave Background as a Probe of the Primordial Black-Hole Abundance

Ryo Saito\textsuperscript{1,2} and Jun’ichi Yokoyama\textsuperscript{2,3}

\textsuperscript{1}Department of Physics, Graduate School of Science, The University of Tokyo, Tokyo 113-0033, Japan
\textsuperscript{2}Research Center for the Early Universe (RESCEU), Graduate School of Science, The University of Tokyo, Tokyo 113-0033, Japan
\textsuperscript{3}Institute for the Physics and Mathematics of the Universe, The University of Tokyo, Chiba 277-8568, Japan

(Received 25 December 2008; published 23 April 2009)

The formation of a significant number of black holes (PBHs) is realized if and only if primordial density fluctuations have a large amplitude, which means that tensor perturbations generated from these scalar perturbations as a second-order effect are also large and comparable to the observational data. We show that pulsar timing data essentially rule out PBHs with $10^2 - 10^4 M_\odot$, which were previously considered as a candidate of intermediate-mass black holes, and that PBHs with a mass range of $10^{20} \text{ to } 10^{26}$ g, which serves as a candidate of dark matter, may be probed by future space-based laser interferometers and atomic interferometers.

DOI: 10.1103/PhysRevLett.102.161101

Primordial black holes (PBHs) are produced when density fluctuations with large amplitudes enter the horizon in the radiation dominated stage of the early Universe with their typical mass given by the horizon mass at that epoch [1,2]. PBHs with a mass smaller than $10^{15}$ g would have been evaporated away by now due to Hawking radiation [3]. The abundance of these light holes has been constrained by big-bang nucleosynthesis [4], gamma-ray background [5], etc.

Heavier PBHs, on the other hand, can play some astrophysical roles today. For example, they may serve as an origin of the intermediate-mass black holes (IMBHs), which are considered to be the observed ultraluminous x-ray sources, if their mass and abundance lie in the range $M_{\text{PBH}} \sim 10^2 M_\odot - 10^4 M_\odot$ and $\Omega_{\text{PBH}} h^2 \sim 10^{-5} - 10^{-2}$, respectively [6]. PBHs with mass $M_{\text{PBH}} \sim 10^{20} - 10^{26}$ g ($10^{-13} M_\odot - 10^{-7} M_\odot$) [2,7] and the abundance $\Omega_{\text{PBH}} h^2 = 0.1$ [8] can provide an astrophysical origin of dark matter (DM) which satisfies the constraint imposed by gravitational lensing experiments [9,10].

The formation of the relevant number of PBHs on a specific mass scale is realized if the power spectrum of primordial density fluctuations has a peak with amplitude $10^{-2} - 10^{-1}$ on the corresponding scales (see [11,12] for inflation models to realize such spectra). In such a situation the second-order effects are expected to play an important role. For example, they generate non-Gaussianity in the statistical distribution of density fluctuation, and the amount of PBH production could be modified [13]. Such an effect was recently investigated in single-field inflation models, but it turned out that the non-Gaussian effect is negligibly small [14], justifying previous analysis assuming Gaussianity [15].

Second-order effects also generate tensor fluctuations to produce stochastic background of gravitational waves (GWs) from scalar-tensor mode coupling [16,17]. Their amplitude may well exceed the first-order tensor perturbation generated by quantum effect during inflation [18] in the current setup since the amplitude of density fluctuations required for a substantial density of PBHs is so large. Furthermore, the amplitude is expected to exceed that of GWs generated during the PBH collapses since smaller amplitude of density fluctuations suffices to produce the second-order GWs with a relevant amplitude, compared to those necessary for the formation of PBHs.

In this Letter, we show the GWs induced by scalar fluctuations as a second-order effect [16,17] are a useful probe to investigate the abundance of the PBHs. We calculate the spectrum of these second-order GWs in the case that scalar fluctuations have a sufficiently large peak to realize the formation of appreciable numbers of PBHs. As a natural consequence, we find that the spectrum of GWs has a peak on a scale approximately equal to the scale of the peak of the scalar fluctuations. We can therefore obtain information on the abundance of PBHs with the horizon mass when the peak of the peak entered the Hubble radius by observing GWs with the frequency corresponding to the same comoving scale, namely, $10^{-9} - 10^{-5}$ Hz for the IMBHs produced primordially and $10^{-3} - 10^{-1}$ Hz for the dark-matter PBHs. Fortunately, the former band is probed by the pulsar timing observations [19,20] while the latter band can be observed in the future by space-based laser interferometers [21–23] as well as atomic gravitational-wave interferometric sensors (AGISs) [24] for the dark-matter PBHs.

We write the perturbed metric as

$$ds^2 = a(\eta)^2[-e^{2\Phi}d\eta^2 + e^{-2\Psi}(\delta_{ij} + h_{ij})dx^idx^j],$$

including both scalar perturbations, $\Phi$ and $\Psi$, and tensor perturbation, $h_{ij}$, which satisfies $\delta_ih^i_j = h^i_i = 0$ with $h^i_j = \delta^i_k h_{kj}$. We assume the lowest-order tensor perturbations are negligible and incorporate only those generated by the scalar mode as a second-order effect. The relevant part of the second-order Einstein equation therefore reads

$$h^\nu_j + 2\mathcal{H}h^\nu_j - \dot{\delta}^2h^\nu_j = 2P^\mu_{rj}S^r_\nu,$$

PACS numbers: 04.30.Db, 04.70.-s, 98.70.Vc, 98.80.–k
where a prime denotes differentiation with respect to the conformal time, \( \eta \), \( \mathcal{P}^{ij}_s \) represents the projection operator to the transverse, traceless part, and \( \mathcal{H} \equiv a'/a \) \cite{16,17}.

Here, the source term reads

\[
S'_s = 2\alpha'(\Psi_s\partial_s\Psi - \frac{4}{3(1 + w)}\alpha'(\Psi + \mathcal{H}^{-1}\Psi'))
\times \partial_s(\Psi + \mathcal{H}^{-1}\Psi'),
\]

with \( w \equiv p/p \) being the equation-of-state parameter of the background fluid. In practice, only the radiation dominated era is relevant, so we take \( w = 1/3 \) hereafter. We also neglect anisotropic stress, which is expected to give only a small correction \cite{17}, and set \( \Phi = \Psi \) at linear order. Note the source term is second-order with respect to the scalar perturbations and absent at linear order. In order to calculate the induced GWs up to second order, therefore, it is sufficient to use the linear scalar modes. Hence, we only need to solve the linear evolution equation \cite{25},

\[
\Psi'_k(\eta) + \frac{4}{\eta}\Psi'_k(\eta) + \frac{k^2}{3}\Psi_k(\eta) = 0,
\]

for the scalar modes, where \( \Psi_k \) represents a Fourier mode of \( \Psi \). Its nondecaying solution is given by \( \Psi_k(\eta) = D_k(\eta)\Psi_k(0) \) with the transfer function

\[
D_k(\eta) = \frac{3}{(k\eta)^3}\left[\sqrt{3}\frac{k\eta}{k\eta} - \cos\left(\frac{k\eta}{\sqrt{3}}\right)\right].
\]

For our purpose we assume the form of the power spectrum of the initial fluctuations to be approximated by the Dirac delta function with respect to \( \ln(k) \),

\[
\mathcal{P}_\Psi(k) = \frac{k^3}{2\pi^2}(\ln(k/k_p))^2 = \mathcal{A}^2\delta_D(\ln(k/k_p)),
\]

where \( k_p \) and \( \mathcal{A}^2 \) represent the wave number of the peak and (amplitude)^2 \( \times \ln(\text{peak width}) \) of the original spectrum, respectively. With this power spectrum the fractional energy density of the region collapsing into PBHs at their formation time is estimated as

\[
\beta(M_{\text{PBH}}) \sim 0.1 \exp\left( -\frac{\Psi_c^2}{2\mathcal{A}^2} \right).
\]

We define the Fourier modes \( h_k \) by

\[
h_{ij}(x, \eta) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik \cdot x} \left[ h^+_{ij}(\eta)e^{ij}_{jk}(k) + h^\times_{ij}(\eta)e^{ij}_{jk}(k) \right],
\]

where \( e^{ij}_{jk}(k) \) are polarization tensors which are normalized as \( \sum_{ij}e^{ij}_{ij}(k)e^{ij}_{ij}(-k) = 2\delta^{\alpha\beta} \). The Fourier transform of the source term (3) is also defined similarly. We find the source term is constant when \( k_p\eta/\sqrt{3} \ll 1 \), while it decreases in proportion to \( \eta^{-2} \) for \( k_p\eta/\sqrt{3} \gg 1 \). As a result the production of scalar-induced GWs mostly occurs around the time when the peak scale \( k_p^{-1} \) crosses the sound horizon. Using the Green function method one can easily find a formal solution to (2), from which we can evaluate the density parameter of GWs contributed by a logarithmic interval of the wave number around \( k \). It is formally expressed as

\[
\Omega_{\text{GW}}(k, \eta) = \frac{k^3}{12\pi^2\mathcal{H}^2}(|h^+_{ij}|^2 + |h^\times_{ij}|^2).
\]

This is valid for modes well inside the horizon \cite{27}. In the radiation dominated regime it is explicitly given by

\[
\Omega_{\text{GW}}(k, \eta) = \frac{2}{3} \int_0^\eta d\eta_1 \int_0^\eta d\eta_2 \eta_1\eta_2 \sin[k(\eta - \eta_1)]
\times \sin[k(\eta - \eta_2)]S_k(\eta_1, \eta_2),
\]

where we have defined

\[
S_k(\eta_1, \eta_2) = \int_0^\infty d\tilde{k} \int_0^{1/3} d\mu \frac{k^2\tilde{k}^2}{|k - \tilde{k}|^3}
\times (1 - \mu^2)^2 f(\tilde{k}, |k - \tilde{k}|, \eta_1)
\times f(\tilde{k}, |k - \tilde{k}|, \eta_2) \mathcal{P}_\Psi(\tilde{k}) \mathcal{P}_\Psi(|k - \tilde{k}|).
\]

Here \( f(k_1, k_2, \eta) \) is a function written in terms of the transfer function for the scalar modes as follows:

\[
f(k_1, k_2, \eta) \equiv 2D_{k_1}(\eta)D_{k_2}(\eta) - [D_{k_1}(\eta) + \mathcal{H}^{-1}D_{k_1}(\eta)]
\times [D_{k_2}(\eta) + \mathcal{H}^{-1}D_{k_2}(\eta)].
\]

In the mass range of the PBHs of our interest, creation of scalar-induced GWs is terminated well before matter-radiation equality time. After that the energy density of GWs decreases in proportion to \( a^{-4} \). As a result the overall feature of the spectrum of \( \Omega_{\text{GW}}(f, \eta_0) \) today is well described by

\[
\Omega_{\text{GW}}(f)h^2 = 7 \times 10^{-9} \left( \frac{g_{*p}}{10.75} \right)^{-1/3} \left( \frac{\mathcal{A}^2}{10^{-3}} \right)^{2/3} \left( \frac{f}{f_p} \right)^2
\times \left[ 1 - \left( \frac{f}{2f_p} \right)^2 \right]^2 \theta \left( 1 - \frac{f}{2f_p} \right)
\]

\[
= A_{\text{GW}} \left( \frac{f}{f_p} \right)^{2/3} \left[ 1 - \left( \frac{f}{2f_p} \right)^2 \right]^2 \theta \left( 1 - \frac{f}{2f_p} \right),
\]

for \( f \leq f_p \equiv 2f_p/\sqrt{3} \), where we have used the frequency \( f_p \).
$f \equiv 2\pi k/a_0$ instead of the wave number. The above expression has the peak value $\Omega_{GW}(f_{GW})h^2 = (16/27)A_{GW}$ at $f_{GW}$. As expected, the amplitude of the induced GWs exceed its first-order counterpart, $\Omega_{GW}h^2 \sim 10^{-14}$ [27], and those generated during the PBH collapses, $\Omega_{GW}h^2 \sim 10^{-13} (f_{GW}/10^{-5} \text{ Hz})^{-1}$. [28].

Note, however, that the actual spectrum of GWs calculated from (6) has a much larger and sharper peak at $f_{GW}$ in addition to the bulk spectrum (14) due to amplification caused by resonance between the transfer function of the scalar modes and the Green function of the GWs [see Eq. (11)] [16]. Such amplification, called resonant amplification in [16], occurs only if the peak width $\Delta$ of the primordial scalar fluctuation is sufficiently small, $\Delta \ll k_p/2$. Since the resonant growth of the amplitude depends on the detailed shape of the primordial power spectrum around the peak, we do not incorporate it, which yields a conservative bound on the PBH abundance.

We now compare our results with observational constraints. For definiteness we identify $M_{PBH}$ with the horizon mass when the peak scale $k_p^{-1}$ entered the Hubble radius. This is a reasonable approximation even if critical behavior [29] is taken into account [30]. Then $M_{PBH}$ is related to the peak frequency of GWs as

$$f_{GW} = 1 \times 10^{-8} \text{ Hz} \left(\frac{M_{PBH}}{10^{26} \text{ g}}\right)^{-1/2} \left(\frac{g_{*p}}{10.75}\right)^{-1/12}. \quad (15)$$

The pulsar timing observations are sensitive to GWs with $f > 1/T$ where $T$ is the data span. The 7-yr data of observation of PSR B1855+09 gives an upper limit

$$\Omega_{GW}(f)h^2 < 4.8 \times 10^{-9} \frac{f}{(4.4 \times 10^{-9} \text{ Hz})^2}, \quad (16)$$

for $f > 4.4 \times 10^{-9}$ Hz at 90% confidence level [19]. By using this limit, we can constrain the abundance of PBHs with mass $M_{PBH} \leq 1 \times 10^{37} \text{ g} = 5 \times 10^{31} M_\odot$.

Space-based laser interferometers are sensitive to GWs with $10^{-5} \lesssim f \lesssim 10$ Hz, which covers the entire mass range of the PBHs which are allowed to be DM, $10^{20} < M_{PBH} < 10^{26} \text{ g}$. LISA will have its best sensitivity $\Omega_{GW}h^2 \sim 10^{-11}$ at $f \sim 10^{-2} \text{ Hz}$ ($M_{PBH} \sim 10^{24} \text{ g}$), BBO and the ultimate-DECIGO are planned to have sensitivities $\Omega_{GW}h^2 \sim 10^{-13}$ and $\Omega_{GW}h^2 \sim 10^{-15}$, respectively, at $f \sim 10^{-1} \text{ Hz}$ ($M_{PBH} \sim 10^{22} \text{ g}$) [31,32].

Figure 1 shows the energy density of the induced GWs obtained by numerically evaluating (11) and tracing its subsequent evolution up to the present, whose approximate form is given by (14). The left wedge-shaped curve represents the case $k_p = 0.2 \text{ pc}^{-1}$ and $\mathcal{A} = 7 \times 10^{-2}$ corresponding to $M_{PBH} = 6 \times 10^2 M_\odot$ and $\Omega_{PBH}h^2 = 10^{-5}$, while the right wedge-shaped curve depicts the case $k_p = 1 \times 10^6 \text{ pc}^{-1}$ and $\mathcal{A} = 9 \times 10^{-2}$ corresponding to $M_{PBH} = 3 \times 10^{22} \text{ g}$ and $\Omega_{PBH}h^2 = 10^{-1}$. We have also shown the limit imposed by the pulsar timing observation and the planned sensitivity of space-based laser interferometers depicted [31] with the instrumental parameters used in [32] as well as those of AGIS [24] and LIGO [33].

As is seen in Fig. 1, the pulsar timing constraint is so stringent that one cannot achieve $\Omega_{PBH}h^2 \gtrsim 10^{-5}$ for PBHs with $4 \times 10^2 M_\odot \lesssim M_{PBH} \lesssim 5 \times 10^3 M_\odot$, ruling out the major mass range of IMBHs. On the other hand, if pulsar timing experiments should find any nontrivial modulation in the near future, it might be due to the PBHs with mass around $10^2 M_\odot$ [20].

It is clear from Fig. 1 that the future space-based laser interferometers and AGISs can test the feasibility of PBHs being the dominant constituent of the DM. LIGO, on the other hand, has good sensitivity at $f \sim 10^{-1} \text{ Hz}$ [33]. This frequency band corresponds to mass scale $M_{PBH} \sim 10^{16}-10^{18} \text{ g}$. Though the sensitivity of LIGO is too low now and in the near future to detect GWs from the second-order effect associated with PBH formation, we could improve the sensitivity by correlation analysis to reach the desired level to probe PBHs. Because the spectrum has a tail extending to lower frequencies, it may be possible to constrain the abundance of the PBHs with $M_{PBH} < 7 \times 10^{16} \text{ g}$ ($f_{GW} > 6 \times 10^2 \text{ Hz}$), which have evaporated by the present epoch and could contribute to cosmic rays. Further study, however, is necessary in order to obtain the conclusion because there are astronomical sources of GWs in this frequency band too.
Figure 2 (color online). New constraints on the mass spectrum of PBHs imposed by scalar-generated GWs. Dotted line represents the mass range to be constrained by future GW detectors.

In summary, we have calculated the spectrum of the stochastic gravitational-wave background generated as a second-order effect from scalar perturbations which have a spectrum with a high peak to realize the formation of appreciable numbers of PBHs. As a result we have found that PBHs with their mass corresponding to that of IMBHs are already being ruled out because the amplitude of the associated GWs exceeds the limit imposed by the pulsar timing. We have also found that if PBHs with mass $10^{20}$–$10^{26}$ g are dominant constituents of DM, we can easily detect the relevant GWs by future space-based laser interferometers and AGISs. The gravitational waves are a new and powerful probe for the mass spectrum of PBHs.

We thank K. Ichiki for useful comments. This work was supported in part by JSPS Grant-in-Aid for Scientific Research No. 19340054 (J.Y.) and by Global COE Program “the Physical Sciences Frontier,” MEXT, Japan.

[33] http://www.ligo.caltech.edu/