Review:

- Energy & Momentum of gravitational waves:
  - $\bar{t}_{\mu\nu} = -\frac{1}{8\pi G} R^{(2)}_{\mu\nu}\left(h^{(1)}_{\mu\nu}\right)$
  - Non-localizable!
  - Averaged over wave-length, TT gauge:

$$t_{\mu\nu} = \frac{1}{32\pi G} \left\langle h^{TT}_{\alpha\beta,\mu} h^{TT\,\alpha\beta,\nu} \right\rangle$$

- Energy emitted by binary: $P = \frac{32}{5} \frac{G^4 \mu^2 M^3}{a^5}$
- Energy-conservation: $a(t) = a_0\left(1 - \frac{t}{\tau}\right)^{\frac{1}{4}}$

$$\tau = \frac{5}{256} \frac{a_0^4 c^5}{G^3 \mu M^2} = \frac{5}{256\pi^{\frac{8}{3}}} f_{gw}^{-\frac{8}{3}} M_{ch}^{-\frac{5}{3}} G^{-\frac{5}{3}} c^5$$

$$\simeq 3.0\,\mathrm{sec}\left(\frac{100\,\mathrm{Hz}}{f_{gw}}\right)^{8/3} \cdot \left(\frac{M_\odot}{M_{ch}}\right)^{5/3}$$

- Absorption of GW in matter: Attenuation length: $l_{Att} = \frac{c^3}{8\pi G \eta}$
- Antenna response: round trip travel time difference:

$$\Delta\delta T(\omega) = D(\omega, n_x) h_{xx}(\omega) - D(\omega, n_y) h_{yy}(\omega)$$

$$D(\omega, n_x) = \frac{i}{2\omega}\left[\frac{1 - e^{i\omega(1-n_x)T}}{(1-n_x)} - e^{2i\omega T} \frac{1 - e^{-i\omega(1+n_x)T}}{(1+n_x)}\right]$$

- Noise:
  - Random arrival noise: $x(t) = \sum_i \delta(t - t_i)$ ; $\langle x(t) \rangle = n$

$$\Rightarrow S_x^1 = \sqrt{2n}$$

  - Def of 1-sided vs 2-sided PSD; ASD
  - Shot noise $S_P^1 = \sqrt{2h\nu P}$ ; $S_I^1 = \sqrt{2eI}$

*simple Michelson

interferometer*: $S^{(1)}_{x_D} = \frac{1}{k}\sqrt{\frac{h\nu}{2P_{in}}}$ Power incident on BS

  - Rad. Press. noise*: $S^{(1)}_{x_D} = \frac{2}{m\omega^2 c}\sqrt{2h\nu P_{in}}$
  - Experimental convention: Power $P = |\psi|^2$

## The quantum mechanical picture

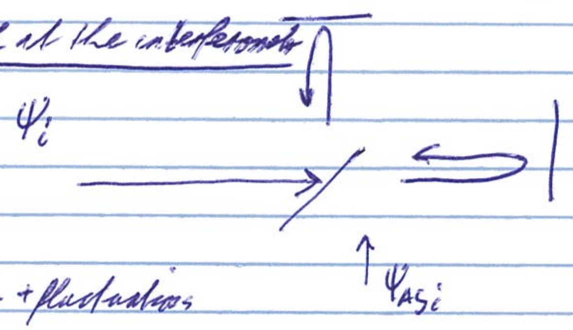
Recall: for rad-press noise I mentioned that the 2 arms are uncorrelated. Why? If we are dealing with intensity fluctuations, why are the not common?

Do we need a full QM picture? An interferometer (at its operating point) is a linear system for field fluctuations $\Rightarrow$

$$\boxed{\text{The QM expectation values follow the classical equations}}$$

$\Rightarrow$ We can analyse the whole interferometer as it reacts to classical noise, and at the end we set the incident noise equal to the QM noise!

So let's look at the interferometer




$\psi_i$ = carrier + fluctuations

$\psi_{AS}$ = fluctuations only

* Rad press. Force: $F_x - F_y = \frac{2}{c}(P_x - P_y)$

$$P_x = \frac{1}{2}|\psi_i + \psi_{AS}|^2 = \frac{1}{2}\left[|\psi_i|^2 + |\psi_{AS}|^2 + 2\mathrm{Re}\,\psi_i\psi_{AS}^*\right]$$

$$P_y = \frac{1}{2}|\psi_i - \psi_{AS}|^2 = \frac{1}{2}\left[\qquad\qquad - 2\mathrm{Re}\,\psi_i\psi_{AS}^*\right]$$

$$\Rightarrow \underline{\underline{F_x - F_y}} = \frac{2}{c}\cdot 2\,\mathrm{Re}(\psi_i\psi_{AS}^*)$$

the AS port matters!

Aside:

## Linear quantum mechanics

· Heisenberg equation of motion for quantum operator $\mathbb{Y}$ ($\mathbb{Y}$ can be $\mathbb{X}, \mathbb{P}, \mathbb{a}_1, \mathbb{a}_2$ (the two quadratures of light)

$$\Rightarrow -i\hbar \frac{d}{dt}\mathbb{Y} = [\mathbb{H}, \mathbb{Y}] \qquad \underbrace{\left( + \frac{\partial \mathbb{Y}}{\partial t} \right)_H}_{\substack{\text{explicit time} \\ \text{dependency} \\ =0 \text{ for all} \\ \text{our examples}}}$$

Linear equation of motion:

$\Leftrightarrow$

$\mathbb{H}$ is quadratic in $\mathbb{Y}$ and its dual variable $\mathbb{P}_Y$ $([\mathbb{Y}, \mathbb{P}_Y] = i\hbar)$

$\Leftrightarrow$

$[\mathbb{H}, \mathbb{Y}]$ is linear in $\mathbb{P}_Y$

$$\Rightarrow -i\hbar \frac{d}{dt}\langle \mathbb{Y} \rangle = \langle [\mathbb{H}, \mathbb{Y}] \rangle = \frac{\partial H(\langle \mathbb{P}_Y \rangle)}{\partial \langle \mathbb{P}_Y \rangle}$$

$\Rightarrow$ expectation value follows classical equation of motion!

$\Rightarrow$ QM behaves identical to an ensemble of classical trajectories with the same uncertainties.

Back to my interferometer

Similarly, for shotnoise:

$$\Psi_{AS} = \Psi_i \frac{1}{2}\left[e^{ik_0x} - e^{-ik_0x}\right] + \Psi_{AS}\frac{1}{2}\left[e^{ik_0x} + e^{-ik_0x}\right] = \Psi_i\, i \sin k_0x + \Psi_{AS}\cos k_0x$$

$$\underset{(k_0x)\ll 1}{\simeq} \Psi_i\, i\,(k_0x) + \Psi_{AS}$$

$$\Rightarrow P_{AS} = |\Psi_i|^2 (k_0x)^2 + |\Psi_{AS}|^2 + 2\,\mathrm{Re}\, i\, \Psi_i \Psi_{AS}^* (k_0x)$$

quadratic in fluctuations!

$\downarrow$ 0

$\Psi_i^c \cdot \Psi_i^{fluct} (k_0x)^2$

$\Rightarrow$ down by $(k_0x)$ !

$\Psi_i^c \cdot \Psi_{AS}$

carrier fluctuations

dominant term!

$\Rightarrow$ Both shot and radiation pressure noise are given by dark port fluctuations!

- But for shot: $\propto -2\,\mathrm{Im}\,\Psi_i \Psi_{AS}^*$

  " rad : $\propto 2\,\mathrm{Re}\,\Psi_i \Psi_{AS}^*$

So lets us be explicit about $\Psi_i$ & $\Psi_{ASi}$:

$$\Psi_i = e^{i\omega t}\Big(\sqrt{P_{in}} \quad + i\,\phi(t) \quad + A(t)\Big)$$

phase fluctuations   amplitude fluctuations

&

$$\Psi_{ASi} = e^{i\omega t}\Big(0 \quad + i\,\varphi(t) \quad + a(t)\Big)$$

no carrier!

Note: • $\phi$, $A$, $\varphi$ & $a$ are random functions

• their amplitude spectral density is defined through:

e.g. for $\phi(t)$: $\quad \phi(f) := \int_0^T e^{-i2\pi f t}\,\phi(t)\,dt$

$$\langle \phi(f)\cdot\phi(f')\rangle = \delta(f-f')\cdot S^2_{\phi\phi}(f)$$

↳ 2-sided power spectral density (PSD)

$$S^1_{\phi\phi} = 2S^2_{\phi\phi}$$

↳ 1-sided PSD

$$S^1_{\phi} := \sqrt{S^1_{\phi\phi}}$$ amplitude spectral density (ASD) of $\phi$

To linear order in fluct.:

$\Rightarrow$ • $2\,\mathrm{Re}\,\Psi_i\Psi^*_{AS} = 2\,\mathrm{Re}\,\sqrt{P_{in}}\cdot[-i\varphi(t)+a(t)] = \sqrt{P_{in}}\cdot 2a(t)$

• $-2\,\mathrm{Im}\,\Psi_i\Psi^*_{AS} = -2\,\mathrm{Im}\,\sqrt{P_{in}}\,[-i\varphi(t)+a(t)] = \sqrt{P_{in}}\cdot 2\varphi(t)$

Thus we find:

· Rad. press. noise: $S'_x(f) = \frac{1}{m\omega^2} \cdot S'_F(f) = \frac{1}{m\omega^2} \cdot \frac{2}{c} \cdot \underbrace{2\,\mathrm{Re}\,\varphi_i \varphi_{AS}^*}_{\sqrt{P_{in}} \cdot 2 \cdot S'_{a(f)}}$

$$= \frac{2}{m\omega^2 c} \sqrt{4 \cdot P_{in}} \cdot S'_{a(f)}$$

· Shot noise: $S'_x(f) = \frac{\partial x}{\partial P} \cdot S'_p(f)$

$$S'_p(f) = -2\,\mathrm{Im}\,\varphi_i \varphi_{AS}^*(k\Delta x) = \boxed{\sqrt{4 \cdot P_{in}}\; S'_{\varphi(f)} \cdot (k \cdot \Delta x)}$$

How big are $S'_\phi$, $S'_A$, $S'_\varphi$, $S'_a$ ? $\Leftarrow$ Just the input condition $\Leftrightarrow$ the only QM that is needed!

· They can be big, but are at least bigger or equal to quantum fluctuations!

Note: · A (ideal) laser beam is a coherent state; i.e. a vacuum state dispaced to a non-zero amplitude!

$\Rightarrow S'_\varphi = S'_\phi \;;\; S'_a = S'_A$

· For the quantum vacuum: $S'_\varphi = S'_a$

· How big are they? Calibrate on coherent beam: · Power $P_{in}$

· Shot noise $S'_p = \sqrt{2h\nu P_{in}}$

$\Rightarrow P(f) = |\varphi_i(f)| = \underset{\substack{\uparrow \\ \text{no fluct!}}}{P_{in}} + 2\sqrt{P_{in}} \cdot A(f)$

$\Rightarrow \boxed{S'_A(f) = \sqrt{\frac{h\nu}{2}}} = S'_\phi = S'_\varphi = S'_a$

quantum fluct. for (unsqueezed) vacuum state!

<u>Note:</u> This picture is very powerful:

· The get the quantum noise of any interferometer configuration

· i) calculate the classical response to phase & amplitude fluctuations

· ii) For every <u>open port</u> send in

- uncorrelated phase & amplitude fluctuations of size $S'_\varphi(f) = S'_a(f) = \sqrt{\frac{h\nu}{2}}$

- squeezed vacuum: $S'_\varphi(f) = \sqrt{\frac{h\nu}{2}} \cdot e^{-\xi}$

& $S'_a(f) = \sqrt{\frac{h\nu}{2}}\, e^{\xi}$ with the right phase relative to the input beam

⇒ iii) after propagating all these fluctuations to the readout, sum them up incoherently (in Power).

⇒ <u>we get the quantum limit of our interferometer configuration!</u> i.e. the full ASD $\Psi'_x(f)$ that includes shot noise & rad. press. noise.

# Electromagnetic field quantization: A simple view

Recall: Maxwell eqn: $\Box A_{\underset{\uparrow}{s}} = 0$

polarization

Our simple normalization $P(t) = |\psi(t)|^2 \Rightarrow A_s(t) \propto \psi(t)$

& $\Box \psi(t) = 0$

Note: $\psi$ is complex since we have E & B field

$E \Leftrightarrow \mathrm{Re}\,\psi$

$B \Leftrightarrow \mathrm{Im}\,\psi$

• Split $\psi(t)$ into modes: $\psi(t) := \int_{-\infty}^{\infty} d\tilde{f}\; \psi_{\tilde{f}}(t)$

with $-\Delta \psi_{\tilde{f}} = (2\pi \tilde{f})^2 \psi_{\tilde{f}} \equiv \tilde{\omega}^2 \psi_{\tilde{f}}$

Note: the integral contains pos & neg frequencies (*)

$\Rightarrow \left(-\partial_t^2 + \tilde{\omega}^2\right)\psi_{\tilde{f}}(t) = 0$

$\Rightarrow \psi_{\tilde{f}}(t) = e^{-i\tilde{\omega} t}\,\psi_{\tilde{f}}(0)$ ⇒ only pos freq. solution (neg freq. already counted, see (*))

⇒ Fourier transform of $\psi(t)$:

$$\boxed{\psi(f) = \int_{-\infty}^{\infty} d\tilde{f} \int_0^T dt\; e^{i2\pi f t}\, e^{-i\tilde{\omega} t}\, \psi_{\tilde{f}}(0) = \int d\tilde{f}\;\; \delta(f-\tilde{f})\,\psi_{\tilde{f}}(0) = \psi_f(0)}$$

Now Maxwell's eqn. for each mode is a harmonic oscillator!

Quantization of a Harmonic Oscillator:

$df = \frac{1}{T}$

Total Energy: $H = \int_0^T dt\,|\Psi(t)|^2 = \int_{-\infty}^{\infty} |\Psi_f(0)|^2\, df$

Define: $\sqrt{df}\,\tilde{\Psi}_f \equiv \sqrt{\frac{\omega}{2}}\,(x + ip)$

$\Rightarrow H = \sum_i \frac{\omega}{2} p_i^2 + \frac{\omega}{2} x_i^2$ $\Rightarrow$ set of harmonic oscillators with "$m = \frac{1}{\omega}$"

$\Updownarrow$

$x$ & $p$ symmetric

$$\left.\begin{aligned} \Rightarrow \dot{x} &= \frac{\partial H}{\partial p} = \omega p \\ \dot{p} &= -\frac{\partial H}{\partial x} = -\omega x \end{aligned}\right\} \Leftrightarrow (-\partial_t^2 + \tilde{\omega}^2)\tilde{\Psi}_f = 0 \text{ \& only pos. freq.}$$

$\Rightarrow$ quantization $[x, p] = i\hbar \Rightarrow \Delta x \cdot \Delta p \geq \frac{\hbar}{2}$ (= for min. uncertainty)

in groundstate/symmetry: $\Delta x = \Delta p = \sqrt{\frac{\hbar}{2}}$

$$\Rightarrow \left\langle df \cdot \left(\mathrm{Re}\,\tilde{\Psi}_f\right)^2 \right\rangle = \frac{\tilde{\omega}}{2} \cdot \langle x^2 \rangle = \frac{\hbar\tilde{\omega}}{4}$$

or $\left\langle (\mathrm{Re}\,\Psi_f)^2 \right\rangle = T\,\frac{\hbar \tilde{f}}{4} \Rightarrow \left\langle \mathrm{Re}\,\Psi_f \cdot \mathrm{Re}\,\Psi_{f'} \right\rangle = \delta(f - f')\,\frac{\hbar\tilde{f}}{4}$

$$\text{But } \left\langle \mathrm{Re}\,\Psi(f) \cdot \mathrm{Re}\,\Psi(f') \right\rangle = \left\langle \mathrm{Re}\,\Psi_f \cdot \mathrm{Re}\,\Psi_{f'} \right\rangle = \delta(f - f') \cdot \underbrace{\frac{\hbar\tilde{f}}{4}}_{S^{(2)}_{\mathrm{Re}\Psi\,\mathrm{Re}\Psi}}$$

$\Downarrow$

$$\Rightarrow \boxed{S'_{\mathrm{Re}\Psi} = \sqrt{\frac{\hbar\tilde{f}}{2}}}$$

Same for $\mathrm{Im}\,\Psi$

## A simple design:

We need to see $h(f) \sim 10^{-21}$ with good SNR

$$\Rightarrow S_h'(f)_{\text{target}} \sim 10^{-23}/\sqrt{Hz}$$

- Shot noise: $S_x' = \frac{1}{k}\sqrt{\frac{\hbar\nu}{2P_{in}}}$

$$\Rightarrow S_h'(f) = \frac{\lambda}{2\pi}\sqrt{\frac{\hbar\nu}{2P_{in}}} \cdot \frac{1}{L} \simeq 10^{-23}/\sqrt{Hz}$$

$$= 5.2 \cdot 10^{-17}/\sqrt{Hz} \cdot \sqrt{\frac{1W \cdot 1m^2}{P_{in} \cdot L^2}} \simeq 10^{-23}/\sqrt{Hz}$$

$$\Rightarrow 5.2 \cdot 10^{3} = \sqrt{\frac{P_{in}}{1W}} \cdot \frac{L}{1km}$$

$\Rightarrow$ for $L = 4km \Rightarrow \underline{P_{in} = 1.7\,MWatt}$ ! ....

- Rad. pres: $S_h'^{\,radpres} = \frac{2}{m\omega^2 c}\sqrt{2\hbar\nu P_{in}}\,\frac{1}{L} = \underline{8 \cdot 10^{-24}} \cdot \underline{\left(\frac{10Hz}{f}\right)^2}$

<u>We will need some tricks...</u>

But before that: We still have to make sure nothing else makes more noise!

Reference: Optical Coatings & Thermal Noise in Precision Measurements (Harry, DeSalvo) ISBN: 9781107003385

# Thermal Noise

1-D system: take a damped oscillator: $m\ddot{x} + \gamma\dot{x} + kx = F_{noise}$

or $Z\dot{x} = F_{noise}$

where $Z = i\omega m + \gamma + \frac{k}{i\omega}$ is the impedance

Assume we have a (for now given) random drive force $F_{noise}$

a) Power dissipation: $P(t) = \underbrace{F_{diss}(t)}_{\gamma v}\, v(t) = \gamma v^2(t)$

$\Rightarrow$ average dissipation $P = \gamma \langle v^2(t) \rangle = \gamma \frac{1}{T}\int_0^T dt\, v^2(t)$

But the equipartion theorem tells us that $\langle p \frac{\partial H}{\partial p} \rangle = \langle p\dot{x} \rangle = m\langle v^2 \rangle = k_B T$ (i.e. $\frac{1}{2}mv^2 = \frac{k_B T}{2}$)

$$\Rightarrow \boxed{P = \gamma \frac{k_B T}{m}}$$

$\Rightarrow$ There needs to be a (random) driving force present to replenish this energy!

Let us assume that this force has a white power spectrum

$$\Rightarrow \langle F_n(f) F_n^*(f') \rangle = \delta(f-f') \underbrace{S_{FF}^2(f)}_{\text{(2-sided) Power spectrum} \Rightarrow \text{assume indep. of } f} \qquad \Rightarrow \langle F_n(f) F_n^*(f) \rangle = \overset{\text{observation time}}{T} \cdot S_{FF}^2$$

The drive power is

$$\Rightarrow \underline{P_{drive}} = \frac{1}{T}\int_0^T dt \langle v(t) \cdot F(t) \rangle \underset{T\,\text{large}}{=} \frac{1}{T}\int df \langle v(f) F^*(f) \rangle = \frac{1}{T}\int df\, Z^{-1} \langle FF^* \rangle = \int \mathrm{Re}\, Z^{-1} S_{FF}^2\, df$$

Note: $\int_{-\infty}^{\infty} \mathrm{Re}(Z^{-1})\, df = \frac{1}{2m}$ for our impedance!

$$\Rightarrow \boxed{P_{drive} = S_{FF}^2 \cdot \frac{1}{2m}} \qquad \Rightarrow P_{diss} = P_{drive} \Rightarrow \boxed{S_{FF}^2 = 2k_B T \cdot \mathrm{Re}\, Z}$$

(Pos & neg frequency)

Pos freq. only $\Rightarrow \times 2 \Rightarrow \boxed{S_{FF}^1 = 4k_B T\, \mathrm{Re}\, Z}$

Turns out this is the correct expression even if $\mathrm{Re}(Z)$ is frequency dependent!

$$\Rightarrow \boxed{S_{FF}^1(f) = 4k_B T\, \mathrm{Re}\, Z(f)}$$

one-sided power spectrum (pos. freq. only)

Aside:

What is $\int_{-\infty}^{\infty} \mathrm{Re}\, Z^{-1}\, df = ?$ for $Z = i\omega m + \gamma + \frac{k}{i\omega}$

$$Z^{-1} = \frac{1}{i\omega m + \gamma + \frac{k}{i\omega}}$$

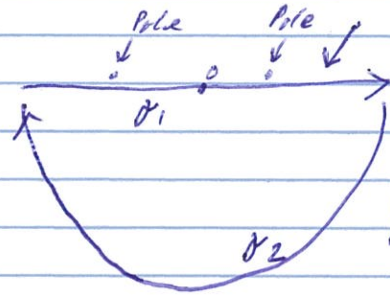
$$\int_{-\infty}^{\infty} \mathrm{Re}\, Z^{-1}\, df \underset{\uparrow}{=} \int_{-\infty}^{\infty} Z^{-1}\, df$$

$\mathrm{Re}\, Z^{-1}$ even
$\mathrm{Im}\, Z^{-1}$ odd

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{\omega}{\omega^2 m - i\gamma\omega - k}$$

complex integral, poles at $\omega = +i\frac{\gamma}{2m} \pm \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m}}$

$$\underset{\omega = z}{=} \int_{-\infty}^{\infty} \frac{dz}{2\pi i} \frac{z}{z^2 m - i\gamma z - k} =: I_1$$




$$\int_{\gamma_1+\gamma_2} \dots dz = 0$$

$$I_2 \underset{\substack{z = re^{+i\phi} \\ dz = z \cdot i\, d\phi}}{=} \int_0^{-\pi} \frac{d\phi}{2\pi} \frac{1}{m - i\frac{\gamma}{z} - \frac{k}{z^2}}$$

$$\underset{r\to\infty}{=} \int_0^{-\pi} \frac{d\phi}{2\pi} \frac{1}{m} = -\frac{1}{2m} \quad \Rightarrow I_1 = -I_2 = \underline{\underline{\frac{1}{2m}}}$$

$$\Rightarrow 4\int_0^{\infty} \mathrm{Re}\, Z^{-1}\, df = \frac{1}{m} \Rightarrow \boxed{4\, \mathrm{Re}(Z^{-1}) \text{ is an effective "inverse mass, 1-sided spectral density"}}$$

# Velocity & Position Power spectrum

• Velocity: $S_{vv}(f) := \frac{1}{T}\langle v(f) \cdot v^*(f) \rangle$

$$= \frac{1}{T} Z^{-1} \langle F F^* \rangle Z^{-1*}$$

$$= Z^{-1} S_{FF}(f) Z^{-1*}$$

$$= 2k_B T \quad Z^{-1}(Z + Z^*) Z^{-1*} = 2k_B T (Z^{-1} + Z^{-1*})$$

$$= 4k_B T \, Re \, Z^{-1}$$

$S_{FF}(f) = 4k_B T \, Re \, Z(f)$ ; $\boxed{S_{vv}(f) = 4k_B T \, Re \, Z^{-1}(f)}$

• Position: $v = i\omega x$ $\quad S_{xx}(f) = \frac{1}{T}\left\langle \frac{v}{i\omega} \frac{v^*}{-i\omega} \right\rangle = \frac{1}{\omega^2} S_{vv}$

$$\Rightarrow \boxed{S_{xx}(f) = \frac{4k_B T \, Re \, Z^{-1}(f)}{\omega^2}}$$

# Example from Electronics: Johnson-Nyquist noise

analogy: Mechanics | Electronics

Impedance: $Z v = F$ | $R \cdot I = U$ ; or

$Z \cdot I = U$

with $Z$ complex impedance (e.g. $i\omega L$, $\frac{1}{i\omega C}$, $R$, or combination)

Power dissipation / drive:

$P = \frac{1}{T} \int F \cdot v \, dt$ $\qquad$ $P = \int U \cdot I \, dt$

i.e.

$F \leftrightarrow U$
$v \leftrightarrow I$
$x \leftrightarrow Q$

$$\Rightarrow \boxed{S'_{UU} = 4 k_B T \cdot \mathrm{Re}\, Z}$$ Johnson-Nyquist noise (Voltage noise)

$$S_{II}^{(1)} = 4 k_B T \,\mathrm{Re}\, Z^{-1}$$ (Current noise)

## Structural vs Viscous damping

So far we assumed that our dissipation force is proportional to velocity: $F = -\gamma V$ $\hat{=}$ viscous damping.

($\gamma$ indep. of frequency.)

We can rewrite our Impedance:

$$Z = i\omega m + \gamma + \frac{k}{i\omega} = i\omega m + \frac{k\left(1 + \frac{i\omega\gamma}{k}\right)}{i\omega}$$

and define $\phi(f) = \frac{2\pi f \gamma}{k}$ $\hat{=}$ $\boxed{\text{loss angle}}$

$= \frac{\omega\gamma}{k}$ (typically $\ll 1$)

& $K_c = K(1+i\phi)$ complex spring constant

$$\Rightarrow \boxed{Z = i\omega m + \frac{K_c}{i\omega}}$$

- For most materials $\phi(f) \sim$ indep of frequency.

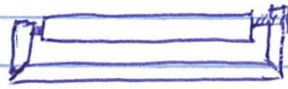
This is called structural damping

(usually $\phi(f) = \text{const}$ was assumed for Mirror coating Brownian noise, but see LIGO-G1700820)

$$\phi(f) = \text{const} \Rightarrow \boxed{\gamma(f) = \frac{k}{\omega}\phi = \frac{k}{2\pi f}\phi}$$

## Example: Thermal noise spectrum of a 1-dim resonance

For example: current aLIGO signal recycling mirror is composite:



clamped mirror

resonance frequency: 3.3 kHz (showed up in displacement spectrum).

**Model:** $Z = i\omega m + \gamma(f) + \frac{k}{i\omega}$ $\qquad Z i\omega x = F_n$

$$\frac{Z}{m} = i\omega + \frac{\gamma(f)}{m} + \frac{\omega_0^2}{i\omega} \quad \text{or} \quad Z = i\omega m + \gamma(f) + \frac{m\omega_0^2}{i\omega}$$

$$Z^{-1} = \frac{1}{\gamma(f) + i\left[\omega m - \frac{m\omega_0^2}{\omega}\right]} = \frac{\gamma - i\left[\omega m - \frac{m\omega_0^2}{\omega}\right]}{\gamma^2 + \left[\omega m - \frac{m\omega^2}{\omega}\right]^2}$$

$$\mathrm{Re}\, Z^{-1} = \frac{\gamma}{\gamma^2 + \left[\omega m - \frac{m\omega^2}{\omega}\right]^2}$$

$$\gamma = \frac{k\phi}{\omega} = \frac{km\phi}{m\omega} = \frac{\omega_0^2 m\phi}{\omega}$$

$$\Rightarrow \boxed{S_{xx}(f) = \frac{4k_BT}{\omega^2}\,\frac{\gamma(f)}{\gamma^2(f) + \left[\omega m - \frac{m\omega_0^2}{\omega}\right]^2}}$$

$$\boxed{= \frac{4k_BT}{\omega^2}\,\frac{1}{m}\,\frac{\omega\omega_0^2\phi}{\left[\omega_0^2\phi\right]^2 + \left[\omega^2 - \omega_0^2\right]^2}}$$

$\Rightarrow$ only $\omega_0$, $\phi$ & $m$ enter

resonance freq. / resonance width / effective mass

for $\omega = \omega_0$: $\quad S_{xx}(f) = \frac{4k_BT}{\omega_0^3}\cdot\frac{1}{m\phi(f_0)}$ $\quad\Rightarrow$ measure $\phi$ on resonance...

for $\omega \ll \omega_0$ $\quad \boxed{S_{xx}(f) = \frac{4k_BT}{\omega_0^2\,\omega}\,\frac{\phi(f)}{m}}$ $\quad \propto \frac{1}{f}$ for structural ($\phi = \text{const}$)

$\propto$ const for viscous ($\phi \propto f$)

$\Rightarrow$ predict thermal noise below resonance

Generalization to multiple degrees of freedom:

Impedance matrix: $\sum_j Z_{ij} \dot{x}_j = F_i$

or simply

$$\mathbf{Z}\dot{\mathbf{x}} = \mathbf{F} \quad \text{(vector notation)}$$

$S_{\dot{x}\dot{x}} = 4k_B T \,\mathrm{Re}\, Z^{-1}$ generalizes to

transpose & conjugate

$$S_{\dot{x}_i \dot{x}_j} = 2k_B T \left( \mathbf{Z}^{-1} + \mathbf{Z}^{+-1} \right)_{ij}$$

similarly $S_{F_i F_j} = 2k_B T \left( \mathbf{Z} + \mathbf{Z}^{+} \right)_{ij}$

⇒ these are thermal noise powerspectra (auto-correlations) $(i=j)$

and cross-powerspectra (cross-correlations) $(i \neq j)$

⇒ All we need to know is the (multi-dim) mechanical response $Z_{ij}$, and we immediately know the thermal noise!

e.g. $Z_{ij}$ is very precisely known & measured for the aLigo quadruple pendulum!

⇒ their thermal noise can be calculated!

Dimension: 3 translation + 3 rotation for each stage ⇒ $4 \cdot 6 = 24$

still finite...

## The Levin Approach to calculating thermal noise

So far we discussed systems with a finite number of discrete degrees of freedom. We are however also interested in continuous systems, such as e.g. the Brownian thermal noise of mirrors.

Out of the $\infty$ degrees of freedom, we care about One particular DoF: Whichever the interferometer reads out!

We can define: $e = \sum_i f_i x_i = f^T x$ as this D.o.F.

(Note: $f_i$ is a projection function that may also have additional units)

We care about: $S_{ee}(f) = \frac{S_{\dot{e}\dot{e}}}{\omega^2} = \frac{2k_BT}{\omega^2} \underbrace{f^{+}(Z^{-1} + Z^{+-1})f}$

the matrix element we care about! $=: M$

We can calculate $M$ in a different way!

Let us drive the system with a general force vector

$F_i = F$

$\Rightarrow$ The system will dissipate

$$P_{diss} = \frac{1}{T}\int_0^T F^+ \dot{x}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} F^+ (Z^{-1}) F\, df$$

T large

$$= \frac{1}{T}\int_0^{\infty} F^+ (Z^{-1} + Z^{+-1}) F\, df$$

$Z(-f) = Z^*(f)$
$F(-f) = F^*(f)$

pos freq. only

**Thus, if we pick** $F = F_0 \cdot f \cdot \cos(\omega t)$

$F_0$: some arb. constant; $f$: projection function

$$\Rightarrow \underline{P_{diss}} = \frac{F_0^2}{4} \cdot \underbrace{f^+ (Z^{-1} + Z^{+-1}) f}_{M}$$

same matrix element!

$$\Rightarrow \boxed{S'_{ee}(f) = \frac{2k_BT}{\omega^2} \cdot \frac{4P_{diss}}{F_0^2} = \frac{8k_BT\, P_{diss}(f)}{\omega^2 F_0^2}}$$

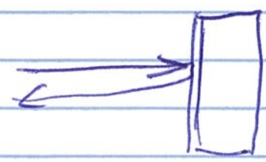
$\Rightarrow$ We can calculate $P_{diss}$ in time domain using e.g. a finite element model!

## Example: Coating Brownian Noise

Use Levin's approach! • Which degree-of-freedom actually matters?

⇒ average phase readout by the laser beam!

$\rho$: transverse coordinates



$$\Psi_{out}(\rho) = \Psi_{in}(\rho)\, e^{2ikx(\rho)}$$

• $\Psi_{out}$ then beats against a reference beam $\Psi_{ref}$, typically of the same shape!

$$e^{i2k\langle x\rangle} = \int d^2\rho \; \Psi_{ref}^*(\rho)\, \Psi_{in}(\rho)\, e^{i2kx(\rho)}$$

• For most applications $\Psi_{ref}^*\Psi_{in}$ is a Gaussian intensity profile:

$$I(\rho) = \frac{2P}{\pi w^2}\, e^{-\frac{2\rho^2}{w^2}}$$

⇒ We thus choose $F(\rho) = \frac{2F_0}{\pi w^2}\, e^{-\frac{2\rho^2}{w^2}} \cos\omega t$

i.e. $f(\rho) = \frac{2}{\pi w^2}\, e^{-\frac{2\rho^2}{w^2}}\, d^2\rho$

& $e = \int \frac{2}{\pi w^2}\, e^{-\frac{2\rho^2}{w^2}}\, x(\rho)\, d^2\rho$

⇒ Procedure: apply pressure field $p = F(\rho)$, calculate mechanical dissipation

• For semi-infinite plane: find Green's function for elastic problem

(note: static approx ⇔ below test mass resonance)

⇒ $x(\rho) = \int G(\rho,\rho')\, p(\rho')\, d^2\rho'$ ; $G(\rho,\rho') = \frac{1-\sigma^2}{\pi E_0}\, \frac{1}{|\rho-\rho'|}$

⇒ $U_{max} = \frac{1}{2}\int p(\rho)\cdot x(\rho)\, d^2\rho$

Note: Dissipated power is directly related to the maximal elastic energy:

e.g. ~~of~~ for a spring $F = +k(1+i\phi)x$ ; $F = F_0 \cos\omega t$

$\Rightarrow x = \frac{F_0}{k}\cos\omega t - \phi\frac{F_0}{k}\sin\omega t$

$$\Rightarrow P_{diss}\cdot T = \oint_{1\,cycle} F\,dx \qquad \dot{x} = -\frac{F_0\omega}{k}\sin\omega t - \phi\frac{F_0\omega}{k}\cos\omega t$$

$$= \int_0^T F\dot{x}\,dt = \int_0^T \phi\frac{F_0^2\omega}{k}\cos^2\omega t = \omega\, T\, \phi \underbrace{\frac{F_0^2}{2k}}_{U_{max}}$$

$$\Rightarrow \omega\,\phi\, U_{max} = P_{diss}$$

$$\Rightarrow \boxed{P_{diss} = 2\pi f\ \phi\ U_{max}}$$

$$\Rightarrow \boxed{S'_{ee} = \frac{8k_BT}{\omega}\ \frac{U_{max}}{F_0^2}\cdot\phi}$$

What is $U_{max}$? Since $\phi_{substrate} << \phi_{coating} \simeq 4\cdot10^{-4}$

$$\left(\frac{\phi_c}{\phi_s} \simeq 10^5 \ldots\right)$$

$\Rightarrow$ Only Energy stored in Coating matters:

$$U^c_{max} = \underset{\substack{\uparrow\\ \text{coating thickness}}}{\delta_c}\ \underbrace{\frac{1}{\pi w^2}}_{\text{beam area}}\ (1+\sigma)(1-2\sigma)\cdot\frac{F_0^2}{E_0}\cdot\Omega_1$$

$\Omega_1$: corrections of $O(1)$ if coating & substrate have different Young's modulus $E_0$

Coating Brownian

$$\Rightarrow \boxed{S'_{coating} = \frac{8k_BT\,(1+\sigma)(1-2\sigma)}{\omega\ \pi w^2\ E_0}\ \delta_c\cdot\phi\cdot\Omega_1}$$

Substrate Brownian: $w\,\phi_s \simeq 6cm\cdot10^{-5}\phi_c = 0.6\mu m\cdot\phi_c << \delta_c\cdot\phi_c$

$w$: Beam Size; $\delta_c$: ~few μm

$\Rightarrow$ Substrate Brownian is less than coating Brow.

<u>Add numbers:</u>

$k_B = 1.38 \cdot 10^{-23} \frac{m^2\, kg}{s^2 K}$ $\qquad L = 4000 m$

$T = 300 K$

$\sigma_{SiO_2} = 0.17$

$E_{SiO_2} = 72.8\ GPa$

$w = 0.06\ m$

$\delta_c \simeq 4 \cdot 10^{-6} m$

$\phi \simeq 4 \cdot 10^{-4}$

$\Omega \sim 1$

$$\Rightarrow \boxed{S'_{cB} \simeq 2\cancel{4} \cdot 10^{-24} / \sqrt{Hz} \cdot \sqrt{\frac{100 Hz}{f}}}$$

## Other forms of thermal noise

We have seen that dissipation leads to a fluctuation of a related variable. (e.g. mechanical dissipation ⇒ fluctuations in position, strain, pressure, ... )

The same is true for any dissipation; e.g. thermal diffusion ⇒ fluctuations in $T$, $\underset{\uparrow \text{heat}}{\delta Q} = T dS$

⇒ as long as our readout couples to those fluctuations, we will see an associated noise!

for example

| | | |
|---|---|---|
| $p$ | $, dx$ | Brownian noise |
| $T$ | $, dS$ | Thermo-elastic noise |
| $\underset{\uparrow \text{chemical potential}}{\mu}$ | $, \underset{\uparrow \text{charge carrier density}}{dn}$ | Charge carrier density noise in a semiconductor. |

- Levin's approach still works!

i.e. we care about $e = \underset{\uparrow \text{readout dependence on that field}}{\vec{f}}{}^{T} \underset{\uparrow \text{temp. fluctuation field}}{\delta \vec{T}}$

⇒ Drive with:

Entropie (Heat) $\vec{\delta S} = \frac{\delta \vec{Q}}{T_0} = S_0 \cdot \vec{f} \cos \omega t$

& calculate dissipation $\boxed{P_{diss} = \frac{1}{T} \oint T \dot{S} \, dt}$

$\boxed{\& \, T\dot{S} = \int dV \, \frac{\kappa}{T} (\vec{\nabla} T)^2}$

$$\Rightarrow \boxed{S_{ee} = \frac{8 k_B T}{\omega^2} \frac{P_{diss}}{S_0^2}}$$

Thermo-elastic noise · matters in coatings w/ low Brownian noise

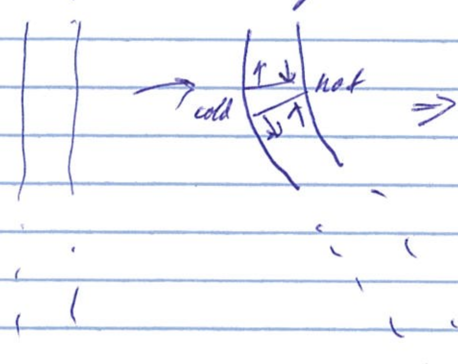
· 2 coupling mechanisms:

- expansion of bulk material

- expansion & index of refraction change in coating

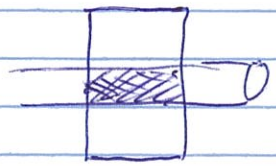
⇒ both depend on the same temperature fluctuations, and are coherent.

· matters in suspensions (fiber bending)




⇒ heat flow ⇒ dissipation

## Example: transmissive thermorefractive noise



Noise picked up by a laser beam going through a slab of material (e.g. glass)

define $\beta_{eff} = \frac{dn}{dT} + (n-1)\,\alpha$

↑(constrained) linear expansion coefficient

$\Rightarrow$ for a small cube: $\Delta x = \beta_{eff}\cdot dz \cdot \delta T(r,z)$

$|dz|$

$\Rightarrow$ integrate, with beam intensity

$$\Delta x = \beta_{eff}\, a \int_0^a dz \int d^2r\; \delta T(r,z)\cdot q(r,z)$$

with readout function $q(r,z) = \frac{1}{a}\frac{2}{\pi w^2}\, e^{-\frac{2r^2}{w^2}}$

$$\Rightarrow \int_0^a dz \int d^2r\; q(r,z) = 1$$

$\Rightarrow$ Levin: Drive the system with heat (entropy) (per unit volume)

$$dq = T\,ds = T\,F_0 \cos(\omega t)\cdot q(r,z)$$

& solve heat diffusion eqn: $c\rho\,\delta\dot{T} \underbrace{-K\Delta\delta T}_{+K\nabla\cdot\vec{j}} = d\dot{q}$

diffusion length scale: $l_{th} = \sqrt{\frac{K}{\omega c\rho}}$

typically $w \gg l_{th} \Rightarrow \Delta$-term is small

$$\Rightarrow \delta T = \frac{dq}{c\rho} = \frac{T F_0}{c\rho}\cos(\omega t)\cdot q(r,z)$$

$$\Rightarrow \vec{\nabla}_r \delta T = -\frac{4r}{w^2}\cdot \delta T$$

**Dissipation:** $\dot{s} = \frac{-\vec{\nabla}\cdot\vec{j}}{T}$ (from heat diffusion eqn)

$$\Rightarrow \dot{S} = -\int dV \frac{\vec{\nabla}\cdot\vec{j}}{T} = +\int dV\, \vec{j}\cdot\vec{\nabla}\left(\frac{1}{T}\right) \quad + \text{surf term}$$

usually small

but see PRD 90, 043013

$$\Rightarrow \dot{S} = -\int dV \frac{1}{T^2}\, \vec{j}\cdot\vec{\nabla}T$$

$$= +\int dV \frac{K}{T^2}\, \vec{\nabla}T\cdot\vec{\nabla}T$$

$$\Rightarrow \dot{Q} \equiv P_{diss.} = T\dot{S} = \int dV \frac{K}{T}(\nabla T)^2$$

$$= \int_0^a da \int d^2r\, \frac{K}{T}\cdot\left(\frac{4r}{w^2}\right)^2\cdot\left(\frac{\tau F_0}{c\varsigma}\right)^2 \cos^2\omega t\, \underbrace{q^2(r,z)}_{\left(\frac{1}{a}\frac{2}{\pi w^2}\right)^2 e^{-\frac{4r^2}{w^2}}}$$

$$\boxed{= \dots = \frac{2K\tau F_0^2}{c^2\varsigma^2}\cdot\frac{1}{a}\frac{1}{\pi w^4} = P_{diss}}$$

$$\Rightarrow S_{\delta x\delta x}(f) = \frac{8k_B T}{\omega^2}\frac{P_{diss}}{F_0}$$

$$\Rightarrow \boxed{S'_{\delta x\delta x}(f) = \frac{16\, k_B K T^2}{\pi c^2\varsigma^2 w^4 \omega^2}\, a\cdot\beta_{eff}}$$

## Back to our simple design

- For a shot noise target of $10^{-23}/\sqrt{Hz}$ we needed $P_{in} = 1.7MW$!

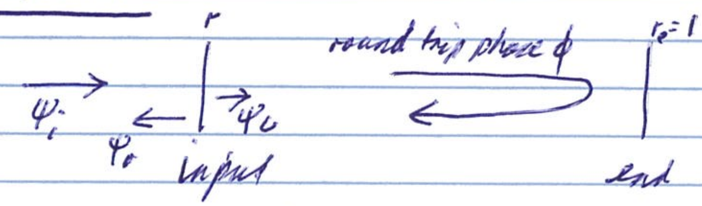
  $\Rightarrow$ Rad. Pressure noise: $\sim 8\cdot 10^{-26}/\sqrt{Hz}\cdot\left(\frac{100Hz}{f}\right)^2$ $\Rightarrow$ some headroom!

  & dominant thermal noise: Brownian noise: $\sim 2\cdot 10^{-24}/\sqrt{Hz}\,\sqrt{\frac{100Hz}{f}}$

- Reduce Power requirements (at cost of some rad. press. noise):

### Fabry-Perot cavities:

1-d calculation:




$r^2+t^2=1$: $\quad \psi_c = t\,\psi_i + r e^{-i\phi}\psi_c \Rightarrow \psi_c = \frac{t}{1-re^{-i\phi}}\psi_i$

$$\psi_r = \Big[\frac{t^2 e^{-i\phi}}{1-re^{-i\phi}} - r\Big]\psi_i$$

sign important!

Note: $\left|\frac{\psi_r}{\psi_i}\right| = \left|\frac{(1-r^2)e^{-i\phi}}{1-re^{-i\phi}} + \frac{-r+r^2e^{-i\phi}}{1-re^{-i\phi}}\right|$

$$= \left|\frac{e^{-i\phi}-r}{1-re^{-i\phi}}\right| = \left|e^{-i\phi}\frac{1-re^{+i\phi}}{1-re^{-i\phi}}\right| = 1$$

$$\boxed{\begin{aligned}\phi &= 2kL = 2\pi\frac{f}{FSR}\\ FSR &= \frac{c}{2L}\end{aligned}}$$

On resonance ($\phi = 2\pi n$): $\psi_c = \frac{t}{1-r}\psi_i \Rightarrow P_c = P_{in}\cdot\frac{(1-r^2)}{(1-r)^2} = P_{in}\frac{1+r}{1-r} = G\,P_{in}$

$\Rightarrow$ Power gain: $\boxed{G = \frac{1+r}{1-r}}$

Phase sensitivity: $\boxed{\frac{\partial\psi_r}{\partial\phi}\cdot\frac{1}{\psi_{in}} = -i\frac{t^2(1-r)+t^2r}{(1-r)^2} = -i\frac{t^2}{(1-r)^2} = -i\,G}$

$\Rightarrow$ Phase sensitivity increased $\propto$ power gain!

Define $\boxed{\text{Finesse } \mathcal{F} := \frac{\pi}{2}\cdot G}$

## Line shape & Finesse

• Line shape: $\left|\frac{\psi_t}{\psi_{in}}\right|^2 = \frac{t^2}{(1-r\cos\phi)^2+(r\sin\phi)^2} = \frac{t^2}{1+r^2-2r\cos\phi}$

for what $\phi$ do we get $\frac{1}{2}$ the power?

$$1+r^2-2r\cos\phi = 2(1+r^2-2r)$$

$$-2r\cos\phi = 1-4r+r^2$$

$$\cos\phi = 2-\frac{r}{2}-\frac{1}{2r} \underset{\substack{r=1-\varepsilon \\ \frac{1}{r}=1+\varepsilon+\varepsilon^2\ldots}}{\cong} 1-\frac{\varepsilon^2}{2}$$

$$\Rightarrow \phi \simeq \varepsilon \simeq 1-r$$


$$\boxed{\frac{2\phi}{2\pi} = \frac{\phi}{\pi} \approx \frac{1-r}{\pi} \simeq \frac{1-r}{1+r}\,\frac{2}{\pi} \simeq \frac{1}{\mathcal{F}}}$$

<u>Cavity Pole:</u> What does a frequency that is slightly offset from the carrier see? $\Delta f \cdot \frac{2\pi}{FSR} = \phi$ ; $FSR = \frac{c}{2L}$

$$\Rightarrow \frac{\psi_c}{\psi_{in}} = \frac{t}{1 - re^{-i\phi}} \simeq \frac{t}{1 - r + ir\phi} = \frac{t}{1-r} \frac{1}{1 + \frac{ir}{1-r}\phi}$$

$$= \frac{t}{1-r} \frac{1}{1 + \frac{ir}{1-r}\frac{2\pi \Delta f}{FSR}}$$

$$=: \frac{t}{1-r} \frac{1}{1 + i\frac{f}{f_p}}$$

$\Rightarrow$ <u>Cavity pole frequency:</u>

$$\boxed{f_p = FSR \cdot \frac{1-r}{2\pi r}}$$

$\Rightarrow$ signals get filtered with a single pole low-pass!

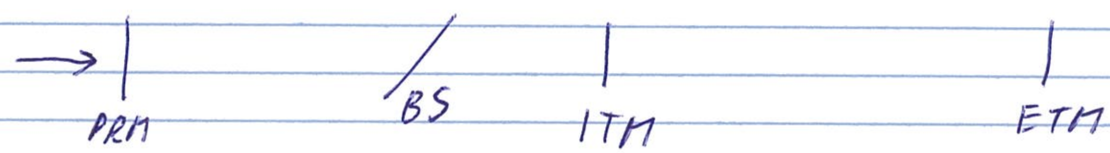
<u>Note:</u> for good reflectors ($r$ close to 1)

$$f_p \simeq \frac{FSR}{2\mathcal{F}}$$

## Power Recycling:

With Fabry-Perot arm cavities we can reduce the input power requirement, but the BS reference power is still equal to the input power.

⇒ We can choose a lower arm cavity finesse, & instead put an additional mirror at the input:




• For the common arm cavity: think of the PRM - ITM distance as a "very thick coating layer"

⇒ helps increase the arm power while allowing to independently set the BS reference power!

Indeed, we want $R_{PRM}$ (PRM power reflectivity) $=$ $R_{IFO}$ (interferometer total reflectivity)

$$= 1 - L_{IFO}$$

total losses in IFO

⇒ ideally No reflected carrier.

In practice some reflected carrier needed for angular control.

## The signal recycling mirror (SRM)

- The same trick can be used on the anti-symmetric port, an SRM changes the differential cavity pole

  ⇒ this allows us to idependently set the reference power levels
  - on the BS
  - at the output (AS) port
  - in the arm cavities

- Note: aLigo lowers the differential cavity finesse (higher differential cavity pole)

  $\hat{=}$ "signal extraction"

  (this is opposite of what the PRM does)

  ↳ lower common cavity pole

  - "signal recycling" would be the opposite (~~lower~~ differential cavity pole)

- The SRM can also be detuned to further shape the sensitivity curve.

## Further topics

- PDH locking
- Wave front sensing & angular control