**Review** · Einstein equations: · parallel transport! $\Delta V^i = -R^i{}_{0j0}\,\Delta x^j \cdot \Delta t = -\Phi_{,ij}\,\Delta x^j \cdot \Delta t$

· Add Newton & Energy-Momentum conservation:

$$\cdot\; G_{\mu\nu} = \bar{R}_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

or

$$R_{\mu\nu} = 8\pi G\, \bar{T}_{\mu\nu}$$

$\bar{A}_{\mu\nu} := A_{\mu\nu} - \frac{1}{2} g_{\mu\nu} A$ (trace-reverse op.)

· Meaning: $\frac{\ddot{V}}{V} = -4\pi G(\rho + 3p)$

· pressure is also a source
· $\rho + 3p$ dictates the Volume evolution

· Linearized Gravity: · $R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[ h_{\nu\alpha,\beta,\mu} - h_{\nu\beta,\alpha,\mu} - h_{\mu\alpha,\beta,\nu} + h_{\mu\beta,\alpha,\nu} \right]$

· invariant under "small" coordinate changes:

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu}\; \xi_{\nu,\mu}$$

· Plane wave solutions: 2 polarizations $h_+, h_\times$

· Effect on 2 separated freely falling particles: $a_x = R^1{}_{001}\,\Delta x = -\frac{\omega^2 h_+}{2}\Delta x$

integrate $\Rightarrow \delta x = \frac{h_+}{2}\Delta x$

· Wave generation in Lorenz gauge: $\bar{h}^{L\mu\nu}{}_{,\nu} = 0 \quad \Rightarrow \Box \bar{h}^L_{\mu\nu} = -16\pi G\, T_{\mu\nu}$

· but wave detection in Transverse Traceless gauge: $[A_{ij}]^{TT} := \left(P_i^k P_j^l - \frac{1}{2} P_{ij} P^{kl}\right) A_{kl}$

$P_{ij} = \delta_{ij} - n_i n_j$

$\vec{n} = \hat{k}$

· Generation of GW: · Radiation field: $L \ll \lambda \ll r$

$$\Rightarrow \bar{h}^L_{\mu\nu} = \frac{4G}{r}\left[\int T_{\mu\nu}\, d^3x\right]^{ret}$$

$$T^{\mu\nu}{}_{,\nu} = 0 \;\Rightarrow\; \bar{h}^L_{ij} = \frac{2G}{r}\frac{d^2}{dt^2} I_{ij}^{ret}$$

$$\Rightarrow h_{ij}^{TT} = \frac{2G}{r}\frac{d^2}{dt^2}\left[I_{ij}^{ret}\right]^{TT} = \frac{2G}{r}\frac{d^2}{dt^2}\left[\mathcal{I}_{ij}^{ret}\right]^{TT}$$

where $I_{ij} = \int d^3x\, \rho\, x^i x^j$ ; $\mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I$

# Energy & Momentum of Gravitational Waves

Recall: • In the Einstein equation

$$G_{\mu\nu}(g_{\mu\nu}) = 8\pi G\, T_{\mu\nu}$$

$\uparrow$

$T_{\mu\nu}$ is only the energy & momentum of matter

• Any self-gravitating energy & momentum from gravitational fields is already included in $G_{\mu\nu}$

(that is why $G_{\mu\nu}$ cannot be linear!)

• In linearized gravity (Lorentz gauge) we have

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu} \qquad \bar{\phantom{x}} \hat{=} \text{ trace reversal}$$

or

$$\Box h_{\mu\nu} = -16\pi G\, \bar{T}_{\mu\nu} \qquad \bar{T}_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T$$

$\Rightarrow$ we need to go to 2nd order.

## So let us go to 2nd order in $h$ or $T$:

Metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (to all orders)

$$h_{\mu\nu} = h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} \quad \text{with } O(h^{(2)}_{\mu\nu}) = O([h^{(1)}_{\mu\nu}]^2)$$

<u>In vacuum</u> $\quad R_{\mu\nu}(g_{\mu\nu}) = 0$ to all orders

$\quad\quad\quad\quad\quad\; R_{\mu\nu}(\eta_{\mu\nu}) = 0$ (Minkowski)

$$R_{\mu\nu}(g_{\mu\nu}) \equiv R_{\mu\nu}(h_{\mu\nu}) = \underbrace{R^{(1)}_{\mu\nu}(h_{\mu\nu})}_{\text{linear in } h_{\mu\nu}!} + \underbrace{R^{(2)}_{\mu\nu}(h_{\mu\nu})}_{\text{quadratic in } h_{\mu\nu}} + \dots$$

$$\Rightarrow 0 = R_{\mu\nu}(h_{\mu\nu}) = R^{(1)}_{\mu\nu}(h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}) + R^{(2)}_{\mu\nu}(h^{(1)}_{\mu\nu}) + \dots$$

$$\underset{\text{linear!}}{=} \underbrace{R^{(1)}_{\mu\nu}(h^{(1)}_{\mu\nu})}_{\text{1st order}} + \underbrace{R^{(1)}_{\mu\nu}(h^{(2)}_{\mu\nu}) + R^{(2)}_{\mu\nu}(h^{(1)}_{\mu\nu})}_{\text{2nd order!}} + \dots$$

$\Rightarrow$ 1st: $R^{(1)}_{\mu\nu}(h^{(1)}_{\mu\nu}) = 0$ ✓

trace-reversed

2nd: $$\boxed{\underbrace{R^{(1)}_{\mu\nu}(h^{(2)}_{\mu\nu})}_{\text{1st order on 2nd order field}} = -\underbrace{R^{(2)}_{\mu\nu}(h^{(1)}_{\mu\nu})}_{\text{2nd order operator on 1st order field}} =: 8\pi G\, \bar{t}_{\mu\nu}}$$

↳ so all we have to do is evaluate $R_{\mu\nu}$ to 2nd order!

$\cdot\, R^{(2)}_{\mu\nu}(h_{\mu\nu}) = \ldots$

<u>to all order</u> $$R_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu,\alpha} - \Gamma^{\alpha}{}_{\alpha\mu,\nu} + \Gamma^{\alpha}{}_{\alpha\gamma}\Gamma^{\gamma}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\nu\gamma}\Gamma^{\gamma}{}_{\alpha\mu}$$

to 2nd order $$= \left[(\eta^{\alpha\lambda} - h^{\alpha\lambda})\Gamma_{\lambda\nu\mu}\right]_{,\alpha} - \left[(\eta^{\alpha\lambda} - h^{\alpha\lambda})\Gamma_{\lambda\alpha\mu}\right]_{,\nu} + \Gamma^{\alpha}_{\alpha\gamma}\Gamma^{\gamma}_{\nu\mu} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\alpha\mu}$$

<u>Note</u>: $\Gamma_{\lambda\nu\mu} \equiv \frac{1}{2}\left(h_{\lambda\mu,\nu} + h_{\nu\lambda,\mu} - h_{\nu\mu,\lambda}\right)$ is <u>exactly</u> 1st order in $h$

$$\Rightarrow \boxed{R^{(2)}_{\mu\nu} = -(h^{\alpha\lambda}\Gamma_{\lambda\nu\mu})_{,\alpha} + (h^{\alpha\lambda}\Gamma_{\lambda\alpha\mu})_{,\nu} + \Gamma^{\alpha}_{\alpha\gamma}\Gamma^{\gamma}_{\nu\mu} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\alpha\mu}}$$

$$\equiv -8\pi G\, \bar{t}_{\mu\nu}$$

<u>done</u>

<u>Caution</u>: $\cdot$ at every point $p$ we can <u>choose</u> $g_{\mu\nu}\big|_p = \eta_{\mu\nu}$ & $\underbrace{g_{\mu\nu,\lambda} = 0}_{\Gamma_{\lambda\nu\mu} = 0}$

$\Rightarrow$ we can achieve $R^{(2)}_{\mu\nu}\big|_p = 0$ by choosing coordinates !?!

$\Rightarrow R^{(2)}_{\mu\nu}$ is <u>not</u> a tensor.

$\Rightarrow$ <u>Non-localizability of Gravitational Energy & Momentum!</u>

Trick: • Average over several wave-length!

$$t_{\mu\nu} \to \langle t_{\mu\nu} \rangle_{\substack{\text{several}\\ \text{wavelengths}}}$$

• avoids non-localizability issue

• permits setting total derivatives to zero: $\langle \frac{\partial}{\partial x^\rho} X \rangle = 0$

• thus also allows partial differentiation ↳ boundary term disappears

$$\langle A\, B_{,\mu} \rangle = -\langle A_{,\mu}\, B \rangle$$

I skip the math here. But in the transverse traceless gauge we get

$$\langle R^{(2)\,TT}_{\mu\nu} \rangle = -\frac{1}{4} \langle (h_{\alpha\beta}^{TT})_{,\mu} (h^{\alpha\beta})_{,\nu} \rangle$$

It's trace is $-\frac{1}{4} \langle h_{\alpha\beta,\mu}\, h^{\alpha\beta,\mu} \rangle = +\frac{1}{4} \langle (\underbrace{\Box h_{\alpha\beta}}_{0}) h^{\alpha\beta} \rangle = 0$

$$\Rightarrow \boxed{t_{\mu\nu} = \frac{1}{32\pi G} \langle h^{TT}_{\alpha\beta,\mu}\, h^{TT\,\alpha\beta}{}_{,\nu} \rangle}$$

For a plane wave:

$$h^{TT}_{\mu\nu} = H_{\mu\nu} \sin k_\gamma x^\gamma \qquad \parallel \hat{e}_z$$

$$t_{\mu\nu} = \frac{1}{32\pi G} \underbrace{k_\mu k_\nu}_{\omega^2 \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}} \underbrace{H_{\alpha\beta} H^{\alpha\beta}}_{2(h_+^2 + h_\times^2)} \underbrace{\langle \cos^2 k_\gamma x^\gamma \rangle}_{\frac{1}{2}}$$

$$\Rightarrow \boxed{t_{\mu\nu} = \underbrace{\frac{\omega^2}{32\pi G} (h_+^2 + h_\times^2)}_{\frac{\pi f^2}{8G}(h_+^2 + h_\times^2)} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu}}$$

# Detecting Gravitational Waves
# Problems 2

## Wave form of a compact binary inspiral

In class we have seen that in the limit of a slowly moving source the gravitational wave strain radiated from a source can be calculated by the famous quadrupole formula

$$h_{\mu\nu}^{TT} = \frac{2G}{r} \ddot{J}_{ij}^{TT}(t-r) \tag{1}$$

where $J_{ij}^{TT}$ is the transverse-traceless component of the reduced quadrupole moment. We have also seen that the energy-momentum pseudo-tensor for gravitational waves is given by

$$t_{\mu\nu} = \frac{1}{32\pi G} \left\langle h_{\alpha\beta,\mu}^{TT} h_{,\nu}^{TT\alpha\beta} \right\rangle , \tag{2}$$

where $\langle ... \rangle$ denotes the average over several wave length. We want to use this to estimate the wave form of a compact binary inspiral to 0th order.

## 1 Kepler motion for a circular binary with masses $m_1$ and $m_2$

Two starts with masses $m_1$ and $m_2$ are orbiting each other in a circular orbit. As usual we define the separation between the stars $a$, the orbital angular frequency $\Omega$, the total mass $M = m_1 + m_2$ and the reduced mass $\mu = (m_1 m_2)/M$. Find the relation between $\Omega$, $M$ and $a$ (Kepler's third law), as well as the orbit of each star. $\vec{x}_1(t)$ and $\vec{x}_2(t)$. (Choose the x-y plane as orbital plane.)

Now find the reduced quadrupole momentum $J_{ij}$ of this binary.

## 2 Total energy emitted

Proof that

$$\begin{aligned} &\textstyle\int_{S^2} d^2\Omega = 4\pi \\ &\textstyle\int_{S^2} n_i n_j d^2\Omega = \frac{4\pi}{3}\delta_{ij} \\ &\textstyle\int_{S^2} n_i n_j n_k n_l d^2\Omega = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) . \end{aligned} \tag{3}$$

Use this to show that

$$\int_{S^2} \dddot{J}_{ij}^{TT} \dddot{J}^{TTij} d^2\Omega = \frac{8\pi}{5} \dddot{J}_{ij} \dddot{J}^{ij} . \tag{4}$$

Now calculate the total power that is emitted in GW by the binary system. Result:

$$P_{\text{emitted}} = \frac{32}{5} \frac{G^4 \mu^2 M^3}{a^5} \tag{5}$$

## 3 The wave form

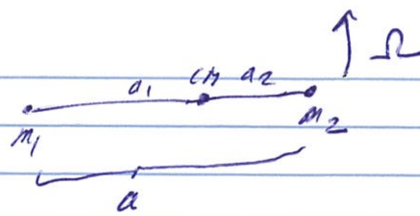
Use energy conservation to find a solution for stars' separation $a(t)$ as a function of time, $a(t) = a_0(1 - t/\tau_0)^{1/4}$. Find the time to merger $\tau_0$ as a function of the gravitational wave frequency $f_{\text{gw}} = 2f = \Omega/\pi$ and the chirp mass $M_{\text{chirp}} = \mu^{3/5} M^{2/5}$. You should get

$$\tau_0 = 3.0 \text{ sec} \left(\frac{100 \text{ Hz}}{f_{\text{gw}}}\right)^{\frac{8}{3}} \left(\frac{M_{\text{sun}}}{M_{\text{chirp}}}\right)^{\frac{5}{3}} \tag{6}$$

For observers on the x-axis (in orbital plane) and on the z-axis (perpendicular to the orbital plane), find the amplitude and frequency for both the plus and cross polarization as a function of time.

## P2

1)




Force equil: $m_1 a_1 \Omega^2 = m_2 a_2 \Omega^2 = \frac{G\mu M}{a^2}$

$M = m_1 + m_2$

$\mu = \frac{m_1 m_2}{M}$

$\mu \cdot M = m_1 \cdot m_2$

i.e. $a_1 m_1 = a_2 m_2$

$$\Rightarrow a \cdot \mu = (a_1 + a_2)\mu = a_1\left(\frac{m_2}{m_2} + \frac{m_1}{m_2}\right)\frac{m_1 m_2}{m_1 + m_2} = a_1 m_1 = a_2 m_2$$

$$\Rightarrow \boxed{\Omega^2 a^3 = GM} \quad \text{Kepler}$$

$$\Rightarrow \vec{x}_1 = a_1 \begin{pmatrix} \cos \Omega t \\ \sin \Omega t \\ 0 \end{pmatrix} \qquad \& \ \vec{x} = a \begin{pmatrix} \cos \Omega t \\ \sin \Omega t \\ 0 \end{pmatrix}$$

$$\vec{x}_2 = -a_2 \begin{pmatrix} \cos \Omega t \\ \sin \Omega t \\ 0 \end{pmatrix}$$

$$\Rightarrow I_{ij} = \int \rho(x)\, x^i x^j d^3x \qquad \text{with } \rho = m_1 \delta^3(x - x_1) + m_2 \delta^3(x - x_2)$$

$$= m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = \underbrace{\left(\frac{m_1 a_1^2}{a^2} + \frac{m_2^2 a_2^2}{a^2}\right)}_{\substack{m_1 a_1 = \mu a \\ \downarrow \\ \mu}} x^i x^j = \mu x^i x^j$$

$$= \mu a^2 \begin{pmatrix} \cos^2 & \sin\cos & 0 \\ \sin\cos & \sin^2 & \\ & 0 & 0 \end{pmatrix}$$

$$I = \mu a^2 \quad \Rightarrow \boxed{\mathcal{J}_{ij} = \mu a^2 \begin{pmatrix} \cos^2 - \frac{1}{3} & \sin\cos & \\ \sin\cos & \sin^2 - \frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix}}$$

$$\mathcal{J}_{ij} = \mu a^2 \begin{pmatrix} \frac{1}{2}\cos + \frac{1}{2} - \frac{1}{3} & \frac{1}{2}\sin & \\ \frac{1}{2}\sin & -\frac{1}{2}\cos + \frac{1}{2} - \frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix}$$

$\cos^2 \Omega = \frac{1}{2}\cos 2\Omega + \frac{1}{2}$

$\sin^2 \Omega = -\frac{1}{2}\cos 2\Omega + \frac{1}{2}$

$\sin\Omega \cos\Omega = \frac{1}{2}\sin 2\Omega$

$$\Rightarrow \dddot{\mathcal{J}}_{ij} = \frac{(2\Omega)^3 \mu a^2}{2} \begin{pmatrix} \cos & \sin & \\ \sin & -\cos & \\ & & 0 \end{pmatrix}$$

2) · $\int d^2\Omega = \underline{\underline{4\pi}}$ ✓

· $\int n_i n_j \, d^2\Omega = 0$ for $i \neq j$ (symmetry)

$\Rightarrow \int n_i n_j \, d^2\Omega = \alpha \delta_{ij}$ $\Rightarrow$ take trace $\Rightarrow 3\alpha = 4\pi$ $\qquad \underline{\alpha = \frac{4\pi}{3}}$

· $\int n_i n_j n_k n_l \, d^2\Omega = 0$ unless indices are pairwise the same.

e.g. $\int n_i^4 \, d^2\Omega = 2\pi \int_0^\pi \cos^4\theta \sin\theta \, d\theta = \frac{4\pi}{5}$

also $\int n_1^4 + n_1^2 n_2^2 + n_1^2 n_3^2 \, d^2\Omega = \int n_i^2 \, d^2\Omega = \frac{4\pi}{3}$

$= \frac{4\pi}{5} \quad \beta \quad \beta \qquad \Rightarrow \underline{\underline{\beta = \frac{4\pi}{15}}}$

where $\int n_i n_j n_k n_l \, d^2\Omega = \beta(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \gamma \delta_{ij}\delta_{kl}\delta_{ik}$

all indices the same $\Rightarrow \gamma = 0$

$\Rightarrow \int J_{ij}^{TT} J_{ij}^{TT} \, d^2\Omega = \int J_{ij} \quad (P_i^k P_j^l - \frac{1}{2} P_{ij} P^{kl}) J_{kl} \qquad P_{ij} = \delta_{ij} - n_i n_j$

$= \int J^{ij} \left[ (\delta_i^k - n_i n_k)(\delta_{jl} - n_j n_l) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta_{kl} - n_k n_l) \right] J_{kl}$

~~$= \int J^{ij} J_{ij} \, d^2\Omega$~~

$= J_{ij} \left( 4\pi - 2\frac{4\pi}{3} + \frac{1}{2}\frac{4\pi}{15} \cdot 2 \right) J^{ij} \quad = 4\pi \frac{6}{15} J_{ij} J^{ij} = \underline{\frac{8\pi}{5} J_{ij} J^{ij}}$

Putting everything to-gether: $\quad \cdot\, h_{ij}^{TT} = \frac{2G}{r}\frac{d^2}{dt^2}\left[\mathcal{I}^{ref}\right]^{TT}$

$$\cdot\; P = \int_{S_2} t^{0i}\, n^i\, r^2 d^2\Omega$$

$$\cdot\; t^{0i} = \frac{1}{32\pi G}\left\langle h_{kj}^{TT\,,0}\, h^{TT\,kj\,,i}\right\rangle$$

$$= \frac{1}{32\pi G}\,\underset{\substack{\uparrow\\ \text{GW freq.}\\ =2\times\text{orbital}\\ \text{frequency}}}{(2\Omega)^2}\left\langle h_{kj}^{TT}\, h^{TT\,kj}\right\rangle$$

$$\cdot\; \left\langle h_{kj}^{TT}\, h^{TT\,kj}\right\rangle = \frac{4G^2}{r^2}(2\Omega)^4\left\langle \left[\mathcal{I}^{ref}\right]^{TT}_{kj}\left[\mathcal{I}^{ref}\right]^{TT}_{kj}\right\rangle$$

$$\cdot\; \mathcal{I}^{ref}_{kj} \underset{\substack{\uparrow\\ \text{no constant parts}}}{=} \frac{\mu a^2}{2}\begin{pmatrix}\cos 2\Omega t & \sin 2\Omega t & 0\\ \sin 2\Omega t & \cos 2\Omega t & 0\\ 0 & 0 & 0\end{pmatrix}_{kj}$$

$$\Rightarrow \mathcal{I}_{kj}\mathcal{I}^{kj} = \frac{\mu^2 a^4}{4}\cdot 2\left(\cos^2 2\Omega t + \sin^2 2\Omega t\right) = \frac{\mu^2 a^4}{2}$$

$$\Rightarrow \int \mathcal{I}^{TT}_{kj}\mathcal{I}^{TT\,kj}\, d^2\Omega = \frac{8\pi}{2\cdot 5}\mu^2 a^4 = \frac{4\pi}{5}\mu^2 a^4$$

$$\Rightarrow \boxed{P = \frac{1}{32\pi G}(2\Omega)^6\cdot 4G^2\cdot\frac{8\pi}{5}\frac{\mu^2 a^4}{2} = \frac{32}{5}\, G\mu^2 a^4\,\Omega^6}$$

$$\cdot\; \Omega^2 = \frac{GM}{a^3} \Rightarrow \Omega^6 = \frac{G^3M^3}{a^9}$$

$$\Rightarrow \boxed{P = \frac{32}{5}\frac{G^4\mu^2M^3}{a^5}}$$

3) <u>orbital energy</u> : $\frac{1}{2}\mu a^2 \Omega^2 - \frac{GM\mu}{a} \underset{\substack{\text{Virial}\\ \text{or}\\ \text{Kepler}}}{=} -\frac{GM\mu}{2a}$

$$P = \dot{E} = \frac{GM\mu}{2a^2}\dot{a} \overset{!}{=} -\frac{32}{5}\frac{G^4\mu^2M^3}{a^5}$$

$$\Rightarrow \boxed{\dot{a}a^3 = -\frac{64}{5}G^3\mu M^2}$$

$$\Rightarrow a(t) = a_0\left(1-\frac{t}{\tau}\right)^{\frac{1}{4}} \quad \Rightarrow \dot{a}a^3 = -\frac{1}{4\tau}a_0^4$$

$$\Rightarrow \frac{1}{\tau} = \frac{256}{5}\frac{G^3\mu M^2}{a_0^4 c^5}$$

use $\Omega^2 = \frac{GM}{a^3}$ $\Rightarrow a^4 = G^{\frac{4}{3}} M^{\frac{4}{3}} \Omega^{-8/3}$ ; $\Omega = 2\pi f_{orb} = \pi f_{gw}$

$$\Rightarrow \boxed{\tau = \frac{5\pi^{-8/3}}{256} f_g^{-8/3} M_{ch}^{-\frac{5}{3}} G^{-\frac{5}{3}} c^5}$$

$$\boxed{\tau = 3\,\text{sec}\left(\frac{100\,\text{Hz}}{f_g}\right)^{+8/3} \cdot \left(\frac{M_\odot}{M_{ch}}\right)^{5/3}}$$

$$M_{ch}^5 = \mu^3 M^2$$

$f_{gw}(t)$ : $(\pi f_g)^2 = \Omega^2 = \frac{GM}{a^3}$ $\qquad a^2 = \frac{G^{\frac{2}{3}} M^{\frac{2}{3}}}{(\pi f_g)^{\frac{4}{3}}}$

$$f_g^2 = \frac{GM}{\pi^2 a^3}$$

$$f_g = \frac{\sqrt{GM}}{\pi} a^{-\frac{3}{2}} = f_{gw}^0 \cdot \left(1 - \frac{t}{\tau}\right)^{-\frac{3}{8}}$$

$$\phi = 2\pi \cdot \int f_g \, dt \qquad + \text{const}$$

$$\Rightarrow \phi(t) = -2\pi f_{gw}^0 \cdot \tau \cdot \frac{8}{5} \left(1 - \frac{t}{\tau}\right)^{\frac{5}{8}}$$

$$\Rightarrow h_{ij}^{TT} = \frac{d^2}{dt^2} \cdot \frac{2G}{r} \frac{\mu a^2(t)}{2} \frac{1}{c^4} \left[\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ \sin\phi & -\cos\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]^{TT}$$

$$\simeq \frac{G\mu}{r} (2\pi f_{gw}(t))^2 a^2(t) \frac{1}{c^4} \left[ \qquad \right]^{TT}$$

$$= -\frac{G\mu}{r} 4\pi^2 \frac{G^{\frac{2}{3}} M^{\frac{2}{3}}}{\pi^{\frac{4}{3}}} \frac{f_{gw}^2}{f_{gw}^{4/3}} \frac{1}{c^4} \left[ \qquad \right]^{TT}$$

$$= -4 \cdot \pi^{\frac{2}{3}} \frac{(GM_{ch})^{5/3}}{r} f_{gw}^{\frac{2}{3}} \frac{1}{c^4} \left[\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ \sin\phi & -\cos\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]^{TT}$$

e.g. $M_{ch} \sim 20 m_{sun}$, $f \sim 300\,\text{Hz}$, $r = 0.84\,\text{pc}$ $(z \simeq 0.2)$

$$\Rightarrow \underline{\underline{h \sim 10^{-21}}}$$

<u>Up to what frequency can we believe this?</u>

- we implicitly assumed $\ddot{\phi} \ll \dot{\phi}^2$ (quasi-circular orbit)
- also once $a(t) \approx \# \frac{2GM}{c^2}$ $\Rightarrow$ lin. gravity no longer good...

$\hookrightarrow$ <u>Note:</u> Termination condition breaks chirp mass degeneracy $\Rightarrow$ we can find $m_1, m_2$

## Absorption of GW in matter

Setup: Assume plane incident wave: $h_{\mu\nu} = A_{\mu\nu}\, e^{ik_\alpha x^\alpha}$

↳ e.g. $h_+$ pol.

- ⇒ this is in TT gauge.
- We in Lorentz gauge:

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}$$

⇒ Since we are only looking for plane waves we can use the TT projection:

$$\left[\Box \bar{h}_{\mu\nu}\right]^{TT} = -16\pi G\, T_{\mu\nu}^{TT}$$

For scatter:

$$\Rightarrow \underbrace{\Box h_{\mu\nu}^{TT}}_{\text{scatter amplitude}} = -16\pi G\, \underbrace{T_{\mu\nu}^{TT}}_{\text{induced quadrupole moment}}$$

- For freely falling dust: $T_{\mu\nu} \sim \rho\, v_i v_k \sim O(h_0^2)$ ⇒ 2nd order!

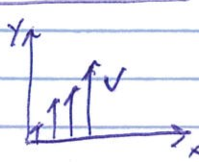
  $v \propto \dot{h}$ ⇒ no $T_{\mu\nu}^{TT}$ due to GW to lin. order...

- For a perfect fluid: no resistance to shearing ... same answer as for dust...

- For material with viscosity $\eta$:

  Def: Rate of shear: $\boxed{\sigma_{ij} := \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) - \frac{1}{3}\, v^{\ell}{}_{,\ell}\, \delta_{ij}}$ Units: Hz

  Example: $v_x = v_z = 0;\ v_y = \alpha x \Rightarrow \underline{\sigma_{12} = \frac{1}{2}\alpha}$




Viscosity $\eta$ is defined as resistance to shear:

$$\boxed{\delta T_{ij} = -2\eta\, \sigma_{ij}}$$

## What is the rate of shear for a passing GW?

$$\delta x^i = \frac{1}{2} h_{ij}^{TT} x^j \quad \Rightarrow \quad v^i = \frac{1}{2} \dot{h}_{ij}^{TT} x^j$$

Def of Riemann:

$$\boxed{v_{i,j} = \frac{1}{2} \dot{h}_{ij}^{TT}}$$

$$a^i = R^i{}_{0 0 j} x^j = -\frac{\omega^2}{2} h_+$$

TT

$$\Rightarrow \boxed{\sigma_{ij} = \frac{1}{2} \dot{h}_{ij}^{TT}} \Rightarrow \delta T_{ij}^{TT} = -\eta \dot{h}_{ij}^{TT}$$

$$\Rightarrow \left(-\partial_t^2 + \partial_z^2\right) h_{ij}^{TT} = +16\pi G \eta \, \partial_t h_{ij}^{TT}$$

$$\left(\omega^2 - k^2\right) = -i\omega \cdot 16\pi G \eta$$

$$k^2 = \omega^2 + i\omega\, 16\pi G \eta$$

$$k = \omega \sqrt{1 + \frac{i 16\pi G \eta}{\omega}} \cong \omega + i 8\pi G \eta + O\left(\frac{(G\eta)^2}{\omega}\right)$$

$$\Rightarrow h_{ij}^{TT} \propto e^{-i\omega t + ikz} \quad \underline{\underline{\propto e^{-8\pi G \eta z}}}$$

$$\Rightarrow \text{attenuation length: } \underline{\underline{\ell_{ATT} = \frac{c^3}{8\pi G \eta}}}$$

$$= 3.14 \cdot 10^9 \frac{1 \text{ pascal-sec}}{\eta}$$

$\cdot$ Hubble distance

$$= 3.14 \cdot 10^9 \text{ Hubble distances} \times \frac{1 \text{ pa-sec}}{\eta}$$

e.g. molten glass: $\eta \simeq 10^3$ pa-sec ...

(Similar: for elastic materials: Shear: $\Sigma_{ij} = \int \sigma_{ij}\, dt = \frac{1}{2} h_{ij}^{TT}$

$$\delta T_{ij} = -2\mu \, \Sigma_{ij} = -\mu h_{ij}^{TT}$$

shear modulus

$$\Rightarrow \Box h_{ij}^{TT} = 16\pi G \mu h_{ij}^{TT}$$

$$\Rightarrow v_{phase} = \frac{\omega}{k} = 1 + 8\pi G \frac{\mu}{k^2}$$

$$v_{group} = \frac{\partial \omega}{\partial k} = 1 - 8\pi G \frac{\mu}{k^2} \quad (c c) \quad )$$

This concludes our discussion of gravitational waves in General Relativity. We now swich to

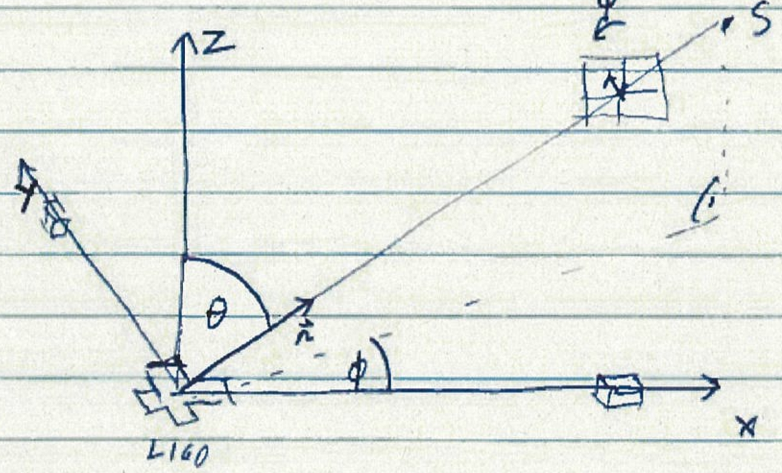
Design constraints for Gravitational Wave detectors

## Detecting GW – the interferometer antenna pattern and frequency response

• We work in TT gauge, because test masses (mirrors) stay at fixed coordinate positions

- certainly true above suspension resonances
- below resonances other noise dominates anyway

We are interested in the interferometer response to a plane wave from an arbitrary direction




We work with detector coordinates $x, y, z$ & source angles $\theta, \phi, \psi$

(For this we ignore any source alignment angles like inclination)

In source coordinates & in TT gauge we have

$$h_+(t,\vec{r}) = h_+(t+z)$$
$$h_\times(t,\vec{r}) = h_\times(t+z)$$

$$h_0 = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_\times & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ (only the spatial part of $h_{\mu\nu}$)

coordinate transform: $\vec{x}_0$ source coordinates
$\vec{x}$ detector coordinates

$$x = R^{\dagger} x_0 \qquad (\text{i.e. } d\vec{x} = R^T d\vec{x}_0)$$

$$\Rightarrow \boxed{h = R^{\dagger} h_0 R}$$

(since $dx^{\dagger} h\, dx = dx_0^T h_0\, dx_0$)

with $R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & \\ -\sin\phi & \cos\phi & \\ & & 1 \end{pmatrix}$ ; $R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$

$$R_z(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & \\ -\sin\psi & \cos\psi & \\ & & 1 \end{pmatrix}$$

$$\Rightarrow \boxed{R = R_z(\psi)\, R_y(\theta)\, R_z(\phi)}$$

$\Rightarrow h_{ij}$ has all 9 components

$$\Rightarrow \boxed{h_{ij} = h_{ij}(t + \vec{n}\cdot\vec{r})}$$

<u>Light propagation in the detector</u>

- We need the null-trajectories for

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij}(t + \vec{n}\cdot\vec{r}))\, dx^i dx^j$$

$\Rightarrow$ for light traveling along the x-direction

$$dt^2 = \left[1 + h_{xx}\left(t + n_x \cdot x \cdot c\right)\right] dx^2$$

$\Rightarrow$ coordinate velocity $v_x = \frac{dx}{dt} = \pm \frac{1}{\sqrt{1 + h_{xx}(t + n_x \cdot x)}} = \pm\left(1 - \frac{1}{2} h_{xx}(t + n_x \cdot x)\right)$

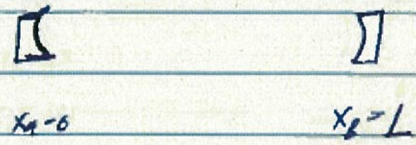
$\Rightarrow$ trajectory of photon $x(t) = x_0 + \int_{t_0}^{t} v_x(t', x')\, dt'$

more convenient in inverse form: $\frac{1}{v_x} = \pm\left(1 + \frac{1}{2} h_{xx}(t + n_x \cdot x)\right)$

$$t(x) = t_0 + \int_{x_0}^{x} \frac{dx'}{v_x(t', x')}$$

$$\Rightarrow \boxed{t(x) = t_1 + x - x_0 + \frac{1}{2}\int_{x_0}^{x} h_{xx}\left(t_0 + x' - x_0 + n_x x'\right) dx'}$$

## Photon round-trip time




· unperturbed propagation time $T = L$

· forward propagation, $t$ = end of trip

$$T_1(t) = T + \frac{1}{2}\int_0^L h_{xx}(t - L + x + n_x \cdot x)\, dx$$

· similarly, the return trip takes

$$T_2(t) = T + \frac{1}{2}\int_0^L h_{xx}(t - x + n_x \cdot x)\, dx$$

$$\Rightarrow T_{r.t.}(t) = T_2(t) + T_1(t-T) \quad (\text{to 1st order } T_2(t) = T)$$

$$\Rightarrow \boxed{\delta T_{r.t.}(t) = \frac{1}{2}\int_0^L \left[h_{xx}(t - 2T + x + n_x \cdot x) + h_{xx}(t - x + n_x \cdot x)\right] dx}$$

"change in round-trip time $\propto$ integral of strain along unperturbed light path"

· This can be expressed in Fourier domain, e.g. $h_{xx}(v) = h_{xx_0} e^{-i\omega v}$

$$\underbrace{\Rightarrow \boxed{\delta T_{r.t.}(\omega) = \frac{1}{-2i\omega}\left[\frac{1 - e^{i\omega(1-n_x)T}}{(1-n_x)} - e^{2i\omega T}\,\frac{1 - e^{-i\omega(1+n_x)T}}{(1+n_x)}\right]}}_{\delta T_{r.t.}(\omega) \;=\; D(\omega, n_x)} \cdot \underbrace{h_{xx_0} e^{-i\omega t}}_{h_{xx}(\omega)}$$

$$\delta T_{r.t.}(\omega) = D(\omega, n_x) \cdot h_{xx}(\omega)$$

Similarly, for the other arm we get

$$\delta T^y_{r.t.}(t) = \frac{1}{2}\int_0^L \left[ h_{yy}(t-2T+y+n_y\cdot y) + h_{yy}(t-y+n_y y) \right] dy$$

$$\delta T^y_{r.t.}(\omega) = \frac{1}{2i\omega}\left[ \frac{1-e^{i\omega(1-n_y)T}}{(1-n_y)} - e^{2i\omega T}\frac{1-e^{-i\omega(1+n_y)T}}{(1+n_y)} \right] h_{xx}(\omega)$$

$$\delta T^y_{r.t.}(\omega) = D(\omega, n_y)\, h_{yy}(\omega)$$

· For the travel time difference, we thus get

$$\Delta\delta T(t) = \frac{1}{2}\int_0^L \left[ h_{xx}(t-2T+\xi+n_x\xi) - h_{yy}(t-2T+\xi+n_y\cdot\xi) + h_{xx}(t-\xi+n_x\xi) - h_{yy}(t-\xi+n_y\cdot\xi) \right] d\xi$$

$$\boxed{\Delta\delta T(\omega) = D(\omega, n_x)\, h_{xx}(\omega) - D(\omega, n_y)\, h_{yy}(\omega)}$$

response of Michelson interferometer (no arm cavities)

· This results in a laser light phase difference $\Delta\phi$ ($\omega_0$: laser freqency)

$$\boxed{\Delta\phi = \omega_0 \cdot \Delta\delta T(\omega)}$$

· Remarks: · at D.C. ($\omega=0$): $\boxed{D(\omega, .) \underset{\omega\to 0}{\to} T}$ $\Rightarrow \boxed{\Delta\delta T = T(h_{xx} - h_{yy})}$

· But at A.C. the interferometer is sensitive to the integral of the strain along the light path. (Obvious in TT-gauge, but true in any gauge)

· In contrast, mirror displacement noise only enters at discrete times $N\cdot T$ (reflections).

## Antenna response of a Michelson Interferometer (no arm cavities)

$$\Delta\delta T(\omega) = \sum_{ij} \left[ D(\omega, n_x)\, \hat{e}_{x_i} \hat{e}_{x_j} - D(\omega, n_y)\, \hat{e}_{y_i} \hat{e}_{y_j} \right] h_{ij}(\omega)$$

$$= \sum_{ij} \left[ D(\omega, n_x)\, (R\hat{e}_x)_i (R\hat{e}_x)_j - D(\omega, n_y)\, (R\hat{e}_y)_i (R\hat{e}_y)_j \right] h_{0ij}(\omega)$$

↑ in source frame

where $n_x = n_x(\theta,\phi) = \sin\theta\cos\phi$

$n_y = n_y(\theta,\phi) = \sin\theta\sin\phi$

$R = R_z(\psi)\, R_y(\theta)\, R_z(\phi)$

at D.C.

$$\Delta\delta T = T \sum_{ij} \left[ (R\hat{e}_x)_i (R\hat{e}_x)_j - (R\hat{e}_y)_i (R\hat{e}_y)_j \right] h_{0ij}$$

$$= 2T \left[ F_+(\psi,\theta,\phi)\, h_+ + F_\times(\psi,\theta,\phi)\, h_\times \right]$$

Note: • $F_\times(\psi,\theta,\phi) = F_+(\psi + \frac{\pi}{4}, \theta, \phi)$

• $F_+(0,0,0) = 1$

• ⚠ - At non-zero frequency the antenna response does not factorize into "$D(\omega)$" * "$F_{+,\times}$"

- Instead, each arm gets its own frequency response $D(\omega, n_{x,y})$

- This becomes important for LIGO at several kHz

• Fabry-Perot arm cavity response is not included here.

# Noise Limitations of Laser Interferometers



**Shot noise:** Naive picture: random arrival of photons!

Note: define $x(t) = \sum_i \delta(t - t_i)$ for random arrival times $t_i$

$$\Rightarrow \langle x(t) \rangle := \frac{1}{T}\int dt\, x(t) = \frac{1}{T} \sum_{i;\, t_i \in [0,T]} =: \frac{\#}{T} = \underline{\underline{n}} \text{ average rate}$$

$$\Rightarrow \langle x(t)\, x(t+\Delta t) \rangle = \Big\langle \sum_{i,j} \delta(t-t_i)\, \delta(t - t_j + \Delta t) \Big\rangle$$

$$= \Big\langle \sum_i \delta(t-t_i)\, \delta(t-t_i+\Delta t) \Big\rangle + \underbrace{\Big\langle \sum_i \sum_{i \neq j} \delta(t-t_i)\, \delta(t-t_j+\Delta t) \Big\rangle}_{\text{chance of random coincidence} \to 0}$$

$$= \sum_{i;\, t \in [0,T]} \frac{1}{T}\, \delta(\Delta t)$$

$$= n\, \delta(\Delta t)$$

$$\Rightarrow \langle x(f)\, x^*(f') \rangle = \iint e^{2\pi i f t} e^{-2\pi i f' t'} \langle x(t)\, x(t') \rangle\, dt\, dt' = \int dt\, e^{2\pi i (f-f')t}\, n$$

$$= n\, \delta(f - f')$$

$\Rightarrow$ 2-sided Power Spectral Density $S^2_{xx}(f) = n$

one-sided $\Rightarrow S^1 = 2 S^2$

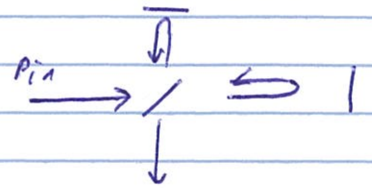
$\Rightarrow$ <u>Amplitude spectral density:</u> $\underline{S^1_x = \sqrt{S^1_{xx}} = \sqrt{2n}}$

<u>for Power:</u> $P = h\nu x \quad \Rightarrow \quad \boxed{S^1_P = \sqrt{2 (h\nu) \langle P \rangle}}$

<u>for current:</u> $I = e x \quad \Rightarrow \quad \boxed{S^1_I = \sqrt{2 e I}}$

Photo-effect: 1 electron per photon

## The Simple Michelson interferometer




for small GW frequencies ($f_{gw} \ll \frac{1}{T}$) ← light 1-way travel time

Light travel time difference:

$$\Delta\delta T = 2T\left[F_+ \cdot h_+ + F_\times h_\times\right]$$

Convention: Field amplitude A in $\sqrt{\text{Watt}}$ : $|A|^2 = \text{Power}$

Definition: Differential arm length $\boxed{x_{DARM} := L_x - L_y} = \frac{c}{2}\Delta\delta T = L\left[F_+ h_+ + F_\times h_\times\right]$

Field at anti-symmetric (AS) port:

$$\psi_{AS} = \psi_i \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\left[e^{i2kL_x} - e^{i2kL_y}\right] = \sqrt{P_{in}} \cdot \frac{1}{2}\underbrace{e^{i2kL}}_{\sim 1}\left[e^{ikx_D} - e^{-ikx_D}\right]$$

2 passes through BS; $\psi_i^2 = P_{in}$

$$= \sqrt{P_{in}} \cdot i \sin(kx_{DARM})$$

⇒ Power on DC-PD: $P_{AS} = |\psi_{AS}|^2 = P_{in} \cdot \sin^2(kx_D)$

⇒ Position sensitivity: $\frac{\partial P_{AS}}{\partial x_D} = P_{in} \cdot 2k \sin(kx_D)\cos(kx_D)$

Where on the fringe shold we operate? Shot noise! $S_P' = \sqrt{2(h\nu) \cdot P_{AS}}$

$$= \sqrt{2h\nu P_{in}} \cdot \sin(kx_D)$$

⇒ Displacement sensitivity: $S_x' = S_P' \cdot \frac{\partial x_D}{\partial P}$

$$= \sqrt{\frac{2h\nu}{P_{in}}}\,\frac{1}{2k\cos(kx_D)}$$

⇒ Close to fringe: $\boxed{S_x' = \frac{1}{k}\sqrt{\frac{h\nu}{2P_{in}}}}$ ← Power incident to BS

## Radiation Pressure noise

Also in the simple picture: each photon reflection off the end mirror transfers the momentum $\frac{2h\nu}{c}$

• The end mirror sees a force $F(t) = \frac{2h\nu}{c} \underbrace{x(t)}_{\sum_i \delta(t-t_i)}$

⇒ Force noise amplitude spectral density:

$$S'_F = \sqrt{2\left(\frac{2h\nu}{c}\right)^2 \cdot n} = \sqrt{\frac{8h\nu}{c^2} P_{end}} = \sqrt{\frac{4h\nu}{c^2} P_{in}}$$

$P_{end} = \frac{1}{2} P_{in}$

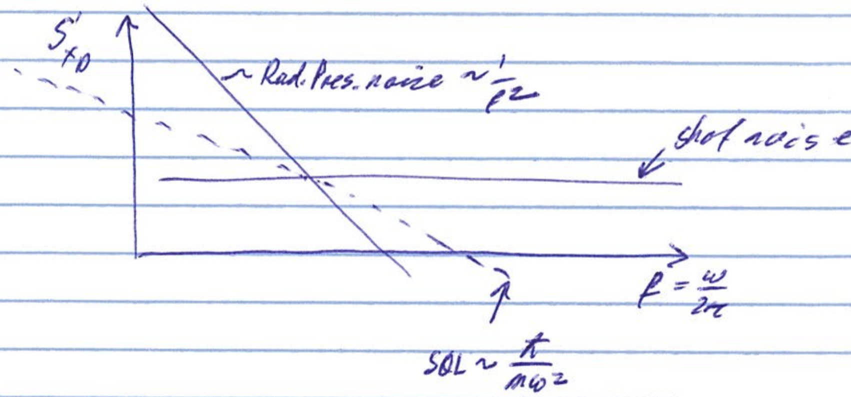
$\omega$: audio freq.

$$\Rightarrow S'_x = \frac{1}{m\omega^2} S'_F = \frac{1}{m\omega^2}\sqrt{\frac{4h\nu}{c^2} P_{in}}$$

• Both mirrors see uncorrelated noise ⇒ sum the power:

⇒ Radiation pressure noise in $x_{DARM} = L_x - L_y$:

$$\boxed{S'_{x_D}(\omega) = \frac{2}{m\omega^2 c}\sqrt{2h\nu P_{in}}}$$




Standard quantum limit (SQL): at each frequency: for some $P_{in}$ the sum of radiation pressure & shot noise has a minimum ⇒ SQL.