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Review

- **Special Relativity:** • $c = \text{const}$ ⇒ Lorentz transform $\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$

$\gamma = \frac{1}{\sqrt{1-v^2}}$

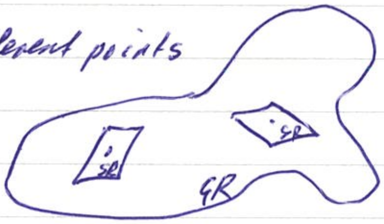
  - Special Relativity in uniformly accelerated frame

⇒ space-time interval:

$$-\Delta\tau^2 = -(1 + a\bar{x})^2 d\bar{t}^2 + d\bar{x}^2 \Rightarrow g_{\alpha\beta} = \begin{pmatrix} -(1+a\bar{x})^2 & 0 \\ 0 & 1 \end{pmatrix}$$

- **General Relativity:** • locally: special Relativity
  - tidal forces relate different points

⇒ Gravity is Geometry



- Geodesic equation:

$$\nabla_{\dot\gamma} \dot\gamma = 0 \Leftrightarrow \ddot\gamma^\mu + \Gamma^\mu_{\alpha\beta} \dot\gamma^\alpha \dot\gamma^\beta = 0$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\nu\beta,\alpha} + g_{\alpha\nu,\beta} - g_{\alpha\beta,\nu} \right)$$

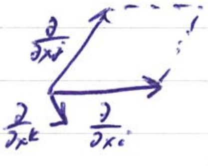
- Parallel transport $\Leftrightarrow \nabla_{\dot\gamma} v = 0 \Leftrightarrow \dot v^\mu + \Gamma^\mu_{\alpha\beta} \dot\gamma^\alpha v^\beta = 0$

Notation: $v^\mu{}_{;\nu} \dot\gamma^\nu = 0$

- in 2D: Gauss Curvature ⇔ Parallel transport along closed curve γ

in 3D: Riemann Curvature: just pick 2-D sub-plane!

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}$$



$$\Rightarrow R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\nu\beta,\mu} - \Gamma^\alpha{}_{\mu\beta,\nu} + \Gamma^\alpha{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\sigma} \Gamma^\sigma{}_{\mu\beta}$$

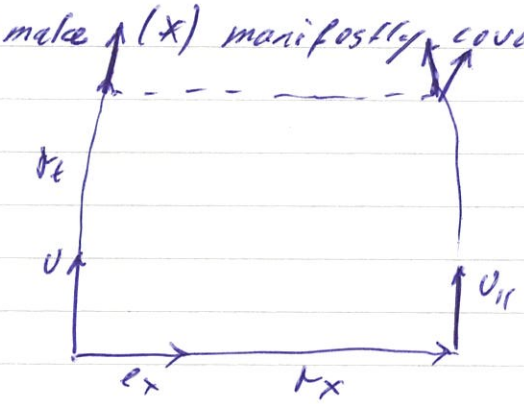
Einstein equation

· We start in a regime where Newton is valid: $\Delta\phi = 4\pi G \rho$ (*)

$a = -\nabla\phi$

$\Rightarrow p = 0$, $v \ll c$, $\phi \ll c^2$

We want to make (*) manifestly covariant:



· start with 4-velocity; at rest w.r.t. to the surrounding dust ($\rho, p=0$)

Compare: (spatial transport $\parallel e_x$ after time evolution)

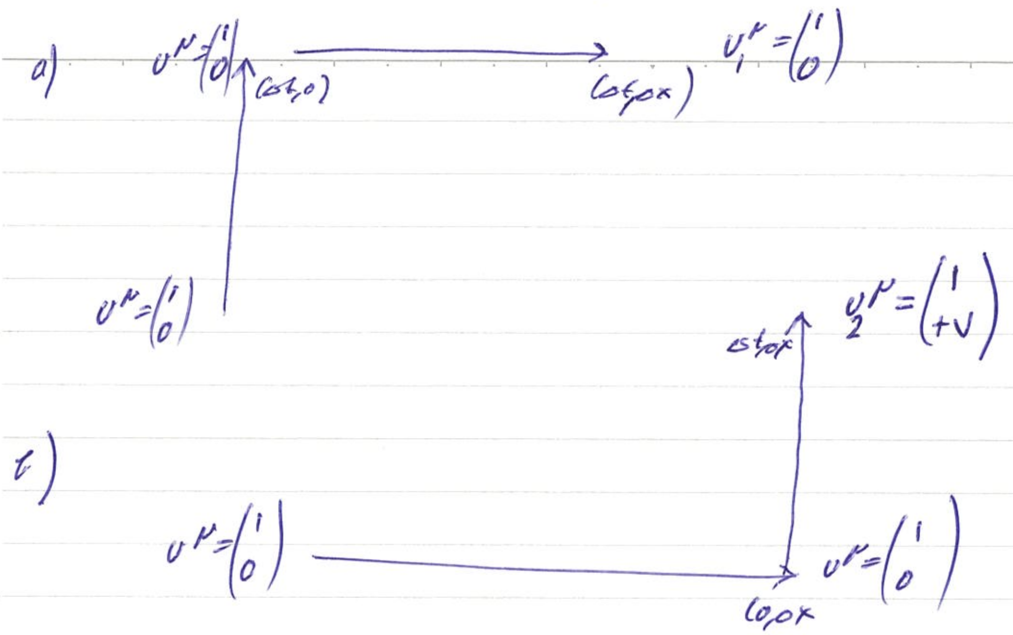
$-$

(time evolution after spatial transport)

$$\Rightarrow \boxed{\Delta v^\mu = -\Delta x \cdot \Delta t \cdot R^\mu{}_{010}}$$

$$= R^\mu{}_{001} \Delta x \Delta t$$

· Now we do the same thing with Newton:



where $v^i = a^i \cdot \Delta t = -\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^1}\phi \cdot \Delta t \cdot \Delta x^1$

$$\Rightarrow \Delta v^\mu = v_1^\mu - v_2^\mu = \begin{pmatrix}0\\-v^i\end{pmatrix} = \begin{pmatrix}0\\ \Delta t \Delta x^1 \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^1}\phi\end{pmatrix}$$

$$\Rightarrow \boxed{\text{We conclude: } \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\phi = R^i{}_{0j0}}$$

i.e. the Riemann ~~cur~~ curvature <u>exactly</u> encodes the <u>physical aspect</u> of the gravitational field; tidal effects!

Moreover: $\Delta\phi = \sum_{i=1,2,3} R^i{}_{0i0} \equiv R^\mu{}_{0\mu 0}$

$=: R_{00}$ (Ricci curvature, trace of Riemann curvature

$$\Rightarrow \boxed{R^\mu{}_{0\mu 0} = 4\pi G \rho}$$

$p = 0, v \ll c, \phi \ll 1$

Ricci Curvature

We can sum over a pair of indices (or contract) the Riemann Curvatu

Ricci $R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$ (just as we did before)

Note: $R^{\alpha}{}_{\beta\gamma\nu}$ has a number of symmetries. As a result the Ricci curvature is the only non-trivial contraction of the Riemann tensor (up to a sign).

Ricci-scalar: trace of Ricci-curvature: $R = R^{\mu}{}_{\mu} = R^{\alpha\mu}{}_{\alpha\mu}$

$\Rightarrow$ We thus have for $p=0, v \ll c, \phi \ll 1$: $R_{00} = 4\pi G T_{00}$

$\uparrow\;\uparrow$
ρ Energy-momen tensor

$\Rightarrow$ 1st guess: $R_{\mu\nu} = 4\pi G T_{\mu\nu}$ $\forall \mu,\nu$!

Problem: Energy-momentum conservation: $T^{\mu\nu}{}_{;\nu} = 0$

(recall E.M. $j^{\mu}{}_{;\mu} = 0 \Leftrightarrow$ charge conserv

$\Rightarrow R^{\mu\nu}{}_{;\nu} = 0$

identity that is always true $\rightarrow \|$

$\frac{1}{2} R_{;\mu} = 0 \Rightarrow T_{,\mu} = 0 \Rightarrow$ The trace of $T^{\mu\nu}$ is constant over space & time!

$\Rightarrow \rho - 3p = \text{const}$ ↯

obviously wrong...

Solution: Einstein-Curvature

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$\Rightarrow \boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}}$$ Einstein equation

normalize with Newton

Note: Non-linear $\Leftrightarrow$ self-gravity!

The meaning of Einstein's eqn.

We saw $R_{00} = 4\pi G(\rho \qquad)$ for $p=0$

now ⇒ $R_{00} = 4\pi G(\rho + 3p)$ for $p \neq 0$

Why? : ⌜ $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$

⇒ <u>trace</u>: $-R = 8\pi G T$

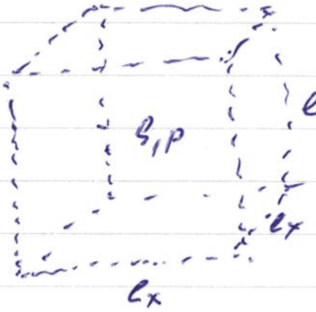
$$\Rightarrow R_{\mu\nu} = 8\pi G \left[T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right]$$

$$\Rightarrow R_{00} = 4\pi G [\rho + 3p] \qquad ; T = -\rho + 3p \; ⌟$$

⇒ a) Pressure also acts as a source for gravity!

Example

- Imagine a box, with its surface traced out by test particles, all initially at rest & freely floating:



$$V = l_x \cdot l_y \cdot l_z$$

initial conditions: $\dot{l}_x = 0$
$\dot{l}_y = 0$
$\dot{l}_z = 0$

- There might be some matter with ρ & p in that box.

Tidal forces:

$$\ddot{l}_x = -R^1{}_{010}\, l_x$$
$$\ddot{l}_y = -R^2{}_{020}\, l_y$$
$$\ddot{l}_z = -R^3{}_{030}\, l_z$$

$$\Rightarrow \dot{V} = \dot{l}_x l_y l_z + l_x \dot{l}_y l_z + l_x l_y \dot{l}_z \underset{\text{initially}}{=} 0$$

$$\ddot{V}\Big|_{t=0} = \ddot{l}_x l_y l_z + l_x \ddot{l}_y l_z + l_x l_y \ddot{l}_z$$

$$= l_x l_y l_z \left(\frac{\ddot{l}_x}{l_x} + \frac{\ddot{l}_y}{l_y} + \frac{\ddot{l}_z}{l_z} \right) = -V\left(R^1{}_{010} + R^2{}_{020} + R^3{}_{030}\right)$$

$$= -V R_{00}$$

$$= -4\pi G(\rho + 3p)\, V$$

$$\Rightarrow \boxed{\frac{\ddot{V}}{V} = -4\pi G(\rho + 3p)}$$

true for any such box! } equivalent to Einstein equation!

$\Rightarrow$ 6) "The expansion of the volume of any set of particles initially at rest is proportional to $-(\rho + 3p)$"

Electro-Magnetism vs General Relativity

	EM	GR
Local symmetry:	Phase rotations (charge)	Lorentz boosts (~~displacement~~ Boost & rotation
	$\psi_i \to e^{iq_i\varphi(x)}\psi_i$	$V^\mu \to \Lambda^\mu{}_\nu(x)\, V^\nu$
	local! i.e. $\varphi(x)$	local! i.e. $\Lambda^\mu{}_\nu(x)$
Covariant derivative to compare different locations:	gauge-covariant derivative	(Levi-Civita) covariant derivative
	$D_\mu\psi_i \equiv (\partial_\mu - iq_iA_\mu)\psi_i$	$\nabla_\mu \frac{\partial}{\partial x^\nu} = \Gamma^\lambda{}_{\mu\nu}\frac{\partial}{\partial x^\lambda}$ $\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\lambda}v^\lambda$
Physical field (gauge field)	$[D_\mu, D_\nu]\psi_i = -iq_i[\partial_\mu A_\nu - \partial_\nu A_\mu]\psi_i$ $= -iq_i\, \underset{\uparrow}{F_{\mu\nu}}\, \psi_i$ $\begin{pmatrix} 0 & E_x & E_y & E_z \\ & 0 & B_z & -B_{\cancel{x}} \\ \text{A.S.} & & 0 & B_x \\ & & & 0 \end{pmatrix}$	$[\nabla_\mu, \nabla_\nu]\frac{\partial}{\partial x^\alpha} = \underset{\uparrow}{R^\beta{}_{\alpha\mu\nu}}\frac{\partial}{\partial x^\beta}$ tidal gravity terms!

Gravitational Waves in

Linearized Gravity

- We want to know what type of wave solutions General Relativity supports. In other words, we are looking for solutions to Einstein's equation in the form:

  - $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; with $|h_{\mu\nu}| << 1$ (linearized Gravity)
  - $h_{\mu\nu} = A_{\mu\nu} \cdot e^{i(-\omega t + \vec{k}\cdot\vec{x})} = A_{\mu\nu}\, e^{i k_\mu x^\mu}$; $k^\mu = \begin{pmatrix} \omega \\ 0 \\ 0 \\ k \end{pmatrix}$; $k_\mu = \begin{pmatrix} -\omega \\ 0 \\ 0 \\ k \end{pmatrix}$

i.e. we pick a plane wave moving in +z direction

- Christophel symbol:

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}\left(g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda} \right)$$

$$\downarrow \qquad \downarrow$$

$$\underbrace{\frac{1}{2}\eta^{\alpha\lambda}}_{\substack{\text{only } 0^{th} \\ \text{order} \\ \text{needed}}} \underbrace{\left(h_{\lambda\nu,\mu} + h_{\mu\lambda,\nu} - h_{\mu\nu,\lambda} \right)}_{\text{is already 1st order in } h}$$

$\Rightarrow \Gamma^{\alpha}{}_{\mu\nu}$ 1st order in h

- Riemann Curvature

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\nu\beta,\mu} - \Gamma^{\alpha}{}_{\mu\beta,\nu} + \underbrace{\Gamma\,\Gamma - \Gamma\,\Gamma}_{\text{2nd order in } h!}$$

$$= \frac{1}{2}\Big[h^{\alpha}{}_{\beta,\nu,\mu} + h_{\nu}{}^{\alpha}{}_{,\beta,\mu} - h_{\nu\beta}{}^{,\alpha}{}_{,\mu}$$

$$- h^{\alpha}{}_{\beta,\mu,\nu} - h_{\mu}{}^{\alpha}{}_{,\beta,\nu} + h_{\mu\beta,}{}^{,\alpha}{}_{,\nu} \Big]$$

$=0$

$$= \frac{1}{2}\Big[h_{\nu}{}^{\alpha}{}_{,\beta,\mu} - h_{\nu\beta}{}^{,\alpha}{}_{,\mu}$$

$$- h_{\mu}{}^{\alpha}{}_{,\beta,\nu} + h_{\mu\beta}{}^{,\alpha}{}_{,\nu} \Big]$$

finally, the Ricci curvature

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \frac{1}{2}\left(h_{\nu}{}^{\alpha}{}_{,\mu,\alpha} + h^{\alpha}{}_{\mu,\alpha,\nu} - h_{,\mu,\nu} - \Box h_{\mu\nu} \right)$$

and for completeness:

$$R \equiv R^{\mu}{}_{\mu} = h^{\mu\nu}{}_{,\mu,\nu} - \Box \overset{\overset{\text{trace}, h^{\mu}{}_{\mu}}{\swarrow}}{h}$$

⇒ <u>Einstein curvature</u>

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R$$

$$= \frac{1}{2}\Big(h_{\nu}{}^{\alpha}{}_{,\mu,\alpha} + h^{\alpha}{}_{\mu,\alpha,\nu} - h_{,\mu,\nu} - \Box h_{\mu\nu}$$
$$- \eta_{\mu\nu} h^{\alpha\beta}{}_{,\alpha,\beta} + \eta_{\mu\nu} \Box h \Big)$$

Why is this <u>still so complicated</u>?

- Wave eqn.
- ensure that $G_{\mu\nu,}{}^{\nu} = 0$
- Remaining freedom to pick coordinates, gauge symmetry ...

Linear Gravity and Gauge Symmetry

Recall: · We are allowed to choose a new set of coordinates $\tilde{y}^\alpha$.

The metric will change as $\tilde{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{y}^\alpha} g_{\mu\nu} \frac{\partial x^\nu}{\partial \tilde{y}^\beta}$

- This is also true for Linear Gravity - as long as we dont violate $|\tilde{h}_{\alpha\beta}| << 1$.
- Define $\xi^\mu(x^\mu) :=: [\underset{\substack{\uparrow \\ \text{new coordinates}}}{\tilde{y}^\mu(x^\mu)} - \underset{\substack{\nwarrow \\ \text{old coordinates}}}{x^\mu}]$ $\Rightarrow x^\mu = \tilde{y}^\mu + \xi^\mu(\tilde{y}^\mu)$

$$\Rightarrow \frac{\partial x^\mu}{\partial \tilde{y}^\alpha} = \frac{\partial(\tilde{y}^\mu + \xi^\mu)}{\partial \tilde{y}^\alpha} = \delta^\mu_\alpha + \xi^\mu_{,\alpha}$$

$\Rightarrow$ Demand that $\boxed{|\xi^\mu_{,\alpha}| << 1}$

$$\Rightarrow \hat{g}_{\alpha\beta} = \eta_{\alpha\beta} + \tilde{h}_{\alpha\beta} = (\delta^\mu_\alpha + \xi^\mu_{,\alpha})(\eta_{\mu\nu} + h_{\mu\nu})(\delta^\nu_\beta + \xi^\nu_{,\beta})$$

$$= \underbrace{\eta_{\alpha\beta}}_{0^{th}\text{ order}} + \underbrace{h_{\alpha\beta} + \xi_{\alpha,\beta} + \xi_{\beta,\alpha}}_{1st\text{ order}} + \text{higher order}$$

$$\Rightarrow \boxed{\tilde{h}_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}}$$

Linearized coordinate change $\hat{=}$ symmetry!

How do $\Gamma^\alpha_{\mu\nu}$, $R^\alpha_{\beta\mu\nu}$, $R_{\mu\nu}$, R, $G_{\mu\nu}$ change under this transformatio

- $$\delta\Gamma^\alpha_{\mu\nu} = \hat{\Gamma}^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\nu} = \frac{1}{2}\left[\xi^\alpha_{,\nu,\mu} + \xi_\nu{}^{,\alpha}{}_{,\mu} + \xi^\alpha_{,\mu,\nu} + \xi_\mu{}^{,\alpha}{}_{,\nu} - \xi_{\mu,\nu}{}^{,\alpha} - \xi_{\nu,\mu}{}^{,\alpha}\right]$$

$$\searrow 0 \swarrow$$

$$\searrow 0 \swarrow$$

$$= \underline{\underline{\xi^\alpha_{,\mu,\nu}}}$$

- $$\delta R^\alpha_{\beta\mu\nu} = \delta\Gamma^\alpha_{\nu\beta,\mu} - \delta\Gamma^\alpha_{\mu\beta,\nu}$$

$$= \xi^\alpha_{,\nu,\beta,\mu} - \xi^\alpha_{,\mu,\beta,\nu} = 0 \quad \Rightarrow \delta R_{\mu\nu} = 0$$

$$\delta R = 0$$

$$\delta G_{\mu\nu} = 0$$

⇒ Compare to Electro-Magnetism:

EM	GR	
A_μ	$\Gamma^\alpha_{\mu\nu}$	⇒ gauge-dependent
$\vec{E}, \vec{B}$ (or $F_{\mu\nu}$)	$R^\alpha_{\beta\mu\nu}$	⇒ physical quantities independent of gauge

Plane Gravitational Waves

We are looking for a vacuum-solution ($R_{\mu\nu}=0$) of the form

$$h_{\mu\nu} = H_{\mu\nu}\, e^{i k_\alpha x^\alpha} \qquad k^\mu = \begin{pmatrix} \omega \\ 0 \\ 0 \\ k \end{pmatrix}$$

$$\Rightarrow R_{\mu\nu}=0 \quad \Rightarrow 0 = \frac{-1}{2}\left(H_\nu{}^\alpha k_\mu k_\alpha + H^\alpha{}_\mu k_\alpha k_\nu - H k_\mu k_\nu - k^2 H_{\mu\nu} \right)$$

What constraints are there for $H_{\mu\nu}$, k^μ ?

a) Use our gauge freedom!: $h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$

i) $h_{00} \to h_{00} + 2i\xi_0 k_0 \qquad \Rightarrow \underline{\xi_0 \overset{!}{=} \frac{-h_{00}}{2ik_0}}$

$\Rightarrow \tilde{h}_{00} = 0$

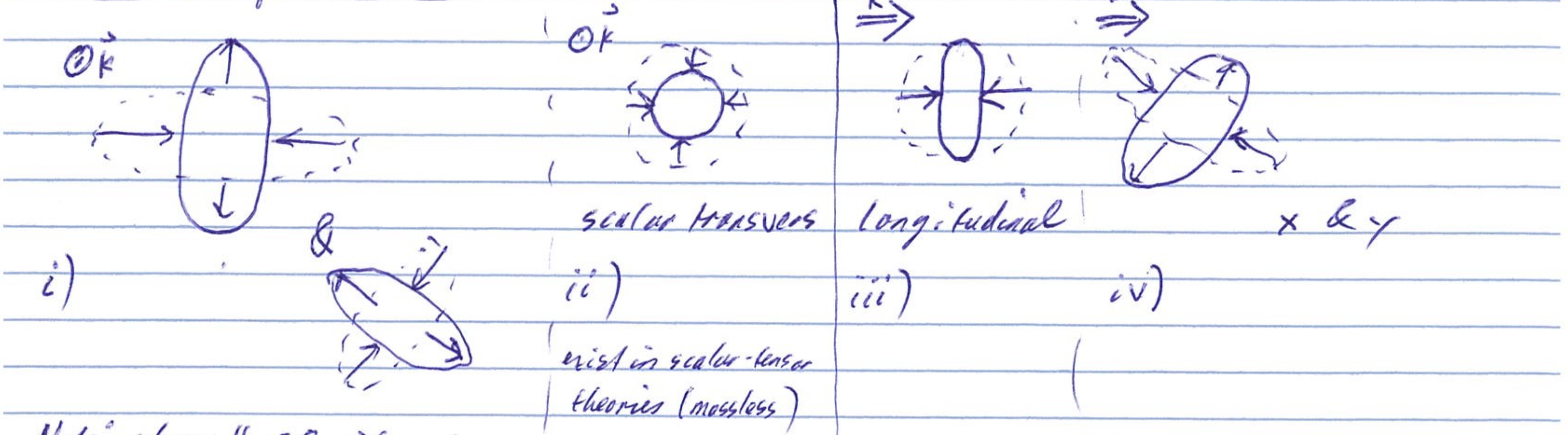
ii) $h_{0\nu} \to h_{0i} + i\xi_0 k_i + i\xi_i k_0 \qquad \Rightarrow \underline{\xi_i \overset{!}{=} \frac{-(h_{0i} + i\xi_0 k_i)}{ik_0}}$

$\Rightarrow \tilde{h}_{0\nu} = 0$

$\Rightarrow$ synchronous gauge

b) Possible modes in a metric theory:

$$H_{\mu\nu} = \underbrace{\begin{pmatrix} 0 & & & \\ & h_+ & 0 & \\ & 0 & -h_+ & \\ & & & 0 \end{pmatrix} ; \begin{pmatrix} 0 & & & \\ & 0 & h_\times & 0 \\ & h_\times & 0 & 0 \\ & & & 0 \end{pmatrix}}_{} ; \begin{pmatrix} 0 & & & \\ & h_s & 0 & \\ & 0 & h_s & \\ & & & 0 \end{pmatrix} ; \begin{pmatrix} 0 & & & \\ & O & & \\ & & & \\ & & & h_\ell \end{pmatrix} ; \begin{pmatrix} 0 & & & \\ & O & & h_x \\ & & & 0 \\ & h_x & 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & & & \\ & O & & 0 \\ & & & h_y \\ & 0 & h_y & 0 \end{pmatrix}$$



exist in scalar-tensor theories (massless)

Note: · trace $H_{\mu\nu}=0$ ⇒ traceless
· $H_{\mu\nu} k^\nu = 0$ ⇒ transverse
(· $H_{0\nu}=0$ ⇒ synchronous)

· transverse-traceless gauge

i) $R_{\mu\nu}$ becomes h_p:

$$R_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & & & \\ & (\vec{k}^2-\omega^2) & 0 & \\ & 0 & -(\vec{k}^2-\omega^2) & \\ & & & 0 \end{pmatrix} h_+$$

$h_\times$:

$$\frac{1}{2}\begin{pmatrix} 0 & & & \\ & 0 & \vec{k}^2-\omega^2 & \\ & \vec{k}^2-\omega^2 & 0 & \\ & & & 0 \end{pmatrix} h_\times$$

e.g. h_+: $R_{11} = -\frac{1}{2}\left[0 + 0 + 0 - k^2 H_{pp} \right] = \frac{k^2}{2} h_+ = \frac{\vec{k}^2-\omega^2}{2} h_+$

ii) e.g. $R_{00} = -\frac{1}{2}\left[\overbrace{-2h_s\omega^2}^{0+0+} - \cancel{(\omega^2+k^2)}\, 0 \right] = \omega^2 h_s \overset{!}{=} 0$

$\Rightarrow$ no wave solution

$$R_{\mu\nu} = \begin{pmatrix} \omega^2 & 0 & 0 & -k\omega \\ 0 & (\vec{k}^2-\omega^2)/2 & 0 & 0 \\ 0 & 0 & (\vec{k}^2-\omega^2)/2 & 0 \\ -k\omega & 0 & 0 & k^2 \end{pmatrix} h_s$$

similar: iii)

$$R_{\mu\nu} = \frac{1}{2}\begin{pmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega^2 \end{pmatrix} h_l$$

$\omega = 0 \Rightarrow$ no wave solution

iv) e.g. ~~for~~ $h_\times$:

$$R_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & k\omega & 0 & 0 \\ k\omega & 0 & 0 & -\omega^2 \\ 0 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \end{pmatrix} h_\times$$

$\Rightarrow \omega = 0 \Rightarrow$ no wave solution

Effect of Gravitational Waves on test particles

We first calculate in the transverse-traceless gauge.

Setup: $h_{\mu\nu} = h_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(kz-\omega t)} = H_{\mu\nu}\, e^{i k_\alpha x^\alpha} \quad k_\alpha = \begin{pmatrix} -\omega \\ 0 \\ 0 \\ k \end{pmatrix}$

a) effect on single test particle: geodesic motion:

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta = 0$$

assume the particle is initially at rest, i.e. $\dot{x}^\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ at $t=0$

$$\Rightarrow \ddot{x}^\mu = \Gamma^\mu_{00}$$

$$\ddot{x}^i = \Gamma^i_{00} =$$

$$= \frac{1}{2}\left(h_{i0,0} + h_{0i,0} - h_{00,i} \right) = \underline{\underline{0}}$$

$$\Rightarrow \dot{x}^\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ \underline{\text{always!}}$$

i.e. test particles do not interact with Grav. Wave!

b) Light traveltime

Ok

- along x-arm:

$$-d\tau^2 = 0 = \underset{-1}{g_{00}}\, dt^2 + \underset{1+h_+}{g_{11}}\, dx^2$$

$\Rightarrow$

$$dt^2 = (1+h_+)\, dx^2$$

$$\underline{dt = \left(1 + \frac{h_+}{2}\right) dx} \quad \& \text{ y-arm: } dt = \left(1 - \frac{h_+}{2}\right) dy$$

assuming $\omega \ll \frac{2\pi}{\tau_{\text{light}}} \Rightarrow \Delta t_x = \frac{h_+}{2} L \quad \& \quad \Delta t_y = -\frac{h_+}{2} L$

difference $\Rightarrow \underline{\underline{\Delta t = \Delta t_x - \Delta t_y}} = \underline{\underline{h_+ L}}$

Note: While both calculations a) & b) were gauge-dependent, the result is not: The separation between inertial test particles is affected by the gravitational wave!

Example: Choose a different gauge:

set $x^{new} = x^{old}\left(1 - \frac{h_+}{2}x\right)$
$y^{new} = y^{old}\left(1 + \frac{h_+}{2}x\right)$ $\Rightarrow$ pick: $\xi^{\mu}_{0} = \frac{h_+}{2}\begin{pmatrix}0\\-x\\y\\0\end{pmatrix}$; where $h_+ = H_+ e^{ik_\alpha x^\alpha}$

$$\Rightarrow h_{+,0} = ik_0 h_+ = -i\omega h_+$$

$$h_{+,3} = ik_3 h_+ = ikh_+$$

$$h_{+,1} = h_{+,2} = 0$$

$$\overset{\xi^0 = 0}{\Rightarrow \xi^{\mu\alpha} = \xi_\mu} \quad \Rightarrow \xi_{\mu,\nu} = \overset{\overset{\nu}{\rightarrow}}{\downarrow\mu}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{i\omega h_+ x}{2} & -\frac{h_+}{2} & 0 & -\frac{ikh_+x}{2} \\ -\frac{i\omega h_+ y}{2} & 0 & \frac{h_+}{2} & +\frac{ikh_+y}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow h_{\mu\nu} = h^{old}_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} = \underline{\underline{\begin{pmatrix} 0 & \omega x & -\omega y & 0 \\ \omega x & 0 & 0 & -kx \\ -\omega y & 0 & 0 & ky \\ 0 & -kx & ky & 0 \end{pmatrix}\frac{ih_+}{2}}}$$

i.e.: $g_{00} = -1$ & $g_{11} = g_{22} = g_{33} = 1$ $\Rightarrow$ light travel time:

$$0 = g_{00}dt^2 + g_{11}dx^2 \Rightarrow \underline{dt = dx}$$

$\Rightarrow$ unchanged!

But: Single particle motion:

$$\ddot{x}^i = -\Gamma^i{}_{00} = -\frac{1}{2}\left(h_{0i,0} + h_{i0,0} - h_{00,i}\right) = -h_{0i,0}$$

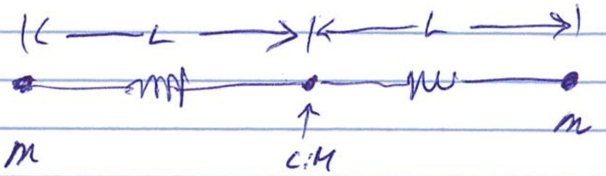
$$\Rightarrow \ddot{x}^1 \underset{\uparrow}{=} -\left[\frac{ih_+}{2}\omega x(-i\omega)\right] = -\frac{\omega^2 h_+ x}{2}$$

$$\underline{\underline{x^1}} = \int dt \int dt\, \ddot{x}^1 = \underline{\underline{\frac{h_+ x}{2}}}$$

similar $\underline{\underline{y = -\frac{h_+ y}{2}}}$ $\Rightarrow \underline{x - y = h_+ L}$!!

$\Rightarrow$ In words: The light travel time to the coordinates $x = L$ is unchanged, but the testparticle now feels a x-depen force.

This gauge is particularly useful for calculating a bar detector response:



equation of motion:

$$m\ddot{x} = -kx - \gamma\dot{x} + F_{ext}$$

or with $\omega_0^2 = \frac{k}{m}$, $\gamma = \frac{\omega_0}{Q}$, $a_{ex} = \frac{F_{ex}}{m}$

$$\ddot{x} = -\omega_0^2 x - \frac{\omega_0}{Q}\dot{x} + a_{ext}$$

$$\Rightarrow -\omega^2 x = -\omega_0^2 x + i\frac{\omega_0 \omega}{Q} x + a_{ext}$$

$$x = \frac{a_{ext}}{\omega_0^2 - \omega^2 - i\frac{\omega_0 \omega}{Q}}$$

where $a_{ex} = -\frac{\omega_g^2 h_+^0}{2} L e^{-i\omega_g t}$

$$\Rightarrow x = \frac{-\frac{\omega_g^2 h_+^0}{2} L}{\omega_0^2 - \omega_g^2 - i\frac{\omega_0 \omega_g}{Q}} \cdot e^{-i\omega_g t}$$

e.g. for $\omega = \omega_g = \omega_0$:

$$\boxed{x = -i\frac{h_+ L}{2} Q e^{-i\omega_g t}}$$

⇒ high mechanical Q (low mechanical loss) needed!

Is there a formulation that never uses gauge-dependent quantities?

Sure! Recall Electromagnetism: Just work with $\vec{E}$ & $\vec{B}$ instead of $\vec{A}$ & V !

⇒ in G.R.: Just work with the Riemann curvature!

recall: $\delta v^i = -R^i{}_{010} \cdot \delta x \cdot \delta t$

$$\delta v_x = R^1{}_{001}\, \delta x \cdot \delta t$$

$$\Rightarrow a_x = R^1{}_{001} \cdot \delta x$$

What is $R^\alpha{}_{\beta\gamma\delta}$?

$$R^1{}_{001} = \frac{1}{2}\Big[h_1{}^1{}_{,0,0} - \overbrace{h_{10}{}^{,1}{}_{,0}}^{0} - \underbrace{h_0{}^1{}_{,0,1}}_{0} + \underbrace{h_{00}{}^{,1}{}_{,1}}_{0} \Big]$$

$$= \frac{1}{2}(-\omega^2) h_+$$

$$\Rightarrow \ddot{x} = -\frac{\omega^2}{2} h_+ \cdot \delta x$$

$$\underset{\text{integrate}}{\Rightarrow} \quad \delta x = \frac{h_+}{2} \delta x = \boxed{\frac{h_+}{2} \cdot L}$$

Aside: There are other non-zero terms for an h_+ wave:

Some calculations show:

$$\underbrace{R_{0101}}_{\text{We generally measure this term.}} = R_{0113} = R_{0220} = R_{0232} = R_{1313} = R_{2332} = \frac{\omega^2}{2} h_+$$

(addional components are connected to the above via asymmetry, or are $= 0$)

for example this one

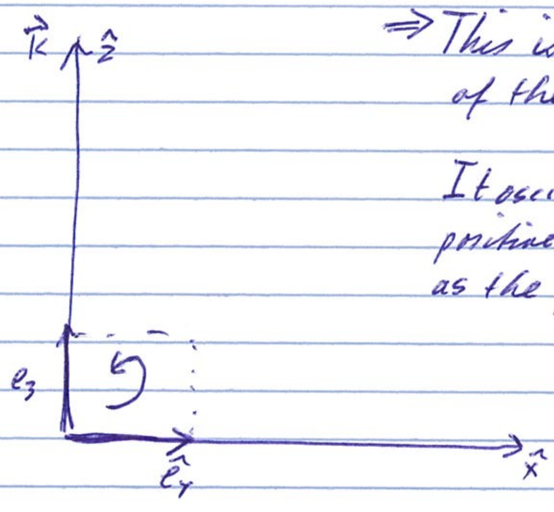
$$R^1{}_{\underbrace{31}3} = \frac{\omega^2}{2} h_+$$

pick 1-3 subplane

parallel transport 3-vector (parallel to $\vec{k}$)

⇒ This is the Gauss-curvature of the 1-3 subplane!

It oscillates, fluctuating between positive & negative curvature as the gravitational wave passes!



Gauge choices

- We have the choice of using any 4-vector field ξ^ν

  $\Rightarrow$ the metric will change as $\tilde{h}_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$

- In Vacuum: We saw the transverse-traceless gauge

$$\begin{array}{ll} \swarrow & \downarrow \\ 0 = h_{\mu\nu}{}^{,\nu} \quad \text{or} \quad 0 = h_{\mu\nu}k^\nu & h = h^\mu_\mu = 0 \end{array}$$

which was also synchronous $(h_{0\nu} = 0)$

in general (i.e. when matter is present) we cannot get all of the above simultaneously.

Common choices are:

A) transverse gauge: $\sum\limits_{i=1,2,3} h^{0i}{}_{,i} = 0$ (only spatial indices)

&

$s^{ij}{}_{,j} = 0$ where $s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right)_{ij \in 1,2,3}$

is the traceless part of the (3×3) spatial part of h_{ij}

Analog to Coulomb gauge in Electromagnetism! $\vec{\nabla}\cdot\vec{A} = 0$

B) Synchronous gauge: $h_{0\nu} = 0$, $\nu \in 0,1,2,3$ $\Rightarrow d\tau^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j$

(That's what we explicitly enforced for plane waves.)

c) Lorenz or harmonic gauge

- choose $h^{L\mu\nu}{}_{,\mu} - \frac{1}{2} h^{L,\nu} = 0$ (also 4 equations)

with this the linearized Einstein curvature is

$$G_{\mu\nu} = -\frac{1}{2}\Box\left(h^L_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^L\right)$$

⇒ introduce the trace-reversed strain

$$\tilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$\Rightarrow \text{Lorenz gauge: } \tilde{h}^{L\mu}{}_{\nu,\mu} = 0$$

$$\Rightarrow G_{\mu\nu} = -\frac{1}{2}\Box\,\tilde{h}^L_{\mu\nu}$$

compare to Electromagnetism: $A^{\mu}{}_{,\mu} = 0$

$$\Rightarrow \boxed{\begin{array}{l} \Box\,\tilde{h}^L_{\mu\nu} = -16\pi G\, T_{\mu\nu} \\ (\text{in vacuum: } \Box\,\hat{h}^L_{\mu\nu} = 0\ ) \end{array}}$$

⚠ in vacuum the transverse-traceless gauge is often used: h^{TT}

$$\Rightarrow \Box\, h^{TT}_{\mu\nu} = 0 \qquad \text{vacuum only!}$$

$h^{TT}_{\mu\nu}$ is different from $h^L_{\mu\nu}$

⇒ We will use Lorenz gauge for wave generation, but transverse-traceless gauge for detection, How do the relate to eachother?

Detecting Gravitational Waves

Problem 1

Linearized Gravity: transverse-traceless and Lorenz gauge

We linerarize gravity by assuming that the metric perturbations are small

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll 1\,. \tag{1}$$

In class we derived the Riemann curvature, Ricci curvature and Ricci scalar up to 1st order:

$$\begin{array}{lcl} R^{\alpha}{}_{\mu\beta\nu} & = & \frac{1}{2}\left[h_{\nu}{}^{\alpha}{}_{,\mu,\beta} - h_{\beta}{}^{\alpha}{}_{,\mu,\nu} + h_{\beta\mu}{}^{,\alpha}{}_{,\nu} - h_{\nu\mu}{}^{,\alpha}{}_{,\beta}\right] \\ R_{\mu\nu} & = & \frac{1}{2}\left[h_{\nu}{}^{\alpha}{}_{,\mu,\alpha} + h_{\mu}{}^{\alpha}{}_{,\nu,\alpha} - h_{,\mu,\nu} - \Box h_{\mu\nu}\right] \\ R & = & +h^{\mu\nu}{}_{,\mu,\nu} - \Box h\,. \end{array} \tag{2}$$

The Einstein curvature, as always, is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R\,. \tag{3}$$

Gauge transform

We have also seen that the Riemann curvature is invariant under gauge transforms

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}\,, \tag{4}$$

i.e. the Riemann curvature has a direct (measurable) physical meaning. The invariance of the Riemann curvature of course also implies the invariance of the Ricci curvature, Ricci scalar and Einstein curvature.

1 Lorenz gauge or harmonic gauge

The Lorenz gauge is defined by

$$h^{L}{}_{\mu}{}^{\nu}{}_{,\nu} = \frac{1}{2}h^{L}{}_{,\mu}\,. \tag{5}$$

Show that this results in the Einstein equation

$$\Box\left(h^{L}{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{L}\right) = -16\pi G T_{\mu\nu}\,. \tag{6}$$

In vacuum, any solution can be written as superposition of plane waves

$$h^{L}{}_{\mu\nu} = h^{L}_{0\,\mu\nu}e^{ik_{\lambda}x^{\lambda}} \tag{7}$$

with $k^{\lambda} = (\omega, \omega\vec{n})$. We can always pick $\vec{n} = \hat{e}_3$. Show that for such a plane wave solution the four Lorenz gauge constraints (equation 5) imply

$$\begin{array}{lcl} h^{L}_{03} & = & -\frac{1}{2}(h^{L}_{00} + h^{L}_{33}) \\ h^{L}_{01} & = & -h^{L}_{31} \\ h^{L}_{02} & = & -h^{L}_{32} \\ h^{L}_{11} & = & -h^{L}_{22}\,. \end{array} \tag{8}$$

2 Trace-reversed Lorenz gauge

Repeat the previous problem with the trace-reversed strain in Lorenz gauge, $\bar{h}^L_{\mu\nu}$:

$$\bar{h}^L_{\mu\nu} = h^L{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^L \,, \tag{9}$$

The Lorenz gauge condition and Einstein equation become

$$\begin{aligned} \bar{h}^{L\,\mu\nu}{}_{,\nu} &= 0 \\ \Box\bar{h}^L_{\mu\nu} &= -16\pi G T_{\mu\nu} \,. \end{aligned} \tag{10}$$

Again looking at a plain wave in 3-direction in vacuum, show that

$$\begin{aligned} \bar{h}^L_{03} &= -\tfrac{1}{2}(\bar{h}^L_{00} + \bar{h}^L_{33}) \\ \bar{h}^L_{01} &= -\bar{h}^L_{31} \\ \bar{h}^L_{02} &= -\bar{h}^L_{32} \\ \bar{h}^L_{33} &= +\bar{h}^L_{00} \,. \end{aligned} \tag{11}$$

3 Transverse-traceless gauge

In vacuum the transverse-traceless gauge is defined by

$$\begin{aligned} h^{TT\mu\nu}{}_{,\nu} &= 0, && \text{transverse} \\ h^{TT\mu}{}_{\mu} &= 0, && \text{traceless} \\ h^{TT}{}_{0\nu} &= 0, && \text{time components are zero} \end{aligned} \tag{12}$$

Show that the corresponding Einstein equation in vacuum is

$$\Box h^{TT}{}_{\mu\nu} = 0 \,. \tag{13}$$

The gauge transform from Lorenz to TT gauge is given by

$$h^{TT}_{\mu\nu} = h^L_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}. \tag{14}$$

In vacuum, any solution can again be written as superposition of plane waves

$$h^{TT}{}_{\mu\nu} = h_0^{TT}{}_{\mu\nu} e^{ik_\lambda x^\lambda} \tag{15}$$

with $k^\lambda = (\omega, \omega\vec{n})$. Show that for a plane wave solution with frequency ω traveling in $\vec{n} = \hat{e}_3$ direction, the vector ξ^μ is given by

$$\xi^\mu = \frac{1}{2i\omega}\left(-h^L_{00}, 2h^L_{01}, 2h^L_{02}, -h^L_{33}\right) \,, \tag{16}$$

which implies

$$\begin{aligned} h^{TT}_{11} &= h^L_{11} \\ h^{TT}_{12} &= h^L_{12} \\ h^{TT}_{22} &= h^L_{22} \\ h^{TT}_{0\nu} &= 0, \quad \nu = (0,1,2,3) \\ h^{TT}_{3\nu} &= 0, \quad \nu = (0,1,2,3) \,. \end{aligned} \tag{17}$$

4 From Lorenz to transverse-traceless gauge ($h^L_{\mu\nu} \rightarrow h^{TT}_{\mu\nu}$)

Show that for a plane wave solution traveling in an arbitrary direction $\vec{n}$, this result (equation 17) for the spatial components can be written as

$$h^{TT}_{ij} = \left(P_i^{\ k} P_j^{\ l} - \frac{1}{2} P_{ij} P^{kl}\right) h^L_{kl} \equiv \left[h^L_{ij}\right]^{TT}, \tag{18}$$

where the projection operator P^{ij} is given by

$$P^{ij} = \delta^{ij} - n^i n^j. \tag{19}$$

5 From trace-reversed Lorenz to transverse-traceless gauge ($\bar{h}^L_{\mu\nu} \rightarrow h^{TT}_{\mu\nu}$)

For plane-wave solutions in arbitrary direction $\vec{n}$, show that the following also is true

$$h^{TT}_{ij} = \left(P_i^{\ k} P_j^{\ l} - \frac{1}{2} P_{ij} P^{kl}\right) \bar{h}^L_{kl} \equiv \left[\bar{h}^L_{ij}\right]^{TT}. \tag{20}$$

Again specializing to $\vec{n} = \hat{e}_3$, show that this implies

$$\begin{aligned}
h^{TT}_{11} &= \tfrac{1}{2}\left(\bar{h}^L_{11} - \bar{h}^L_{22}\right) \\
h^{TT}_{12} &= \bar{h}^L_{12} \\
h^{TT}_{22} &= \tfrac{1}{2}\left(\bar{h}^L_{22} - \bar{h}^L_{11}\right) \\
h^{TT}_{0\nu} &= 0, \quad \nu = (0,1,2,3) \\
h^{TT}_{3\nu} &= 0, \quad \nu = (0,1,2,3),
\end{aligned} \tag{21}$$

which is worth remembering.

PS1

1) a) $h_\mu{}^\nu{}_{,\nu} = \frac{1}{2} h_{,\mu}$ Lorentz gauge

$$R_{\mu\nu} = \frac{1}{2}\Big[\underset{\downarrow}{h_\nu{}^\alpha{}_{,\mu,\alpha}} + \underset{\downarrow}{h_\mu{}^\alpha{}_{,\nu,\alpha}} - h_{,\mu,\nu} - \Box h_{\mu\nu}\Big]$$

$$\underbrace{\frac{1}{2} h_{,\nu,\mu} \qquad \frac{1}{2} h_{,\mu,\nu} \qquad\qquad}_{0} \qquad = -\frac{1}{2}\Box h_{\mu\nu}$$

$\Rightarrow$ trace-reverse operation to go to $G_{\mu\nu}$:

$$G_{\mu\nu} = -\frac{1}{2}\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \right) = 8\pi G T_{\mu\nu}$$

b) explicit constraints: $_{,\mu} \leftrightarrow i k_\mu$ $\quad k_\mu = \omega\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$; $k^\mu = \omega\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$

$$h = h^\mu{}_\mu = -h_{00} + h_{11} + h_{22} + h_{33}$$

$0 = h_{0\nu}k^\nu - \frac{1}{2}h k_0 \quad$ i) $h_{00} + h_{03} + \frac{1}{2}(h) = 0$

$0 = h_{3\nu}k^\nu - \frac{1}{2}h k_3 \quad$ ii) $h_{30} + h_{33} - \frac{1}{2}(h) = 0$

$\boxed{i)+ii)}$ $\quad 2h_{03} = -(h_{00} + h_{33})$

i)-ii) $\quad h_{00} - h_{33} + (-h_{00} + h_{11} + h_{22} + h_{33}) = 0 \quad \Rightarrow h_{11} = -h_{22}$

$0 = h_{\underset{2}{1}\nu}k^\nu - \frac{1}{2}h\underbrace{k_{\underset{2}{1}}}_{0} \qquad h_{\underset{2}{1}0} + h_{\underset{2}{1}3} = 0 \qquad \Rightarrow h_{10} = -h_{13}$

$h_{20} = -h_{23}$

2 a) Trace-reversed! $\Rightarrow -\frac{1}{2}\Box \tilde{h}^L_{\mu\nu} = 8\pi G T_{\mu\nu}$

b)

$$h_{00} + h_{03} = 0 \qquad \Rightarrow -h_{03} = h_{00} = h_{33}$$

$$h_{30} + h_{33} = 0$$

$$h_{10} + h_{13} = 0$$

$$h_{20} + h_{23} = 0$$

3) Transverse-trace-less in vacuum:

$$R_{\mu\nu} = \frac{1}{2}\Big[\underset{\substack{\downarrow\\0}}{h_\nu{}^\alpha{}_{,\mu,\alpha}} + \underset{\substack{\downarrow\\0}}{h_\mu{}^\alpha{}_{,\nu,\alpha}} - \underset{\substack{\downarrow\\0}}{h_{,\mu,\nu}} - \Box h_{\mu\nu}\Big] \qquad \& \; G_{\mu\nu} = R_{\mu\nu}\,!$$

ξ^μ given! (We could get it by enforcing the synchronous condition, as we did before...)

$$\xi_\mu = \frac{1}{2i\omega}\begin{pmatrix} h^L_{00} \\ 2h^L_{01} \\ 2h^L_{02} \\ -h^L_{33} \end{pmatrix} \Rightarrow \xi_{\mu,\nu} = \frac{1}{2}\begin{pmatrix} -h_{00} & 0 & 0 & h_{00} \\ -2h_{01} & 0 & 0 & 2h_{01} \\ -2h_{02} & 0 & 0 & 2h_{02} \\ +h_{33} & 0 & 0 & -h_{33} \end{pmatrix}$$

$$\Rightarrow \xi_{\mu,\nu} + \xi_{\nu,\mu} = \begin{pmatrix} -h_{00} & -h_{01} & -h_{02} & -h_{03} \\ -h_{01} & 0 & 0 & -h_{31} \\ -h_{02} & 0 & 0 & -h_{32} \\ -h_{03} & -h_{31} & -h_{32} & -h_{33} \end{pmatrix}$$

$$\Rightarrow h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^L_{11} & h^L_{12} & 0 \\ 0 & h^L_{21} & h^L_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \; h^{TT}_{\mu\nu} k^\nu = 0 \; \checkmark$$

$$h^{TT}_{0\nu} = 0 \; \checkmark$$

$$h = h^L_{11} + h^L_{22} \underset{\substack{\uparrow\\\text{for plane waves}}}{=} 0 \; \checkmark$$

4) **Note:** $h^L_{\mu\nu} \Rightarrow h^{TT}_{\mu\nu}$ is a projection operator

$$h^{TT}_{\mu\nu} = \underbrace{\begin{pmatrix}0&&&\\&1&&0\\&0&1&\\&&&0\end{pmatrix}}_{\text{spatial part:}} \cdot \left(h^L_{\mu\nu}\right) \begin{pmatrix}0&&&\\&1&&0\\&0&1&\\&&&0\end{pmatrix}$$

$$\begin{pmatrix}1&&\\&1&0\\0&&0\end{pmatrix} = 1\!\!1_{ij} - n_i n_j \qquad \text{where } n = \begin{pmatrix}0\\0\\1\end{pmatrix},\ n \parallel k$$

$$=: P_{ij} \qquad \& \text{ trace} \begin{pmatrix}1&&\\0&1&0\\&&0\end{pmatrix} \cdot \begin{pmatrix}h_{11}&&*\\ *&h_{22}&\\&&*\end{pmatrix}$$

$$\Rightarrow \boxed{\text{equation } 18!} \qquad = \boxed{h_{11} + h_{22} = 0}$$

Note:

$$\underbrace{\left(P_i^{\ k} P_j^{\ l} - \tfrac{1}{2} P_{ij} P^{kl}\right)}_{\text{guarantees that the result is trace-free}} h_{kl}$$

$$\sum_{ij} \left(P_i^{\ k} P_j^{\ l} - \tfrac{1}{2} P_{ij} P^{kl}\right) = 0$$

and rejects any trace of h_{kl}:

$$\left(P_i^{\ k} P_j^{\ l} - \tfrac{1}{2} P_{ij} P^{kl}\right) \delta_{kl} = P_{ij} - \tfrac{2}{2} P_{ij} = 0$$

5) Since the result of the projection is trace-free, it is also valid for the trace-reversed gauge: $\left(P_i^{\ k} P_j^{\ l} - \tfrac{1}{2} P_{ij} P^{kl}\right) \eta_{kl} = 0$

$$\Rightarrow h^{TT}_{ij} = \left(P_i^{\ k} P_j^{\ l} - \tfrac{1}{2} P_{ij} P^{kl}\right) \tilde{h}^L_{kl} =: \left[\tilde{h}^L_{\mu\nu}\right]^{TT}$$

$$\Rightarrow h^{TT}_{ij} = \begin{pmatrix}\tilde{h}^L_{11} & \tilde{h}^L_{12} & 0\\ \tilde{h}^L_{21} & \tilde{h}^L_{22} & 0\\ 0 & 0 & 0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1&0&\\0&1&\\&-&0\end{pmatrix} \cdot \left(\tilde{h}^L_{11} + \tilde{h}^L_{22}\right) \Rightarrow \boxed{\begin{aligned} h^{TT}_{11} &= \tfrac{1}{2}\left(\tilde{h}^L_{11} - \tilde{h}^L_{22}\right)\\ h^{TT}_{22} &= \tfrac{1}{2}\left(\tilde{h}^L_{22} - \tilde{h}^L_{11}\right)\\ h^{TT}_{12} &= \tilde{h}^L_{12}\end{aligned}}$$

Generation of Gravitational waves

- We stay with linearized gravity: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$
- I work in the trace-reversed Lorenz gauge: $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$; $\tilde{h}_{\mu}{}^{\nu}{}_{,\nu} = 0$

$$\Rightarrow \boxed{\Box \tilde{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}}$$

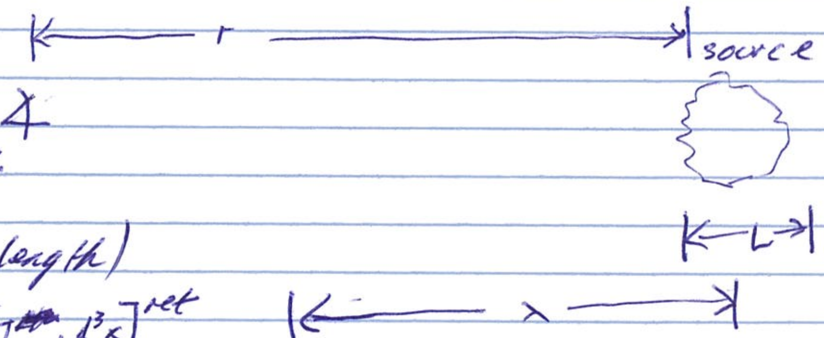
$\Rightarrow$ • Simple wave equation for each component.

• Components do not mix!

$\Rightarrow$ Standard Green's function approach works:

$$\boxed{\tilde{h}_{\mu\nu}(t, x) = 4G \int d^3x' \, \frac{T_{\mu\nu}(\underline{x}', t' = t - |\underline{x} - \underline{x}'|)}{|\underline{x} - \underline{x}'|}}$$

Radiation in the far field:



- We now make 2 assumptions:

i) $L \ll \lambda$ (Source $\ll$ Wavelength)

$$\Rightarrow \int T_{\mu\nu}^{ret} \, d^3x \rightarrow \left[\int T_{\mu\nu} \, d^3x\right]^{ret}$$

ii) $\lambda \ll r$ (Wavelength $\ll$ distance)

$$\Rightarrow \int \frac{T_{\mu\nu}}{|x - x'|} d^3x' \rightarrow \frac{1}{r} \int T_{\mu\nu} \, d^3x'$$

$$\Rightarrow \boxed{\tilde{h}^{L}_{\mu\nu}(t, \underline{x}) = \frac{4G}{r}\left[\int T_{\mu\nu} d^3x\right]^{ret} \equiv \frac{4G}{r}\left[\int T_{\mu\nu} d^3x\right]_{(t-r, \bullet)}}$$

Now: energy conservation : $T^{00}{}_{,0} = -T^{0j}{}_{,j}$

& momentum conservation : $T^{j0}{}_{,0} = -T^{jk}{}_{,k}$

$$\Rightarrow T^{00}{}_{,0,0} = -T^{0j}{}_{,j,0} = T^{jk}{}_{,j,k}$$

$$\Rightarrow T^{00}{}_{,0,0}\, x^j x^k = T^{lm}{}_{,l,m}\, x^j x^k$$

$$= \left(T^{lm}{}_{,l}\, x^j x^k\right)_{,m} - T^{lm}{}_{,l}\, \delta^j_m\, x^k - T^{lm}{}_{,l}\, x^j \delta^k_m$$

$$= \left(\quad \ldots \quad \right)_{,m} - T^{lj}{}_{,l}\, x^k - T^{lk}{}_{,l}\, x^j$$

$$= \left(\qquad \right)_{,m} - \left(T^{lj} x^k\right)_{,l} - \left(T^{lk} x^j\right)_{,l}$$

$$+ T^{kj} \qquad + T^{jk}$$

But $T^{lm} = 0$ outside source region

⇒ total derivatives disappear in integration!

$$\Rightarrow \int T^{00}{}_{,0,0}\, x^j x^k\, d^3x = 2 \int T^{jk}\, d^3x \qquad + \underbrace{\text{surface term}}_{0}$$

$$\Rightarrow \int T^{jk} d^3x = \frac{1}{2} \int T^{00}{}_{,0,0}\, d^3x$$

$$= \frac{1}{2} \frac{d^2}{dt^2}\, I^{jk} \qquad \text{with } I^{jk} = \int \underset{T^{00}}{\underset{\uparrow}{\rho}}\, x^j x^k\, d^3x$$

Thus we find: $\tilde{h}^{L}_{ij} = \frac{4G}{r}\left[\int T_{ij}\, d^3x\right]^{ret}$

$$= \frac{2G}{r}\left[\int T^{00}{}_{,0,0}\, x^j x^k\, d^3x\right]^{ret}$$

$$\boxed{\tilde{h}^{L}_{ij} = \frac{2G}{r}\frac{d^2}{dt^2} I^{ret}_{ij}}$$

⇒ now project to transverse-traceless gauge:

$$\boxed{h^{TT}_{ij} = \frac{2G}{r}\frac{d^2}{dt^2}\left[I^{ret}_{ij}\right]^{TT}}$$ see Problems 1

Note: Define mass quadropole moment as trace-free:

$$\mathcal{J}_{ij} := \int T^{00}\left(x^i x^j - \frac{1}{3} r^2 \delta_{ij}\right) d^3x$$

$$\Rightarrow \left[I_{ij}\right]^{TT} = \left[\mathcal{J}_{ij}\right]^{TT}$$ since $\left(P_i{}^k P_j{}^l - \frac{1}{2} P_{ij} P^{kl}\right)\delta_{ij} =$

Quadropole formula: $\rightarrow \boxed{h^{TT}_{ij} = \frac{2G}{r}\frac{d^2}{dt^2}\left[\mathcal{J}^{ret}_{ij}\right]^{TT}}$

Notes:
- we used $L \ll \lambda \ll r$
- we linearized gravity ⇒ we neglected self-gravity, which has $T^{\mu\nu}_{self} = O(h^2)$
- Both "monopole radiation" $\tilde{h}_{00} = \frac{4G}{r}\int T_{00}\, d^3x$

  & "dipole radiation" $\tilde{h}_{0j} = \frac{4G}{r}\frac{d}{dt}\int T_{00}\, x^j\, d^3x$

  are pure gauge modes, thanks to Energy & Momentum conservation.

(Compare ElectroMagnetism: $A^0 \propto \frac{1}{r}\int \rho\, d^3x$ is gauge mode - thanks to charge conserv