NO. 1

Observing Gravitational Waves

Summer 2017

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			10^{25}–12^{10}	13^{00}–14^{45}	14^{55}–16^{40}	
			2	3	4	
Lectures:	Tue	May 16	x	x		
		May 23	x	x	x	
		May 30	x	x	x	(201a)
		June 6	x	x	x	
		June 13	x	x	x	(student presentation)

Room 201b (except 5/30: 201a)

Web page: granite.phys.s.u-tokyo.ac.jp/ja/?Lecture2017Stefan

or: tinyurl.com/GW2017Ballmer

Evaluation: Student presentations on June 13
(I will ask you to submit your presentation notes for my reference.)

Homework: none, but frequent example problems in class.

Questions: Who had previously a lecture on:
- General Relativity ?
- Special Relativity ?
- General Mechanics / Lagrange equations ?

Special Relativity & Notation

- Classical Mechanics (as known around 1905) is invariant* under Galilei Transformations (GT):

t, x coordinates in reference frame S
t', x' " " " " S'

GT:
$$\begin{cases} x = x' + v \cdot t' \\ t = t' \end{cases} \Rightarrow \text{absolute time!}$$

*example: $F = m \cdot a$; $a = \ddot{x} \underset{GT}{\equiv} \ddot{x}' = a'$ & $F \underset{G.T.}{\equiv} F'$; $m = m'$

- But in the 1860's Maxwell published his set of equations describing Electro-Magnetism. They look nice when written with potentials V & $\vec{A}$:

$$\vec{B} = \vec{\nabla} \wedge \vec{A} \qquad \& \ \vec{E} = -\vec{\nabla} V - \frac{\partial}{\partial t} \vec{A}$$

$\Rightarrow$ using the Lorentz gauge: $\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial}{\partial t} V$

they are
$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta\right)\begin{pmatrix} \frac{V}{c} \\ \vec{A} \end{pmatrix} = -\mu_0 \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$$

Note: Wave equation with propagation speed c! Indep. of reference fr

$\Rightarrow$ Inconsistent with G.T.!

Notation

- $x^0 = c \cdot t$ i.e. we measure time in meters $\Leftrightarrow$ "$c=1$"
- 4-vectors: $j^\mu \equiv \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$ $A^\mu = \begin{pmatrix} V/c \\ \vec{A} \end{pmatrix}$
  - greek indices: $\mu, \nu, \lambda, \kappa$, etc run from $0,1,2,3$
  - latin indices: i, j, k, l, etc. run only $1,2,3$
- Einstein summation convention:
  - repeated indices are <u>summed over.</u> (one upper & one lower index required)

$$\text{e.g. Lorentz gauge} \quad \frac{\partial}{\partial x^\nu} A^\nu = 0$$

  - index is pulled up or down with the metric $\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

$$\text{e.g.} \quad \partial_\nu \partial^\nu \equiv \partial_\mu \eta^{\mu\nu} \partial_\nu = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = -\left(\frac{\partial}{\partial x^0}\right)^2 + \sum_{i=1,2,3} \left(\frac{\partial}{\partial x^i}\right)^2 =: \Box$$

$\hat{=}$ "Wave", or d'Alembert operator

- Indices after comma are derivatives:

$$\text{e.g.} \quad \partial_\mu A^\nu = \frac{\partial}{\partial x^\mu} A^\nu \equiv A^\nu{}_{,\mu}$$

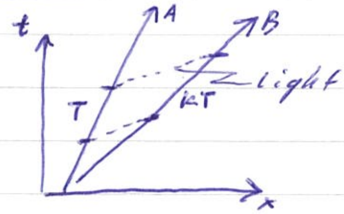
$$\partial^\mu A^\nu = \eta^{\mu\lambda} A^\nu{}_{,\lambda} \equiv A^{\nu,\lambda}$$

$\Rightarrow$ Example: Maxwell eqn: $A^{\mu,\nu}{}_{,\nu} = -\mu_0 j^\mu$

with Lorentz gauge: $A^\nu{}_{,\nu} = 0$

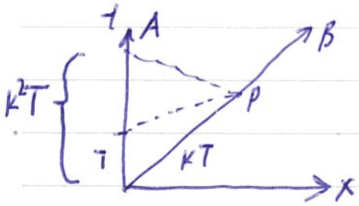
"c=const" ⇒ Lorentz transformation

a) Relativistic Doppler shift: Take 2 clocks, B is moving away from A with velocity v



- clocks sent light pulses at interval T
- pulses are received at intervals $k \cdot T > T$ (v>
- True for both clocks: A→B & B→A!

b) Special case: Radar setup:



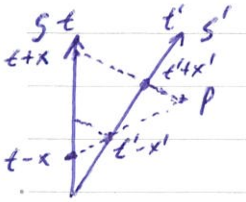
- A at rest, B moves with v
- Radar uses round-trip light propagation time to measure distance:

⇒ the coordinates of Point P are:

$$(t,x) = \left(\tfrac{1}{2}(k^2+1)\,T,\ \tfrac{1}{2}(k^2-1)\cdot T\right) \qquad (c=1)$$

$$\Rightarrow v = \frac{x}{t} = \frac{k^2-1}{k^2+1} \quad \Rightarrow \boxed{k = \sqrt{\frac{1+v}{1-v}}}$$

c) Lorentz transformation



- Two radar stations S & S' measure the coordinates of P ⇒ P: (t, x) & P: (t', x')
- Using the Dopplershift relations we find:

$$\left.\begin{aligned}(t'-x') &= k\,(t-x)\\ k\,(t'+x') &= (t+x)\end{aligned}\right\}\ \text{2 eqn. with 2 variables ⇒ solve!}$$

Lorentz Transformation (LT) ⇒

$$\boxed{\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}} = \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} = \Lambda(\chi) \begin{pmatrix} t' \\ x' \end{pmatrix}$$

with $\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh\chi$; $\gamma v = \sinh\chi$; χ: rapidity

- Note: - we only needed c = const!
  - Perpendicular directions Y & Z are unchanged

Remarks on S.R:

Note: Given 2 events $\begin{pmatrix} t_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$ & $\begin{pmatrix} t_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}$ $\Rightarrow$ $\Delta t = t_2 - t_1$
$\Delta x = x_2 - x_1$
$\Delta y = y_2 - y_1$
$\Delta z = z_2 - z_1$

- Under Galilean physics, the space separation $\Delta x = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ and time separation Δt had both physical meanings
- In relativity this goes away! Only the space-time interval $-\Delta\tau^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ remains as physical (coordinate-independent) quantity.

- This can easily be written with the Minkowski metric:

$$-d\tau^2 = dx^\mu \eta_{\mu\nu} dx^\nu \quad ; \eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Relativistic Mechanics: Kinematics

Mechanics needs some updates:

Kinematics: coordinate difference dx^μ is a 4-vector differential!

i.e. it's "4-lenght" is $-d\tau^2 = dx^\mu \eta_{\mu\nu} dx^\nu$

if. we have a particle moving from x_1^μ to $x_2^\mu = x_1^\mu + dx^\mu$

we can put ourselfs in the particle's instantaneous rest frame: $d\tilde{x}^\mu$: $d\tilde{x}^i = 0$, $d\tilde{x}^0 = d\tau$

$$\Rightarrow -d\tau^2 = d\tilde{x}^\mu \eta_{\mu\nu} d\tilde{x}^\nu$$

$\Rightarrow$ τ is the particle's propper or Eigen-time.

- We can use the propper time to define a 4-velocity:

$$U^\mu = \frac{d}{d\tau} x^\mu = \frac{dt}{d\tau}\frac{d}{dt}\begin{pmatrix} t \\ \vec{x} \end{pmatrix}^\mu = \frac{dt}{d\tau}\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$$

- but $U^\mu U_\mu = \frac{1}{d\tau^2}\left(dx^\mu dx_\mu\right) = -1$

$$\Rightarrow -1 = -\left(\frac{dt}{d\tau}\right)^2 \left(1 - \vec{v}^2\right)$$

$$\Rightarrow \frac{dt}{d\tau} = \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$\Rightarrow U^\mu = \begin{pmatrix} \gamma \\ \gamma v \end{pmatrix}$$

- For m = particle mass $\Rightarrow$ 4-momentum $p^\mu = m U^\mu = \begin{pmatrix} m\gamma \\ m\gamma\vec{v} \end{pmatrix} = \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$

$\Rightarrow \boxed{m^2 = -p^\mu p_\mu = E^2 - \vec{p}^2}$ $\Rightarrow$ E includes rest energy m + kinetic energy

Relativistic Mechanics: Dynamics

Newton 2: $\vec{F} = m\vec{a}$ or $\boxed{\frac{d}{dt}\vec{p} = \vec{F}}$

- remains valid if we use $\vec{p} = m\gamma v$ & coord.-time t
- but NOT manifestly invariant
- Note: $p^\mu u_\mu = -m \Rightarrow \left(\frac{d}{dt}p^\mu\right)u_\mu = 0 \Rightarrow -\frac{dE}{dt} + \frac{d\vec{p}}{dt}\cdot\vec{v} = 0$

$\Rightarrow \dot{E} = \vec{F}\cdot\vec{v} = \text{Power}$

- Lorentz invariant formalism:

$$\frac{d}{d\tau}p^\mu = K^\mu = \begin{pmatrix} K^0 \\ \vec{K} \end{pmatrix}$$

$$\vec{K} = \frac{dt}{d\tau}\frac{d\vec{p}}{dt} = \gamma\vec{F} \quad , K^0 = \frac{dt}{d\tau}\frac{d}{dt}E = \gamma\cdot\text{Power}$$

Note: no external force

$$\Longleftrightarrow$$

$$\frac{d}{d\tau}p^\mu = 0$$

$$\Longleftrightarrow$$

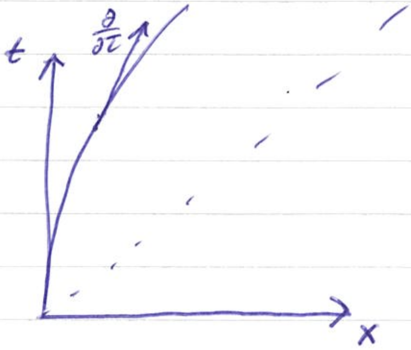
$$\frac{d^2}{d\tau^2}x^\mu = 0$$

$$\Longleftrightarrow$$

"straight line in Minkowski space-time"

Example:

- A rocket is at rest initially ($t=0$), and accelerates with constant acceleration a in its eigenframe.



- Lab frame: (t, x)
- At $t=0$: $x=0$
- What is the trajectory $t(\tau), x(\tau)$?

- $\frac{d}{d\tau} u^\mu = \frac{1}{m} K^\mu =: a^\mu \qquad (*)$
- in eigen-frame: $\bar{a}^\alpha = \begin{pmatrix} 0 \\ a \end{pmatrix} \qquad \bar{u}^\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- Lorentz transform back to lab frame: $u^\mu = \Lambda^\mu{}_\alpha u^\alpha$, $a^\mu = \Lambda^\mu{}_\alpha a^\alpha$

$$\Rightarrow u^\mu = \begin{pmatrix} \cosh\chi \\ \sinh\chi \end{pmatrix} \qquad a^\mu = a\begin{pmatrix} \sinh\chi \\ \cosh\chi \end{pmatrix}$$

$$(*) \Rightarrow \frac{d}{d\tau}\chi = a \quad \Rightarrow \boxed{\chi = a\tau}$$

- trajectory: $\begin{pmatrix} t \\ x \end{pmatrix} = x^\mu = \int_0^\tau u^\mu d\tau = \frac{1}{a}\begin{pmatrix} \sinh a\tau \\ \cosh a\tau - 1 \end{pmatrix}$

The metric

- In Special Relativity the space-time interval is given by

$$-d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$ for any inertial reference frame

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$ is the metric.

- We now want to use non-inertial coordinates, for example coordinates defined by the observer on our rocket.

$$\bar{t} = (t, x)$$
$$\bar{x} = (t, x)$$

- The space-time interval thus is

$$\boxed{-d\tau^2 = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\alpha} d\bar{x}^\alpha \frac{\partial x^\nu}{\partial \bar{x}^\beta} d\bar{x}^\beta = g_{\alpha\beta}\, d\bar{x}^\alpha d\bar{x}^\beta}$$

with $g_{\alpha\beta} = \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \eta_{\mu\nu} \frac{\partial x^\nu}{\partial \bar{x}^\beta}$

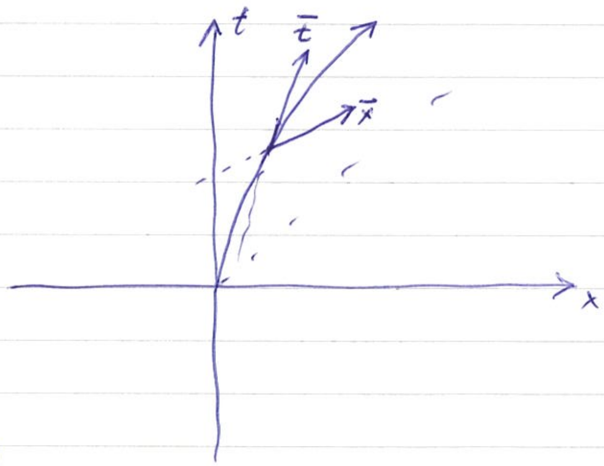
Example: The astronaut's metric

For our rocket we already know:

i) $\begin{pmatrix}\bar t\\ \bar x\end{pmatrix} = \begin{pmatrix}\tau\\ 0\end{pmatrix}$ corresponds to the rocket's world line, with eigentime τ

ii) $\begin{pmatrix}\bar t\\ \bar x\end{pmatrix} = \begin{pmatrix}\tau\\ x\end{pmatrix}$ for a fixed τ are all events that happen simultaneously with $\begin{pmatrix}\tau\\ 0\end{pmatrix}$, as seen by the rocket

iii) The $\bar x$-axis is parametrized in physical length, as measured by the rocket



- We know: $t(\bar t, 0) = \frac{1}{a} \sinh a\bar t$

  $x(\bar t, 0) = \frac{1}{a}(\cosh a\bar t - 1)$

- The direction $\left(\frac{\partial}{\partial \bar t}\right)^\mu$ is given by $\left(\frac{\partial}{\partial \bar t}\right)^\mu = v^\mu = \begin{matrix}\cosh a\bar t\\ \sinh a\bar t\end{matrix}$

- The direction $\left(\frac{\partial}{\partial \bar x}\right)^\mu$ has to be orthogonal to v^μ (ii) & normalized (iii).

$$\left(\frac{\partial}{\partial \bar x}\right)^\mu = \begin{pmatrix}\sinh a\bar t\\ \cosh a\bar t\end{pmatrix}$$

$$\Rightarrow t(\bar t, \bar x) = t(\bar t, 0) + \int_0^{\bar x} \left(\frac{\partial}{\partial \bar x}\right)^0 d\bar x = \left(\frac{1}{a} + \bar x\right) \sinh a\bar t$$

$$x(\bar t, \bar x) = x(\bar t, 0) + \int_0^{\bar x} \left(\frac{\partial}{\partial \bar x}\right)^1 d\bar x = \left(\frac{1}{a} + \bar x\right) \cosh a\bar t - \frac{1}{a}$$

$$\Rightarrow dt = (1 + a\bar{x}) \cosh a\bar{t} \; d\bar{t} + \sinh a\bar{t} \; d\bar{x}$$

$$dx = (1 + a\bar{x}) \sinh a\bar{t} \; d\bar{t} + \cosh a\bar{t} \; d\bar{x}$$

$$\Rightarrow -d\tau^2 = -dt^2 + dx^2 = -(1 + a\bar{x})^2 d\bar{t}^2 + d\bar{x}^2$$

$$\text{or} \quad \boxed{g_{\alpha\beta} = \begin{pmatrix} -(1+a\bar{x})^2 & 0 \\ 0 & 1 \end{pmatrix}}$$

$\Rightarrow$ Metric of <u>Minkowski space-time</u> (special relativity) in <u>accelerated</u> frame.

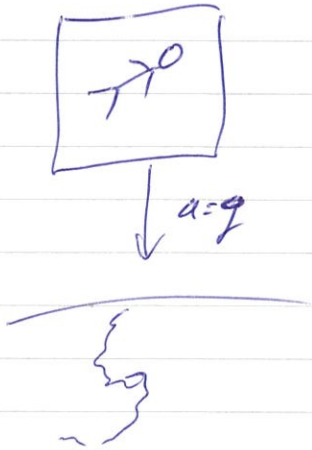
* <u>Particles</u> & <u>light</u> still move on "straight lines", or <u>geodesics</u> (No physics has changed.) We are just describing it in an accelerated frame.

General Relativity in a Nutshell

What is a gravitational field?

- On earth every object feels an acceleration downward,
- But locally* this is equivalent with an accelerated rocket!

  ⇒ If we enclose ourselves in a freely-falling elevator, this effect of the gravitational field goes away!



⇒ Einstein's Equivalence principle

$$\boxed{\text{"It is impossible to detect the existance of a gravitational field with a local experiment."}}$$

⇒ Implication: Gravitational-wave detectors are big (not local).

Using Newton's Gravity: $\ddot{x} = -\nabla\phi(x)$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{acceleration}}}{\ddot{x}^i} = \underset{\substack{\uparrow \\ g=\nabla\phi|_{x=0} \\ =0 \text{ in free fall}}}{-g^i} \underbrace{- \left.\frac{\partial^2\phi}{\partial x^i \partial x^j}\right|_{x=0} \cdot x^j}_{\substack{\text{non-local (higher order in } x\text{) terms} \\ \hat{=} \text{ tidal effects} \\ \hat{=} \text{ "real gravitational field"}}} + O(x^2)$$

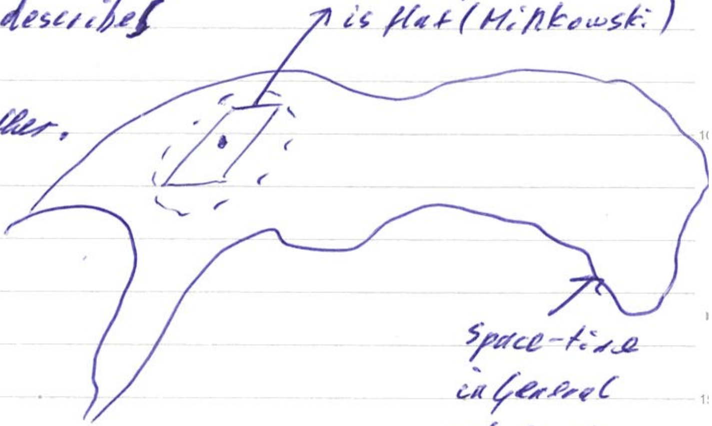
* local: neglecting tidal terms.

General Relativity in a Nutshell (cont.)

- **Idea:** • We describe particles only under the influence of gravity as freely falling.

  - Locally (i.e. excluding tidal terms) special relativity describes all physics. Locally space-time looks like Minkowski.

  - The theory of gravity describes how these locally flat space-time pieces fit together. Gravity is geometry!



⇒ Particles only under the influence of gravity & light follow geodesics of the space-time!

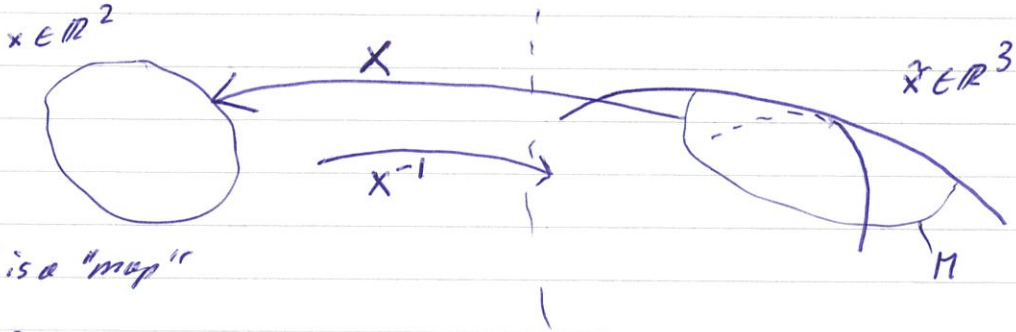
- e.g. orbits are geodesics.
- but orbits intersect themselfs! 

this is impossible in a flat space-time ()!

⇒ In general relativity space-time cannot be flat.

Geometry in 2D!

- To aid our reasoning we look at 2D surfaces embedded in $\mathbb{R}^3$



- X is a "map"
- $\mathbb{R}^3$ has Euklidian metric δ_{ij}
- $\tilde{x}^i = (X^{-1})^i(x^a) \Rightarrow \frac{\partial \tilde{x}^i}{\partial x^a} = \frac{\partial (X^{-1})^i}{\partial x^a}$

$\Rightarrow$ we can pull the Euklidian metric! $g_{ab} = \delta_{ij} \frac{\partial \tilde{x}^i}{\partial x^a} \frac{\partial \tilde{x}^j}{\partial x^b}$

$$\Rightarrow \Big|\ ds^2 = g_{ab}\, dx^a dx^b = \delta_{ij}\, d\tilde{x}^i d\tilde{x}^j$$

Claim: everything we care about is encoded in g_{ab}

Curves & Geodesics

We now add a parametrized curve $\rho(\lambda)$:

- $\tilde{\rho}(\lambda) = X^{-1}\big(\rho(\lambda)\big)$
- Curve length: $L = \displaystyle\int_{\lambda_1}^{\lambda_2} \left| \frac{d\tilde{\rho}^i}{d\lambda} \right| d\lambda = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}\, d\lambda$
- We parametrize $\tilde{\rho}(\lambda)$ (or $\rho(\lambda)$ using g_{ab}) by its length:

$$\left| \frac{d}{d\lambda} \tilde{\rho}^i(\lambda) \right| = 1 \quad ; \text{Notation: } \dot{\tilde{\rho}} \equiv \frac{d}{d\lambda} \tilde{\rho}$$

$$\Rightarrow \dot{\vec{\tilde{\rho}}} \cdot \ddot{\vec{\tilde{\rho}}} = \frac{1}{2} \left(\dot{\vec{\tilde{\rho}}} \cdot \dot{\vec{\tilde{\rho}}} \right)^{\cdot} = 0 \quad \Rightarrow \underline{\underline{\dot{\tilde{\rho}} \perp \ddot{\tilde{\rho}}}}$$

- We define:
  - $\vec{n}(\tilde{x})$: normal to M in $\tilde{x}^\mu$
  - $\vec{S} \equiv \vec{n} \wedge \dot{\vec{\tilde{\rho}}}$ the intrinsic normal

$$\Rightarrow \boxed{\ddot{\vec{\tilde{\rho}}} = k_g \vec{S} + k_n \vec{n}}$$

k_g: geodesic curvature of ρ

k_n: normal curvature of ρ

$$\boxed{k_g = 0 \Leftrightarrow \rho \text{ a geodesic}}$$

- k_g is <u>intrinsic</u>; i.e independent of embedding (determined by $\rho(\lambda)$ & g_{ab})

The geodesic equation

- The curve length is $\int_{\lambda_1}^{\lambda_2} \sqrt{g_{ab}\,\dot{\gamma}^a \dot{\gamma}^b}\, d\lambda$ & is invariant under re-parametrization $\lambda = \lambda(\tilde{\lambda})$. $\Rightarrow$ We always parametrize by length; $d\lambda = \sqrt{g_{ab}\, dx^a dx^b}$.

- Geodesics are given by the shortest possible length:

$$\delta \int \sqrt{g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}}\, d\lambda = 0 \quad \Leftrightarrow \quad \text{geodesic equation}$$

- since $\delta \int \sqrt{g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}} = \frac{1}{2} \int \frac{1}{\sqrt{}}\, \delta\left(g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \right) d\lambda$

  $\Rightarrow$ if we restrict the parametrization to $d\lambda = d(\text{length})$ we can use

$$\boxed{\delta \int g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}\, d\lambda}$$

Reminder Lagrange Equations

$$\delta \int L(q, \dot{q})\, dt$$

$$= \int \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \quad dt$$

$$\delta q = \frac{d q(t,\eta)}{d\eta}$$

$$\delta \dot{q} = \frac{d}{d\eta}\frac{d}{dt} q(t,\eta) = \frac{d}{dt}\delta q$$

$$= \int \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \quad dt$$

$= 0$ for all δq

$$\Rightarrow \boxed{\frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}}$$

Now: $\delta \int g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}\, d\lambda = 0$

$$\Rightarrow L\left(\gamma^a, \frac{d\gamma^a}{d\lambda}\right) = g_{ab}(\gamma^a) \frac{d\gamma^a}{d\lambda}\frac{d\gamma^b}{d\lambda}$$

$$\Rightarrow \frac{\partial L}{\partial \gamma^a} = g_{\alpha\beta,a} \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

$\cdot = \frac{d}{d\lambda}$ $\quad \Rightarrow \frac{\partial L}{\partial \dot\gamma^a} = g_{a\beta}\frac{dx^\beta}{d\lambda} + g_{\alpha a}\frac{dx^\alpha}{d\lambda}$

$$\Rightarrow \frac{d}{d\lambda}\frac{\partial L}{\partial \dot\gamma^a} = \left(g_{a\beta,\alpha} + g_{\alpha a,\beta}\right)\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} + 2 g_{a\beta}\frac{d^2x^\beta}{d\lambda^2}$$

$$\Rightarrow 0 = \frac{d}{d\lambda}\frac{\partial L}{\partial \dot\gamma^a} - \frac{\partial L}{\partial \gamma^a}$$

$$= 2 g_{a\beta}\frac{d^2x^\beta}{d\lambda^2} + \left(g_{a\beta,\alpha} + g_{\alpha a,\beta} - g_{\alpha\beta,a}\right)\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

· Define: $\boxed{\Gamma^c{}_{\alpha\beta} := \frac{1}{2} g^{ca}\left(g_{a\beta,\alpha} + g_{\alpha a,\beta} - g_{\alpha\beta,a}\right)}$

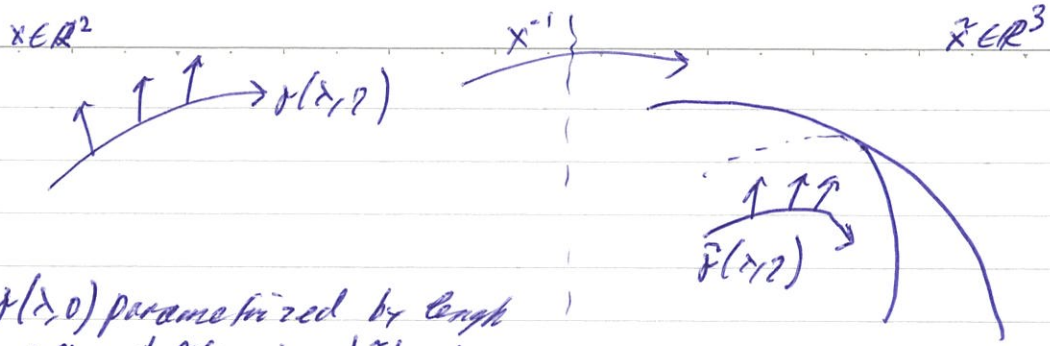
Christoffel Symbols

$\ast \cdot \frac{1}{2}g^{ca}$ $\quad \Rightarrow \quad 0 = \frac{d^2x^c}{d\lambda^2} + \Gamma^c{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$

or $\quad \boxed{\ddot x^c + \Gamma^c{}_{\alpha\beta}\dot x^\alpha \dot x^\beta = 0}$ $\quad$ <u>geodesic eqn. for</u> $x^c = \gamma^c(\lambda)$ $\quad \left(\cdot = \frac{d}{d\lambda}\right)$

$\Rightarrow$ Geodesics are determined by g_{ab}!

Parallel transport & Covariant Derivative



- $\gamma(\lambda,0)$ parametrized by length
  $\Rightarrow \tilde{v} := \frac{d}{d\lambda}\tilde{\gamma}(\lambda,0)$; $|\tilde{v}| = 1$

- $\gamma(\lambda,\eta)$ for λ fixed parametrized by length ; $\vec{V}(\lambda) := \frac{d}{d\eta}\tilde{\gamma}(\lambda,\eta)\Big|_{\eta=0}$

  $\Rightarrow \vec{V}(\lambda)$ is a (normalized) vector field along $\tilde{\gamma}(\lambda,0)$

Def: Covariant derivative (for a vector field) $\nabla_{\vec{v}}\vec{V}$

$$\nabla_{\vec{v}}\vec{V} = \frac{d}{d\lambda}\vec{V}(\lambda) - \left(\frac{d}{d\lambda}\vec{V}, \vec{n}\right)\cdot\vec{n}$$

i.e. $\nabla_{\vec{v}}\vec{V}$ is the orthogonal projection of $\frac{d\vec{V}}{d\lambda}$ onto M

Def: $\vec{V}$ parallel along $\gamma \Leftrightarrow \nabla_{\vec{v}}\vec{V} = 0$

Note: γ is a geodesic $\Leftrightarrow \dot{\gamma}$ is parallel along $\gamma \Leftrightarrow \nabla_{\dot{\gamma}}\dot{\gamma} = 0$

$$\Leftrightarrow$$

$$\frac{d^2}{d\lambda^2}\gamma^c + \Gamma^c{}_{ab}\frac{d}{d\lambda}\gamma^a\frac{d}{d\lambda}\gamma^b$$

Note: $\nabla_{\vec{v}} \vec{v}$ is intrinsic (despite my extrinsic, but intuitive definition)

No proof: In coordinates $\nabla_{\vec{v}} \vec{v}$ is given by

$$\boxed{(\nabla_{\dot\gamma} \vec{v})^c = \frac{d}{d\lambda} v^c + \Gamma^c{}_{ab}\, \dot\gamma^a v^b}$$ (clearly intrinsic)

(no proof, but consistent with $(\nabla_{\dot\gamma} \dot\gamma)^c = \ddot\gamma^c + \Gamma^c{}_{ab}\, \dot\gamma^a \dot\gamma^b$)

In particular, for coordinate vector fields $\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}$:

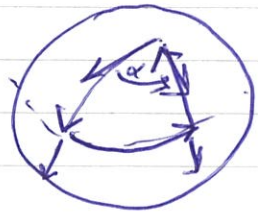
$$\left(\nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b}\right)^c = \underbrace{\frac{\partial}{\partial x^a} \underbrace{\left(\frac{\partial}{\partial x^b}\right)^c}_{\delta^c_b}}_{0} + \Gamma^c{}_{ab}$$

i.e. $$\boxed{\nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b} = \Gamma^c{}_{ab} \frac{\partial}{\partial x^c}}$$

Gauss Curvature

Example: parallel transport on a sphere (radius a)

parallel transport on a closed loop results in a rotation of the vector by α



Triangle area: $\frac{4\pi a^2}{2} \cdot \frac{\alpha}{2\pi} = \alpha a^2$

Define Gauss curvature: $\boxed{K = \lim_{\gamma \to 0} \frac{\alpha}{A_{enclosed}}} \underset{\text{on sphere}}{=} \frac{1}{a^2}$

completely intrinsic curvature measure!

In words: The Gauss Curvature is given by $K = \frac{\alpha}{A}$, where

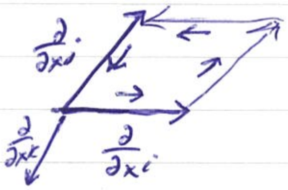
α is the rotation angle of a vector after parallel transport along an infinitesimally closed curve γ, and A is the area encircled by that curve γ.

In 2D K is the only intrinsic curvature measure.

Riemann curvature

All that is left to do is to generalize the Gauss curvature to 4 (3+1) dimensions!

Idea:
- pick 2 vectors that span a 2D-sub-space, say $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}$
- These vectors define an infinitesimal loop:



- pick a 3rd vector $\frac{\partial}{\partial x^k}$, and transport it in a circle. How much did it change?

$\Rightarrow$ The result is again a vector, given by

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} \equiv \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}$$

$\hat{=}$ Riemann curvature

Riemann Curvature in Coordinates

$$R^{\alpha}{}_{\beta\mu\nu} :\!= \left[\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu \right) \frac{\partial}{\partial x^\beta} \right]^\alpha \qquad \left(\nabla_\mu \equiv \nabla_{\frac{\partial}{\partial x^\mu}} \right)$$

What is $\nabla_\mu \nabla_\nu \frac{\partial}{\partial x^\beta}$?

$$\nabla_\mu \left(\nabla_\nu \frac{\partial}{\partial x^\beta} \right) = \nabla_\mu \Big(\underbrace{\underbrace{\Gamma^\alpha{}_{\nu\beta}}_{\text{component}} \underbrace{\frac{\partial}{\partial x^\alpha}}_{\text{basis}}}_{\text{vector!}} \Big)$$

$$= \left(\nabla_\mu \frac{\partial}{\partial x^\alpha} \right) \left(\Gamma^\alpha{}_{\nu\beta} \right) \qquad + \Gamma^\alpha{}_{\nu\beta,\mu} \frac{\partial}{\partial x^\alpha}$$

$$= \left[\Gamma^\lambda{}_{\mu\alpha} \Gamma^\alpha{}_{\nu\beta} + \Gamma^\lambda{}_{\nu\beta,\mu} \right] \frac{\partial}{\partial x^\lambda}$$

$$\Rightarrow \boxed{R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\nu\beta,\mu} - \Gamma^\alpha{}_{\mu\beta,\nu} + \Gamma^\alpha{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\sigma} \Gamma^\sigma{}_{\mu\beta}}$$