Review:
Towards the Fundamental Quantum Limit of Linear Measurement of Classical Signals

Speaker: Yutaro Enomoto

Overview

-- H. Miao et al., PRL 119, 050801 (2017).

-- The quantum Cramér-Rao bound (QCRB) sets a strict limit for the parameter estimation of a quantum system; an interferometer is a quantum system with parameter x(t).

-- Authors applied QCRB to linear measurements of continuous classical signal and derived a condition to achieve the bound.

-- Their general discussion was applied to GW detectors and they found that a test mass can be viewed as a resource for improving the sensitivity.

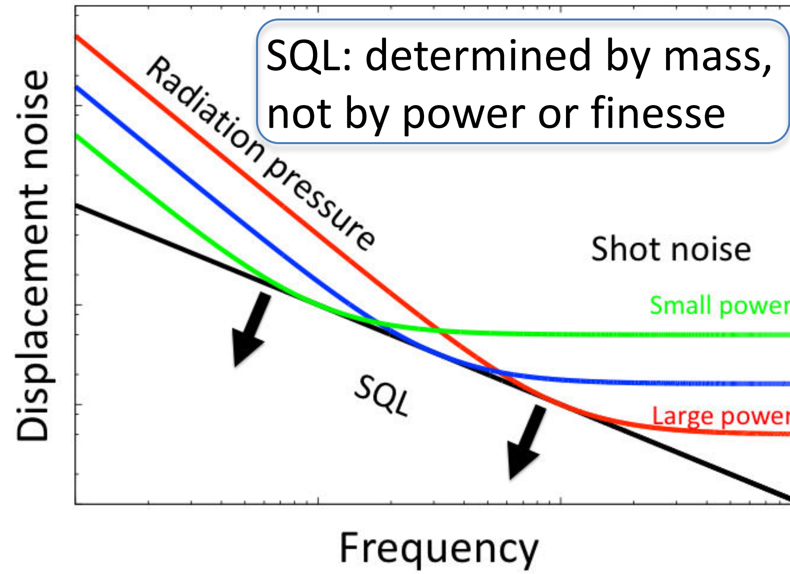
Contents

this paper's contents

1. Introduction

-- Quantum noise in a position measurement (like a GW detector) is one of the “fundamental” noises.

-- It had been considered that the standard quantum limit (SQL) is fundamental and cannot be beaten.



-- It turned out that quantum non-demolition (QND) measurement enables us to beat the SQL.

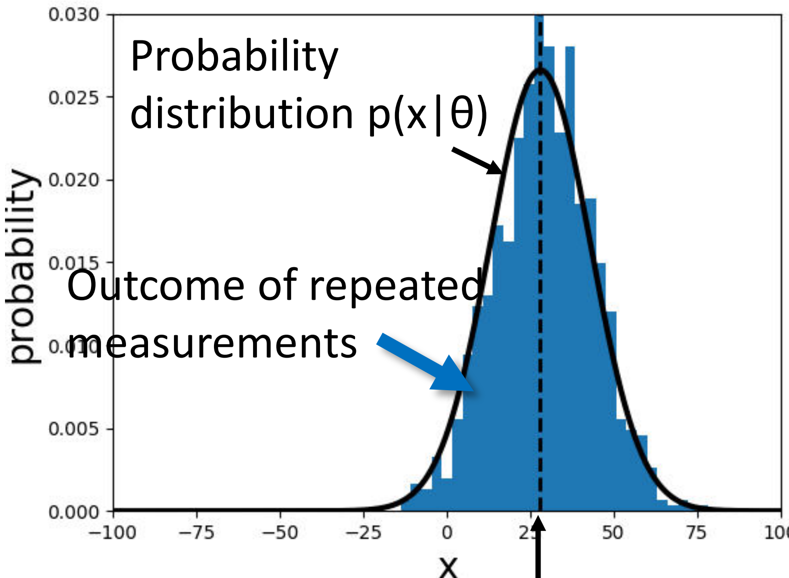
-- The SQL is not the fundamental limit,

then ***<u>what is the fundamental limit of the quantum noise???</u>***

=> Quantum Cramér-Rao bound is fundamental and can’t be beaten.

2. Cramér-Rao bound

-- In estimation theory and statistics, the Cramér–Rao bound expresses a lower bound on the variance of estimators of a deterministic (fixed, though unknown) parameter. (from wikipedia)



??? unknown parameter θ
(mean μ in this example)

Try to ESTIMATE the parameter(s) of the measured system θ

Best achievable estimation error is limited by **Cramér-Rao bound**.
→ cannot be beaten, fundamental

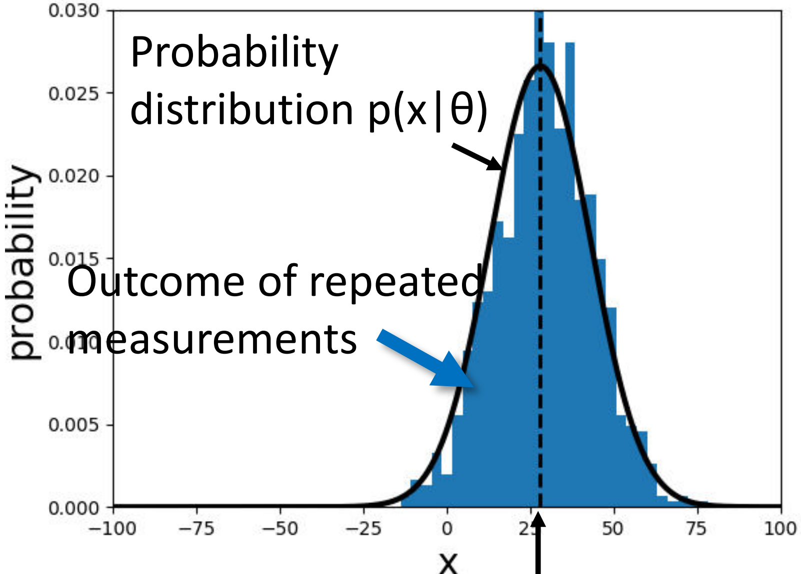
NOTE:
-- θ is a (set of) classical number(s)
-- classical system → (original) CRB
-- quantum system → quantum CRB

Classical case

-- Classical measurement is characterized by probability p(x|θ):

x: measurement outcome, θ: parameter that characterizes p(x)

$$\theta = (\theta_1, \theta_2, \cdots, \theta_d)$$



??? unknown parameter θ
(mean μ in this example)

J: Fisher matrix

$$J_{ij} = \sum_x \frac{1}{p(x|\theta)} \frac{\partial p(x|\theta)}{\partial \theta_i} \frac{\partial p(x|\theta)}{\partial \theta_j}$$

Then this inequality holds:

$$C \geq \frac{1}{N} J^{-1}$$

(C: covariance matrix of θ)

(N: # of repeats of measurement)

or more explicitly

$$(\Delta\theta_i^{\text{est}})^2 \geq \frac{1}{N} (J^{-1})_{ii}$$

Cramér-Rao bound

Quantum case

-- If measurement basis is fixed, probability p(x|θ) is determined.

(Quantum theory predicts probability distribution.)

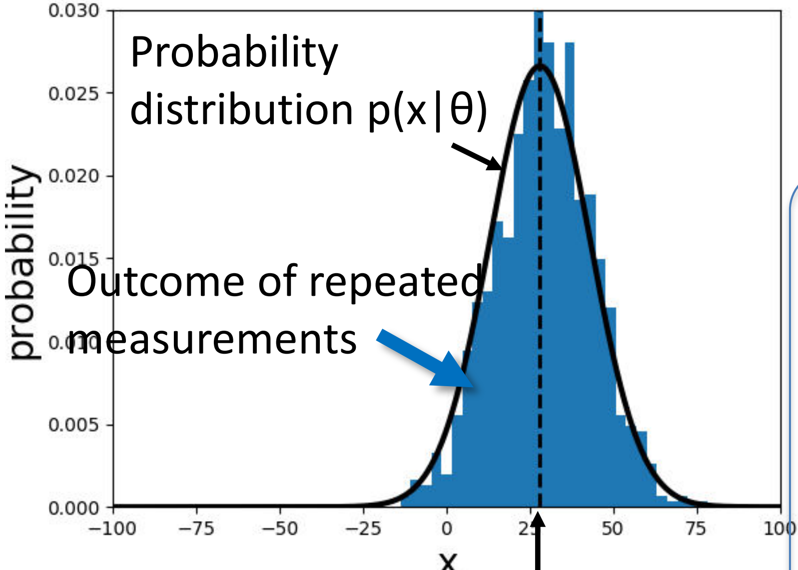
Cramér-Rao bound is applied

-- However, measurement basis is arbitrary.

$\hat{E}_x$: POVM, generalization of projection operator $|x\rangle\langle x|$

$$p(x|\theta) = \mathrm{tr}[\hat{\rho}_\theta \hat{E}_x]$$

$\hat{\rho}_\theta$: density operator (system's state)



Choice of measurement operator $\hat{E}_x$

??? unknown state $\hat{\rho}_\theta$

Measurement-operator-independent inequality will be important!! For example...

S: quantum Fisher matrix

$$S_{ij} = \frac{1}{2}\mathrm{tr}[\hat{\rho}_\theta(\hat{L}_{\theta i}\hat{L}_{\theta j}^\dagger + \hat{L}_{\theta j}\hat{L}_{\theta i}^\dagger)], \quad \frac{\partial \hat{\rho}_\theta}{\partial \theta_i} = \frac{1}{2}\left(\hat{\rho}_\theta \hat{L}_{\theta i} + \hat{L}_{\theta i}^\dagger \hat{\rho}_\theta\right)$$

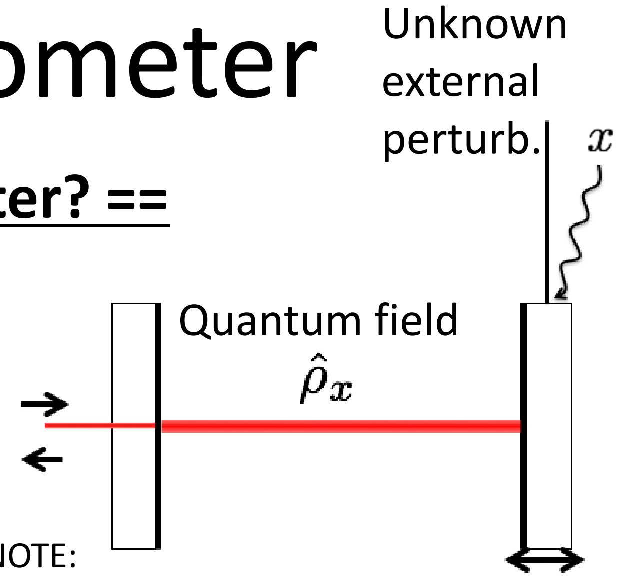
Then, $$(\Delta\theta_i^{\mathrm{est}})^2 \geq \frac{1}{N}(J^{-1})_{ii} \geq \frac{1}{N}(S^{-1})_{ii}$$

Quantum Cramér-Rao bound

NOTE: -- There are an infinite number of quantum Fisher matrix.

-- For more detail on QCRB, see e.g. 量子測定と量子制御

QCRB and interferometer



== How is the QCRB related to interferometer? ==

-- Consider displacement measurement.

Optical field is quantum, and depends on unknown parameter x(t) that we want to estimate.

=> **The QCRB can be applied!!**

NOTE:
From here, x is not measurement outcome but parameter to be estimated. Sorry for confusing notations…

We can set the limit to **the estimation error of x(t)**

||

Quantum noise of the measurement

-- Next questions: $(\Delta x^{\mathrm{est}})^2 \geq \frac{1}{N}(J^{-1}) \geq \frac{1}{N}(S^{-1})$

-- What is the meaning of this inequality???

-- What is S like???

=> **Next a few slides**

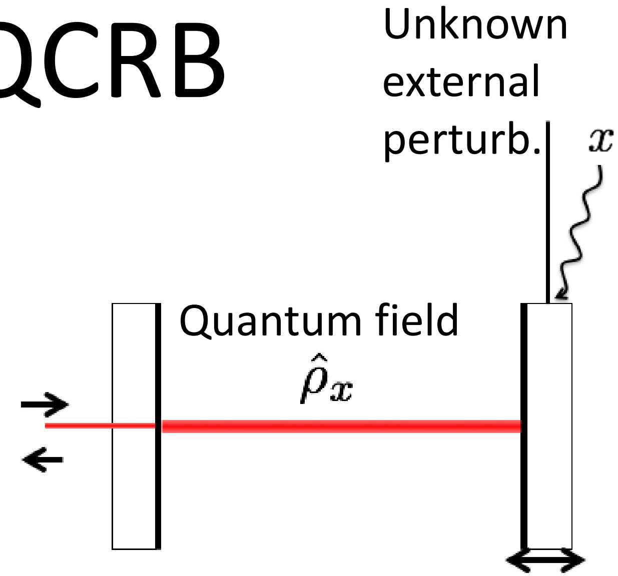
Meaning of the QCRB

the QCRB

$$(\Delta x^{\mathrm{est}})^2 \geq \frac{1}{N}(J^{-1}) \geq \frac{1}{N}(S^{-1})$$

where $S = \frac{1}{2}\mathrm{tr}[\hat{\rho}_x(\hat{L}_x\hat{L}_x^\dagger + \hat{L}_x\hat{L}_x^\dagger)]$

$$\frac{\partial \hat{\rho}_x}{\partial x} = \frac{1}{2}\left(\hat{\rho}_\theta \hat{L}_x + \hat{L}_x^\dagger \hat{\rho}_x\right)$$



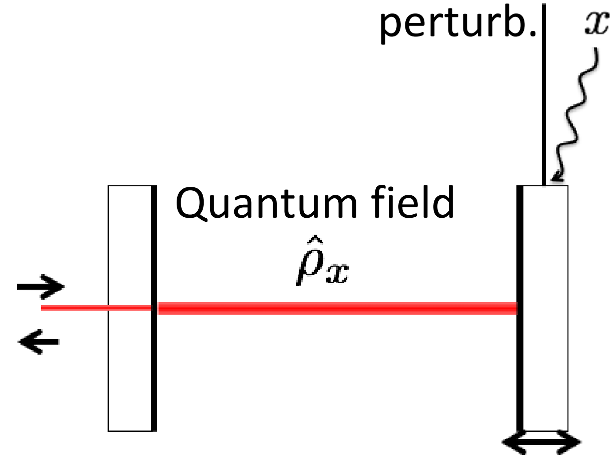
NOTE: the quantum Fisher information S depends on the state $\hat{\rho}_x$

=> This limit depends on, for example:

- **injected laser power**
- **how much the vacuum is squeezed**

cf.) The SQL does not depend on the power.

Single-shot measurement



the QCRB

$$(\Delta x^{\mathrm{est}})^2 \geq \frac{1}{N}(J^{-1}) \geq \frac{1}{N}(S^{-1})$$

where $S = \frac{1}{2}\mathrm{tr}[\hat{\rho}_x(\hat{L}_x\hat{L}_x^\dagger + \hat{L}_x\hat{L}_x^\dagger)]$

$$\frac{\partial \hat{\rho}_x}{\partial x} = \frac{1}{2}\left(\hat{\rho}_\theta \hat{L}_x + \hat{L}_x^\dagger \hat{\rho}_x\right)$$

= An example to see what S, the quantum Fisher matrix, is like =

-- single-shot measurement: an impulse $x/\tau \to x\delta(t)$ is applied,
then try to estimate x => $\hat{H}_{\mathrm{int}} = -\hat{F}x/\tau$ (τ: duration of the impulse)

$$\hat{\rho}_x = \mathrm{e}^{i\hat{F}x/\hbar}\hat{\rho}\mathrm{e}^{-i\hat{F}x/\hbar},\ \frac{d\hat{\rho}_x}{dx} = +\frac{i}{\hbar}\hat{F}\hat{\rho}_x - \frac{i}{\hbar}\hat{\rho}_x\hat{F} = \frac{1}{2}\left(\hat{\rho}_x\hat{L}_x + \hat{L}_x^\dagger\hat{\rho}_x\right),$$ where $\hat{L}_x = -2i\hat{F}/\hbar$

$$(\Delta x^{\mathrm{est}})^2 \geq \frac{\hbar^2}{4\langle \hat{F}^2\rangle}$$

the QCRB

The QCRB is **inversely proportional** to the fluctuation of the observable that couples to x.
=> **in this example, the (radiation pressure) force**

Single-shot measurement

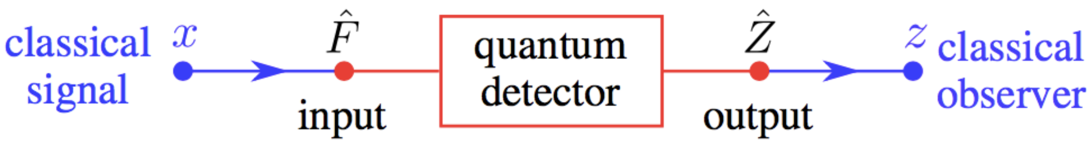
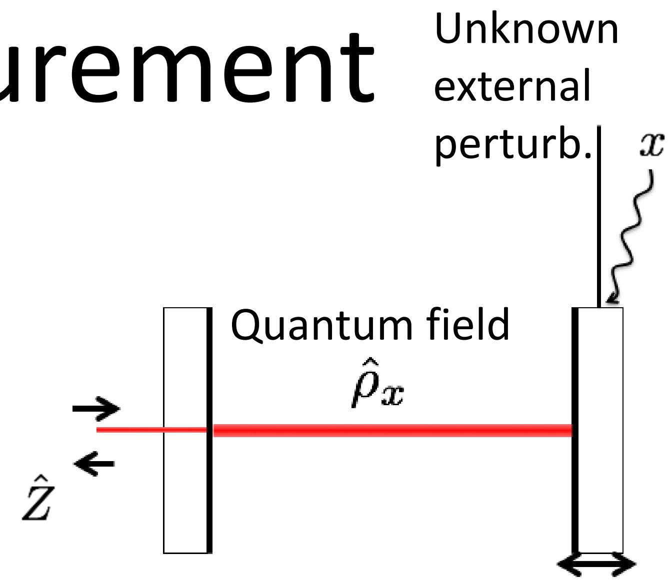


FIG. 1 of the paper



= Authors' approach (linear response theory) =

-- With linear response theory, they derived the conditions to be satisfied in order **<u>for the equality to hold</u>**.

$$\hat{H}_{\rm int} = -\hat{F} x \delta(t)$$

$\hat{Z}$: measured observable (e.g. some quadrature of the output optical field)

unperturbed operators

$$\hat{Z} = \hat{Z}^{(0)} + (i/\hbar)[\hat{Z}^{(0)}, \hat{F}^{(0)}]x = \hat{Z}^{(0)} + \chi_{ZF} x$$

$$\sigma_{xx} \equiv {\rm Tr}[\hat{\rho}_{\rm det}(\hat{x}_{\rm est} - x)^2] = \sigma_{ZZ}/\chi_{ZF}^2 \quad (\hat{x}_{\rm est} = \hat{Z}/\chi_{ZF})$$

Then, $\underbrace{\sigma_{ZZ}\sigma_{FF} - \sigma_{ZF}^2 \geq (\hbar^2/4)\chi_{ZF}^2}_{\text{the Uncertainty Relation}}$ => $\sigma_{xx} \geq \dfrac{\hbar^2}{4\sigma_{FF}} + \dfrac{\sigma_{ZF}^2}{\sigma_{FF}\chi_{ZF}^2} \geq \dfrac{\hbar^2}{4\sigma_{FF}} = \sigma_{xx}^{\rm QCRB}$

Correlation σ_{ZF} needs to be zero.

Summary up to here

-- The QCRB limits the parameter estimation error.

position measurement case → limits the quantum noise level

$$(\Delta x^{\mathrm{est}})^2 \geq \frac{1}{N}(J^{-1}) \geq \frac{1}{N}(S^{-1})$$

-- The QCRB depends on a state of the system, such as **laser power**

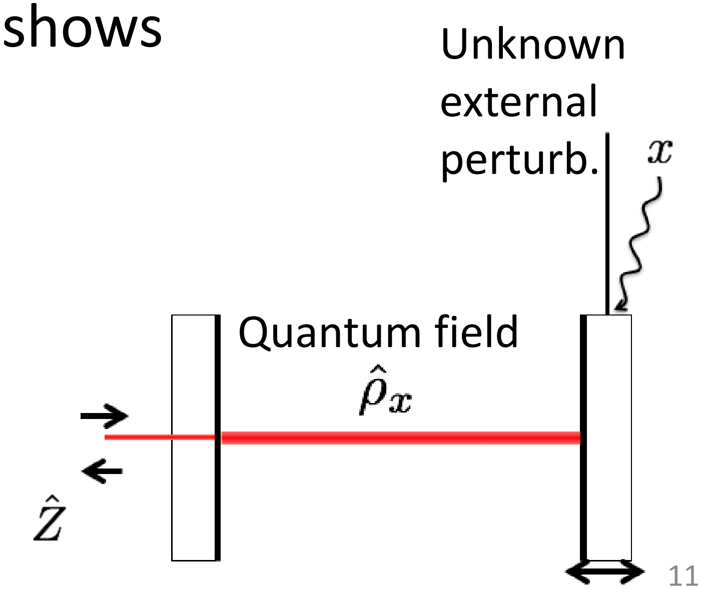
<u>NOT similar to the SQL</u>

-- An example of single-shot measurement shows

$$\sigma_{xx} \geq \frac{\hbar^2}{4\sigma_{FF}} + \frac{\sigma_{ZF}^2}{\sigma_{FF}\chi_{ZF}^2} \geq \frac{\hbar^2}{4\sigma_{FF}} = \sigma_{xx}^{\mathrm{QCRB}}$$

-- The QCRB is **<u>inversely proportional</u>** to the fluctuation of the observable that couples to x, **<u>the (radiation pressure) force</u>** in the example.

-- Correlation σ_{ZF} needs to be zero to achieve the bound



3. What this paper showed

-- Our interest: **continuous measurement of x(t)** $\hat{H}_{\text{int}} = -\hat{F}x(t)$

- output optical field is **sequentially and continuously** measured.
- the number of parameters is **infinite**; x(t) is continuous.

=> Previous discussion cannot be applied directly.

(Not sequential measurement, finite parameter set)

-- For such linear continuous measurement, the QCRB had been derived, but the condition for the equality to hold had not.

$$\sigma_{xx}^{\text{QCRB}}(\omega) = \frac{\hbar^2}{4\bar{S}_{FF}(\omega)}$$

Symmetrized (or one-sided) spectral density

-- This paper derived that condition.

Detector is in a minimum uncertainty state and $\bar{S}_{ZF}(\omega) = 0$ (Backaction to be 0)

Equality condition is important; this will tell us how we should design a detector.

The second condition can be satisfied if $\mathrm{Im}[\chi_{FF}(\omega)] = 0$

3. What this paper showed

Derivation

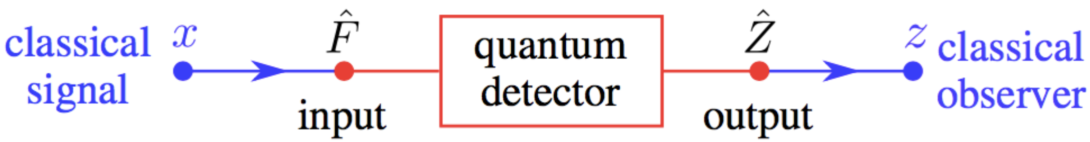
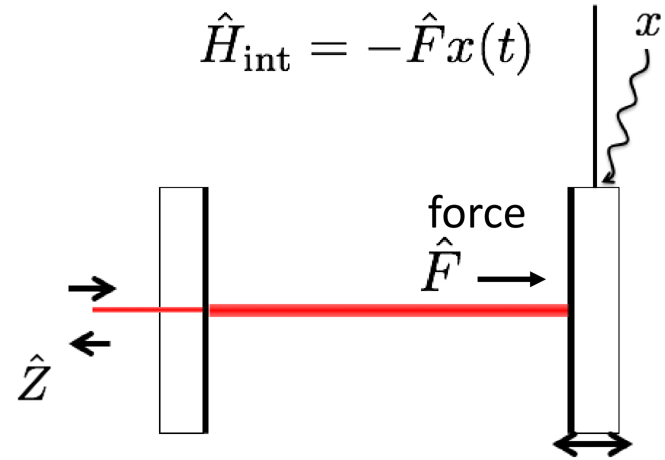


FIG. 1 of the paper



1. With a linear-response approach, like

$$\hat{Z}(t) = \hat{Z}^{(0)}(t) + \int_{-\infty}^{\infty} dt' \chi_{ZF}(t-t')x(t'), \quad \text{where} \quad \chi_{AB}(t,t') \equiv \frac{i}{\hbar}[\hat{A}^{(0)}(t), \hat{B}^{(0)}(t')]\Theta(t-t')$$

and using the <u>**simultaneous (sequential) measurability**</u> condition,

$$[\hat{Z}(t), \hat{Z}(t')] = 0 \quad \forall t, t'$$

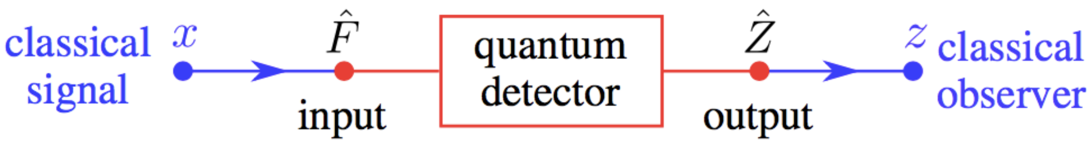
e.g. photo-detection at the output port does not disturb the interferometer

they described the Uncertainty Relation with spectral densities:

$$\bar{S}_{ZZ}(\omega)\bar{S}_{FF}(\omega) - |\bar{S}_{ZF}(\omega)|^2 \geq \frac{\hbar^2}{4}|\chi_{ZF}(\omega)|^2 + \hbar|\mathrm{Im}[\bar{S}_{ZZ}(\omega)\chi_{FF}(\omega) - \bar{S}^*_{ZF}(\omega)\chi_{ZF}(\omega)]|.$$

This was derived in another paper by the first author, [PRA **95** 012103 (2017)]

Derivation



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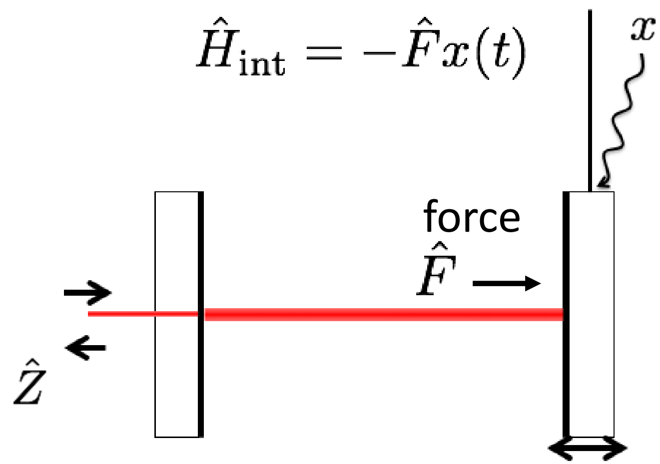


2. Equality condition. If the detector is in a minimum uncertainty state,

e.g. coherent state, squeezed vacuum state

$$\bar{S}_{ZZ}(\omega)\bar{S}_{FF}(\omega) - |\bar{S}_{ZF}(\omega)|^2$$

equality

ZERO (PRA **95** 012103 (2017))

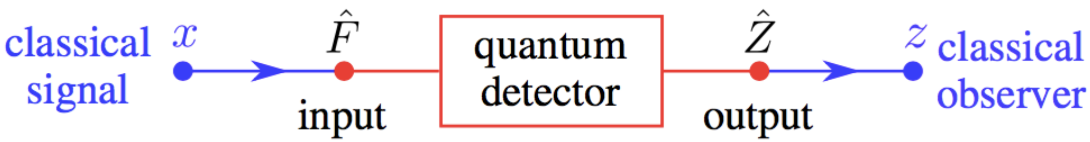
$$\geq \frac{\hbar^2}{4}|\chi_{ZF}(\omega)|^2 + \hbar|\mathrm{Im}[\bar{S}_{ZZ}(\omega)\chi_{FF}(\omega) - \bar{S}^*_{ZF}(\omega)\chi_{ZF}(\omega)]|.$$

On the other hand, $\hat{Z}(t) = \hat{Z}^{(0)}(t) + \int_{-\infty}^{\infty} dt' \chi_{ZF}(t-t')x(t')$ => $\hat{x}_{\rm est}(\omega) = \hat{Z}(\omega)/\chi_{ZF}(\omega)$

Then the Uncertainty Relation turns to be

$$\sigma_{xx}(\omega) \geq \frac{\hbar^2}{4\bar{S}_{FF}} + \frac{|\bar{S}_{ZF}|^2 + \hbar|\mathrm{Im}[\bar{S}_{ZZ}\chi_{FF} - \bar{S}^*_{ZF}\chi_{ZF}]|}{\bar{S}_{FF}|\chi_{ZF}|^2} \geq \frac{\hbar^2}{4\bar{S}_{FF}} = \sigma_{xx}^{\rm QCRB}$$

Derivation



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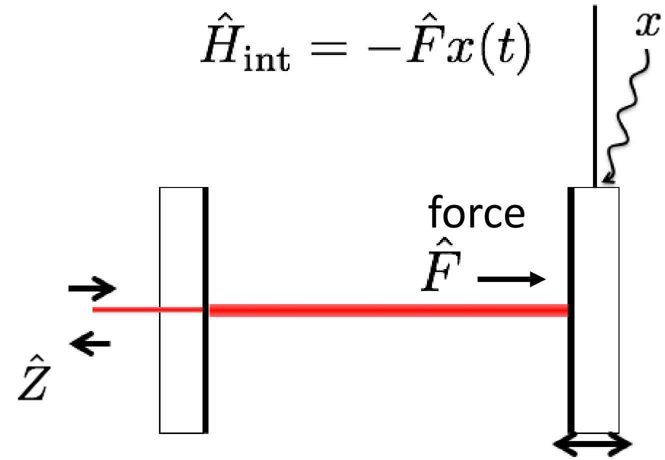


2. Equality condition. If the detector is in <u>a minimum uncertainty state</u>,

e.g. coherent state, squeezed vacuum state

$$\bar S_{ZZ}(\omega)\bar S_{FF}(\omega) - |\bar S_{ZF}(\omega)|^2 \geq \frac{\hbar^2}{4}|\chi_{ZF}(\omega)|^2 + \underline{\hbar|\mathrm{Im}[\bar S_{ZZ}(\omega)\chi_{FF}(\omega) - \bar S^*_{ZF}(\omega)\chi_{ZF}(\omega)]|}.$$

equality

ZERO (PRA **95** 012103 (2017))

On the other hand, $\hat Z(t) = \hat Z^{(0)}(t) + \int_{-\infty}^{\infty} dt' \chi_{ZF}(t-t')x(t')$ => $\hat x_{\mathrm{est}}(\omega) = \hat Z(\omega)/\chi_{ZF}(\omega)$

Then the Uncertainty Relation turns to be

$$\sigma_{xx}(\omega) \overset{=}{\geq} \frac{\hbar^2}{4\bar S_{FF}} + \frac{|\cancel{\bar S_{ZF}}|^2 + \cancel{\hbar|\mathrm{Im}[\bar S_{ZZ}\chi_{FF} - \bar S^*_{ZF}\chi_{ZF}]|}}{\bar S_{FF}|\chi_{ZF}|^2} \geq \frac{\hbar^2}{4\bar S_{FF}} = \sigma_{xx}^{\mathrm{QCRB}}$$

=> This shows the QCRB is achieved if:

$\boxed{\text{Detector is in a minimum uncertainty state}}$ and $\boxed{\bar S_{ZF}(\omega) = 0}$

Derivation

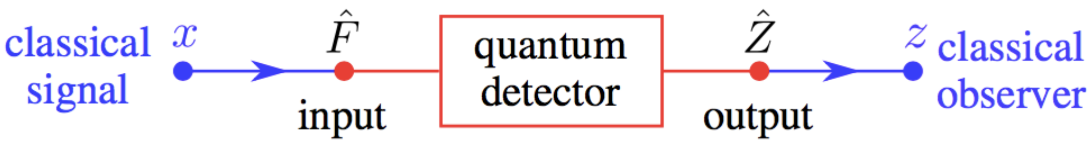
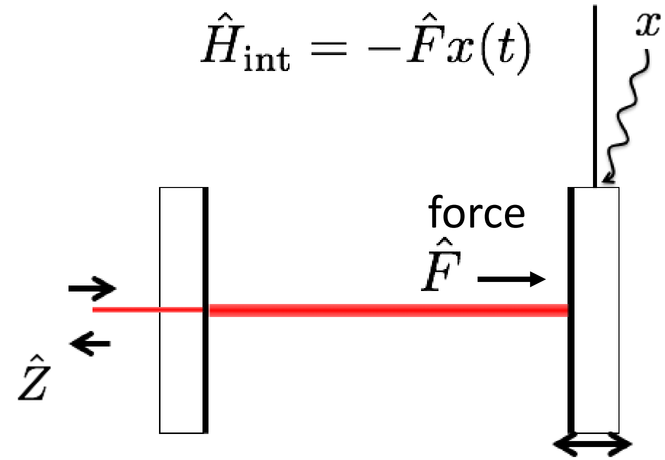


FIG. 1 of the paper



3. Can we achieve $\bar{S}_{ZF}(\omega) = 0$?

=> If $\mathrm{Im}[\chi_{FF}(\omega)] = 0$, we can find the optimal $\hat{Z}$ to achieve it.

They showed $\mathrm{Im}[\chi_{FF}(\omega)] = 0$ <=> $\underline{\hat{Z} = \hat{Z}_1 \sin\theta + \hat{Z}_2 \cos\theta,\ \tan\theta \in \text{Reals}}$

Homodyne phase is real!!

They also showed what happens if $\mathrm{Im}[\chi_{FF}(\omega)] \neq 0$:

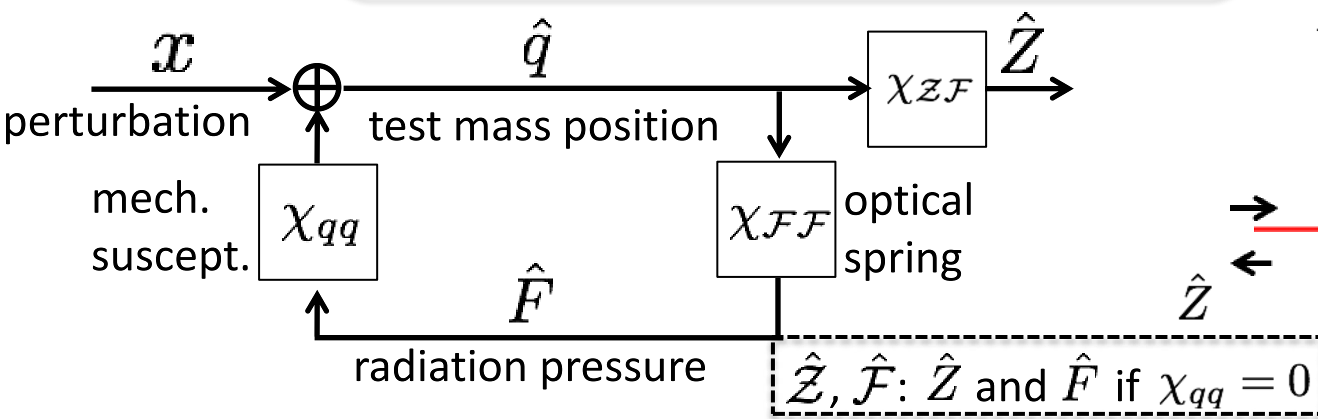
$$\boxed{\sigma_{xx}^{\text{QCRB}} \leq \min \sigma_{xx} \leq 2\sigma_{xx}^{\text{QCRB}}}$$

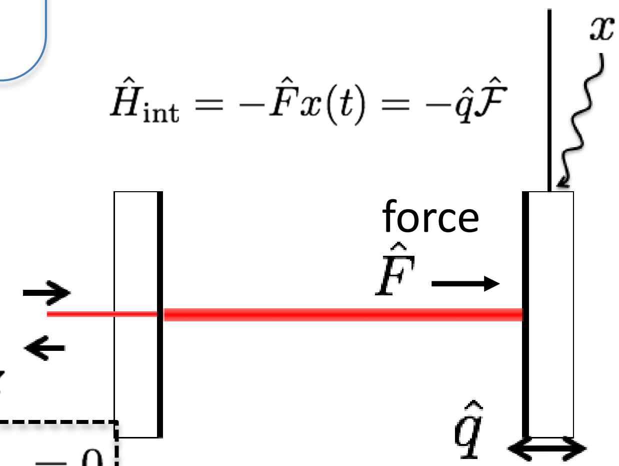
=> **At least, $\sqrt{2}$ times worse sensitivity can be achieved**

4. Application to GW detectors

-- We can re-write the QCRB as:

$$\sigma_{xx} \geq \frac{\hbar^2}{4\bar{S}_{FF}} = \frac{\hbar^2}{4\bar{S}_{\mathcal{F}\mathcal{F}}} |1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}|^2$$



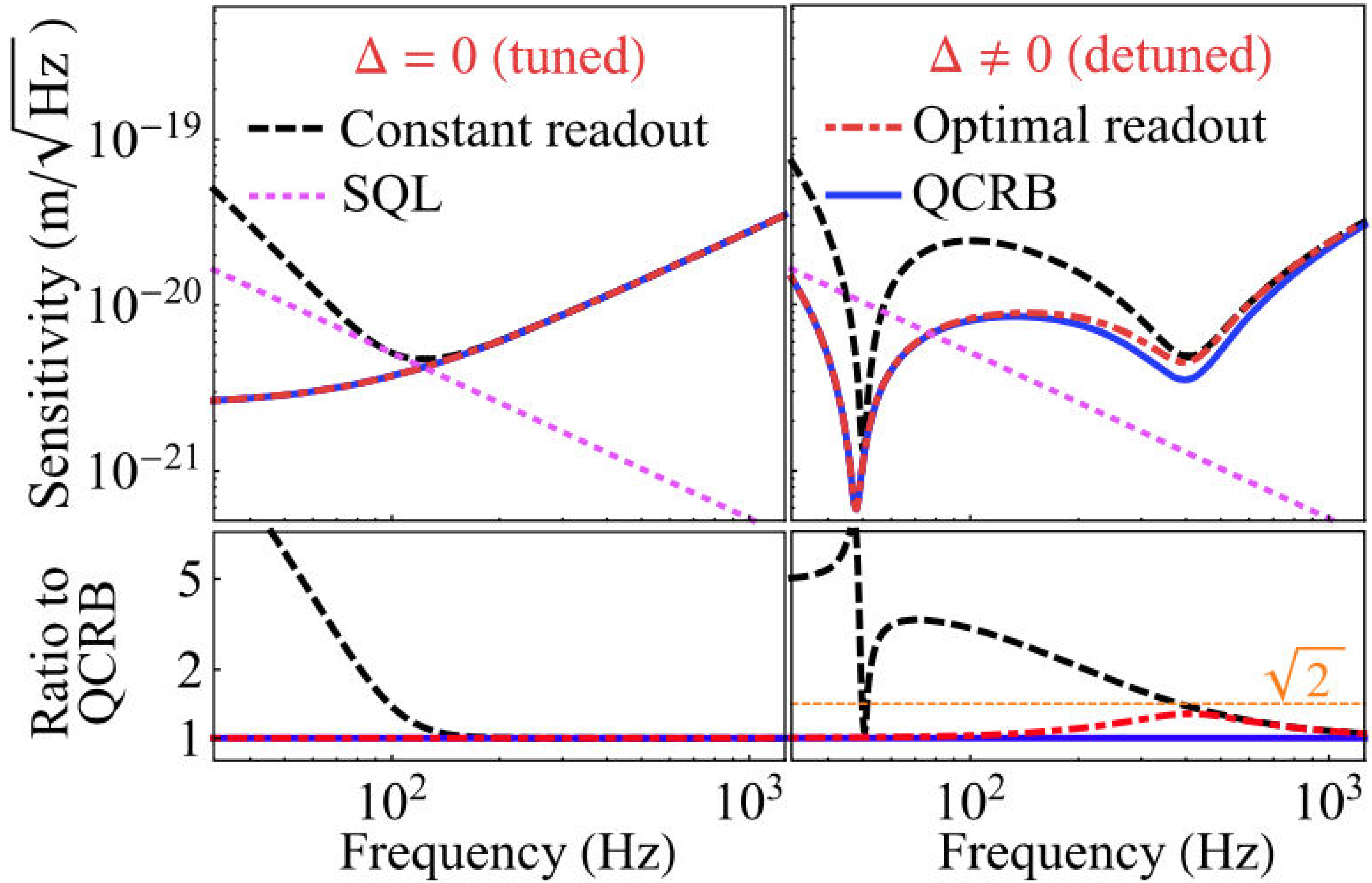


$$\hat{F}^{(0)} = \frac{\hat{\mathcal{F}}^{(0)}}{1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}}, \quad \hat{Z}^{(0)} = \hat{\mathcal{Z}}^{(0)} + \frac{\chi_{\mathcal{Z}\mathcal{F}}\chi_{qq}\hat{\mathcal{F}}^{(0)}}{1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}} \quad \Rightarrow \quad \bar{S}_{FF} = \frac{1}{|1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}|^2}\bar{S}_{\mathcal{F}\mathcal{F}}$$

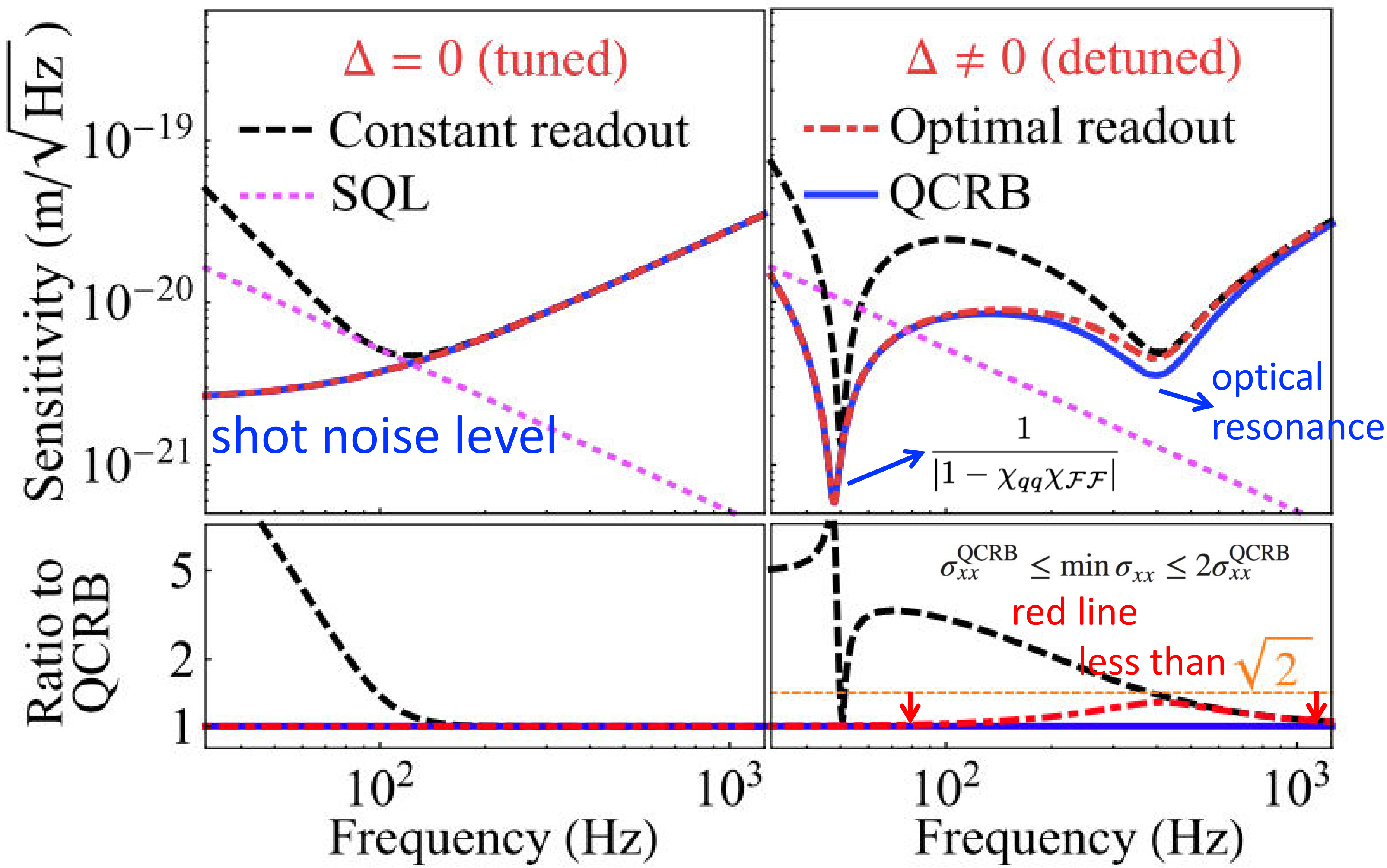
$$= \frac{1}{|1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}|}$$

This implies: **if optomechanical amplification is high, the QCRB can be very low**.

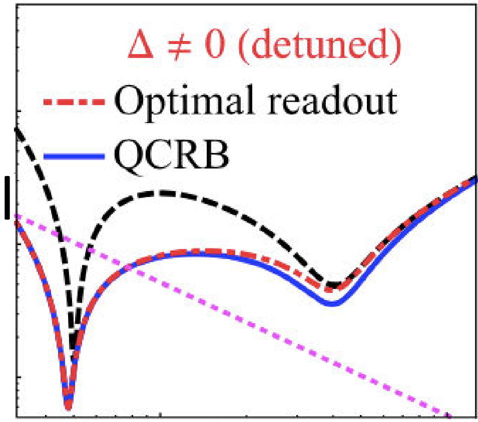
Examples


Δ = 0 (tuned)
Constant readout
SQL
Δ ≠ 0 (detuned)
Optimal readout
QCRB
Sensitivity (m/√Hz)
10^{-19}
10^{-20}
10^{-21}
Ratio to QCRB
5
2
1
$\sqrt{2}$
10^2
10^3
Frequency (Hz)

Examples


$\Delta = 0$ (tuned)
Constant readout
SQL
Sensitivity (m/$\sqrt{\text{Hz}}$)
10^{-19}
10^{-20}
10^{-21}
shot noise level
$\Delta \neq 0$ (detuned)
Optimal readout
QCRB
optical resonance
$\frac{1}{|1-\chi_{qq}\chi_{\mathcal{F}\mathcal{F}}|}$
Ratio to QCRB
5
2
1
$\sigma_{xx}^{\text{QCRB}} \leq \min \sigma_{xx} \leq 2\sigma_{xx}^{\text{QCRB}}$
red line
less than $\sqrt{2}$
10^2
10^3
Frequency (Hz)
10^2
10^3
Frequency (Hz)

5. Discussion



$$\sigma_{xx} \geq \frac{\hbar^2}{4\bar{S}_{FF}} = \frac{\hbar^2}{4\bar{S}_{\mathcal{F}\mathcal{F}}}|1-\chi_{qq}\chi_{\mathcal{F}\mathcal{F}}|^2$$

= Interpretation of the two dips =

Both dips come from positive-feedback-induced optical resonance.

=> **Radiation pressure $\hat{S}_{FF}$ is amplified there**.

- high freq. dip → optical resonance
- Low freq. dip → ponderomotive squeezing/amplification

Test mass acts as **a resource for this amplification**, a medium for squeezing, **rather than a victim of the quantum backaction**.

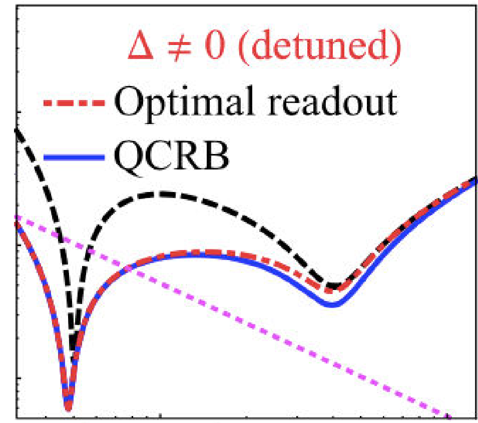
Enhancing the radiation pressure fluctuation somehow will give us a better sensitivity.

5. Discussion

= Why $\sqrt{2}$ times worse? =

Around the high freq. dip, only lower sidebands are resonating in the cavity.

=> **optimal readout will be lower sidebands** like $(\hat{Z}_1 - i\hat{Z}_2)/\sqrt{2}$, but this is **impossible** with ordinary detection scheme.

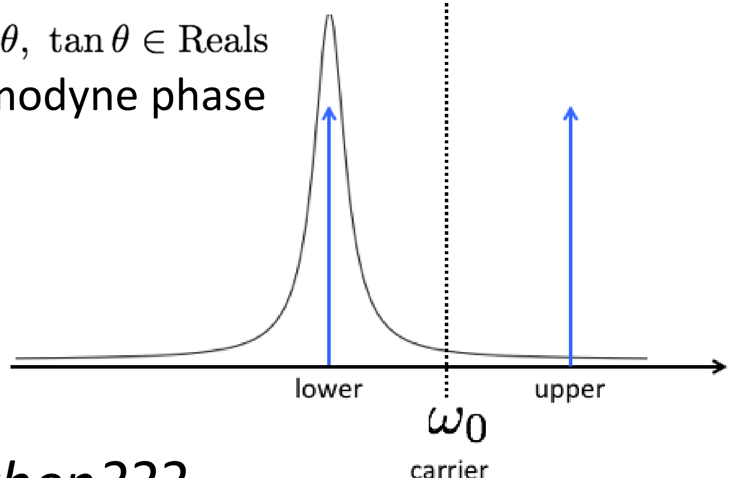


$$\sigma_{xx} \geq \frac{\hbar^2}{4\bar{S}_{FF}} = \frac{\hbar^2}{4\bar{S}_{\mathcal{F}\mathcal{F}}}|1 - \chi_{qq}\chi_{\mathcal{F}\mathcal{F}}|^2$$

Recall this:

$$\hat{Z} = \hat{Z}_1 \sin\theta + \hat{Z}_2 \cos\theta, \ \tan\theta \in \mathrm{Reals}$$

Whether $\hat{Z}$ such that $\bar{S}_{ZF}(\omega) = 0$ is realized with real homodyne phase determines if the QCRB is achieved or not.



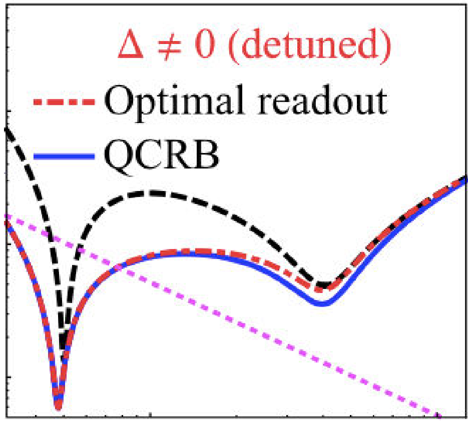
Why complex homodyne phase forbidden, then???

5. Discussion

= Effect of loss =

The quantum Cramér-Rao bound does not come from some trade-off.

=> the limit can ideally be infinitely small



$$\sigma_{xx} \geq \frac{\hbar^2}{4\bar{S}_{FF}} = \frac{\hbar^2}{4\bar{S}_{\mathcal{F}\mathcal{F}}}|1-\chi_{qq}\chi_{\mathcal{F}\mathcal{F}}|^2$$

In reality, there are always losses everywhere.

- perfect backaction evasion is impossible
- squeezing (internal/external) degrades

<u>**Incorporating the effect of losses will be important**</u>

END

Appendix

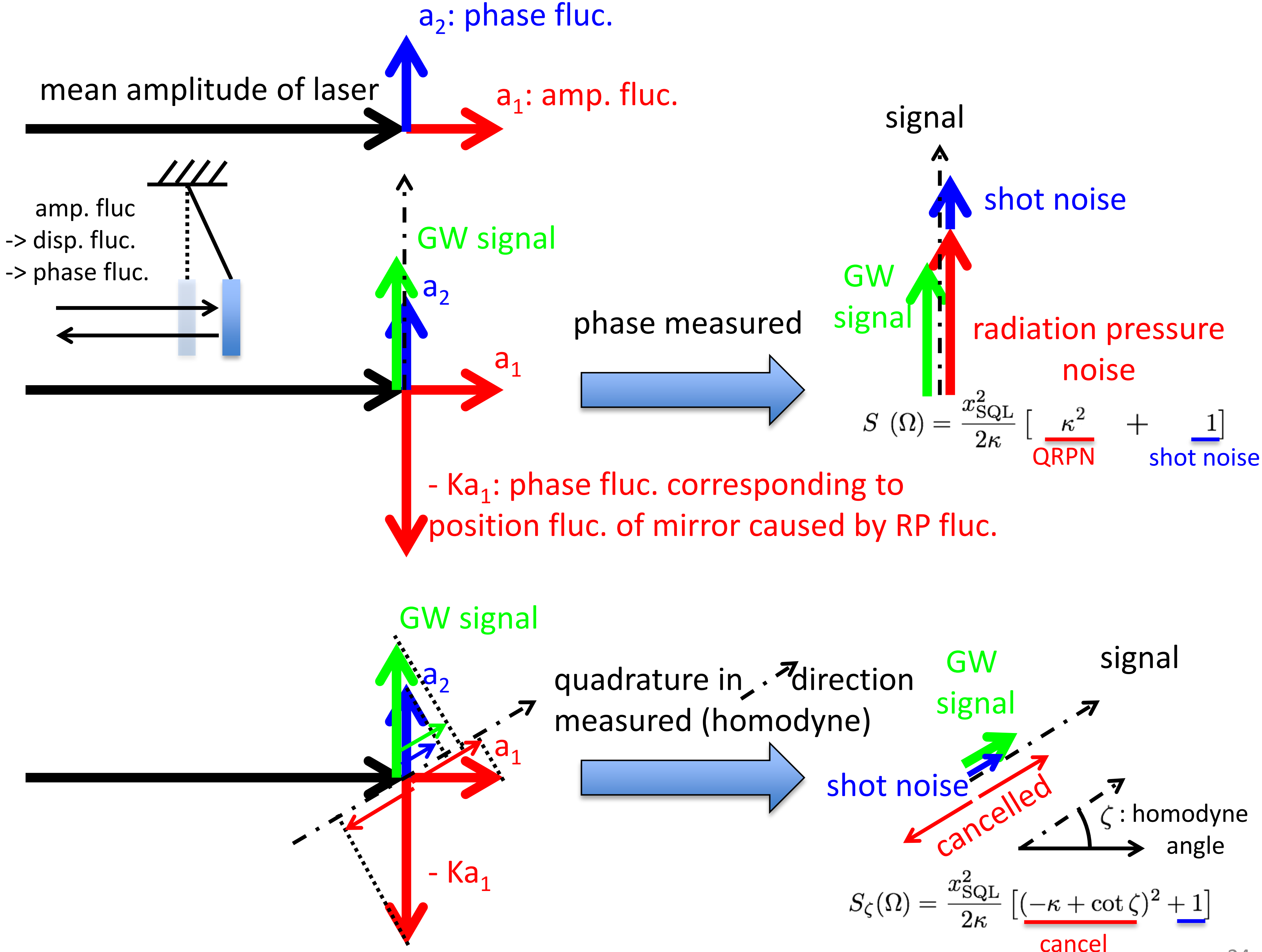

a_2: phase fluc.
mean amplitude of laser
a_1: amp. fluc.
amp. fluc
-> disp. fluc.
-> phase fluc.
GW signal
a_2
a_1
phase measured
signal
shot noise
GW signal
radiation pressure noise
$S(\Omega) = \frac{x_{\mathrm{SQL}}^2}{2\kappa}\left[\underline{\kappa^2} + \underline{1}\right]$
QRPN
shot noise
$-Ka_1$: phase fluc. corresponding to position fluc. of mirror caused by RP fluc.
GW signal
a_2
a_1
$-Ka_1$
quadrature in direction measured (homodyne)
GW signal
signal
shot noise
cancelled
ζ : homodyne angle
$S_\zeta(\Omega) = \frac{x_{\mathrm{SQL}}^2}{2\kappa}\left[\underline{(-\kappa + \cot\zeta)^2} + \underline{1}\right]$
cancel

FIG. 4. The top row shows the QCRB (solid curve) for a LIGO-type GW detector with detuning frequency $\Delta = 0$ (left) and $\Delta/(2\pi) = 400$ Hz (right), and various sensitivity curves for comparison: dashed curve, constant phase quadrature readout; dash-dotted curve, readout quadrature optimized to maximize sensitivity at each frequency; and dotted curve, the SQL $\sqrt{4\hbar/(M\omega^2)}$. The bottom row shows the ratio to the QCRB for selected curves. Other relevant parameters are $M = 40$ kg, $P_{\mathrm{cav}} = 800$ kW, $L_{\mathrm{arm}} = 4$ km, $\gamma/(2\pi) \approx 100$ Hz, and laser frequency $\omega_0/(2\pi) \approx 3 \times 10^{14}$ Hz.