Gravitational Waves as a Probe of Dark Matter Mini-Spike

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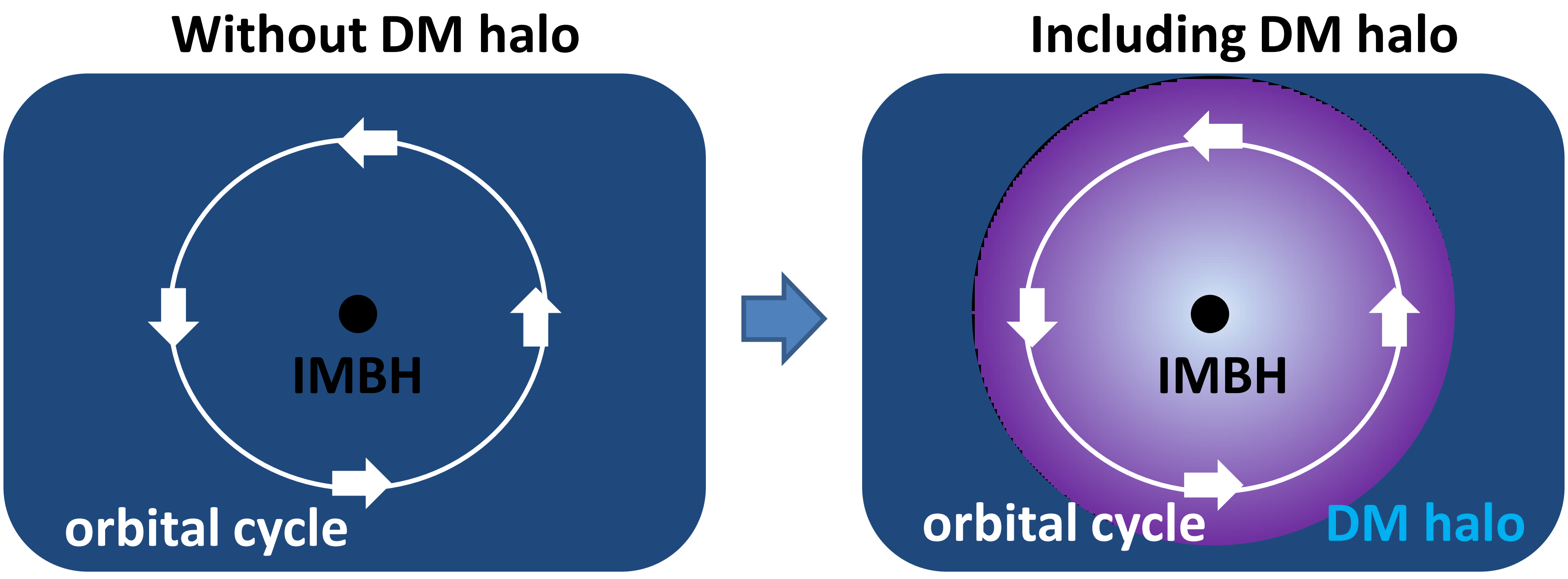
Ref. Phys. Rev. Lett. 110, 221101 (2013)

Contents

§ 1 Introduction

Situation & Motivation

- Consider a binary system formed of a stellar mass particle and intermediate-mass BH (IMBH) surrounded by dark matter (DM) halo.
- Consider the inspiral GW from the binary
- How accurate DM parameters are determined by GW observations ?



Strategy

- Particle movement affected by BH & DM

↓

- Inspiral gravitational wave (GW)

↓

- DM information contained in GW waveform

↓

- Extract DM information by matched filtering

↓

We found DM information can be determined very accurately by GW observations.

§ 2 Dark Matter Distribution

Dark Matter Distribution around Massive Black Hole

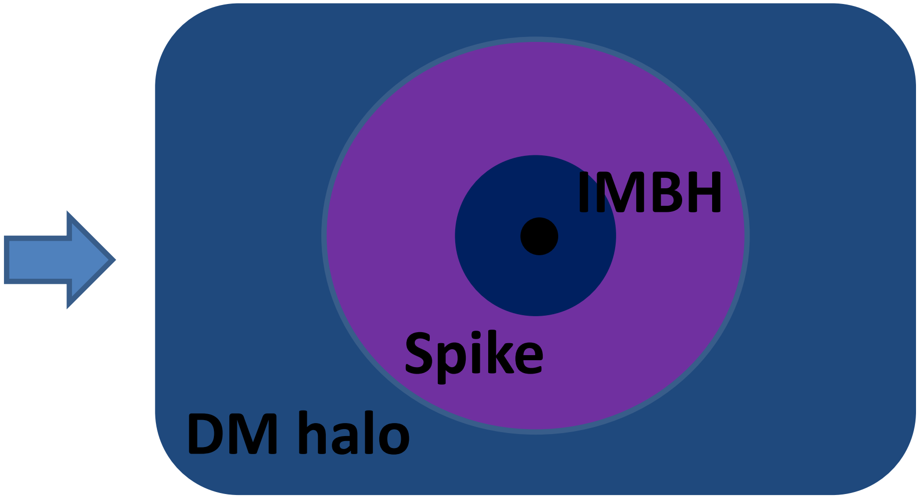
- Suggested first by Gondolo & Silk (1999)
- Adiabatic growth of IMBH creates high DM region.
- ρ : DM density

This region is called DM mini-spike

$$\rho_i(r) \propto r^{-\gamma} \quad (0 \leq \gamma \leq 2)$$

$$\rho_f(r) \propto r^{-\alpha} \left(\alpha = \frac{9-2\gamma}{4-\gamma}\right)$$

initial profile : ρi (r)



final profile : ρf (r)

Dark Matter Distribution around Massive Black Hole

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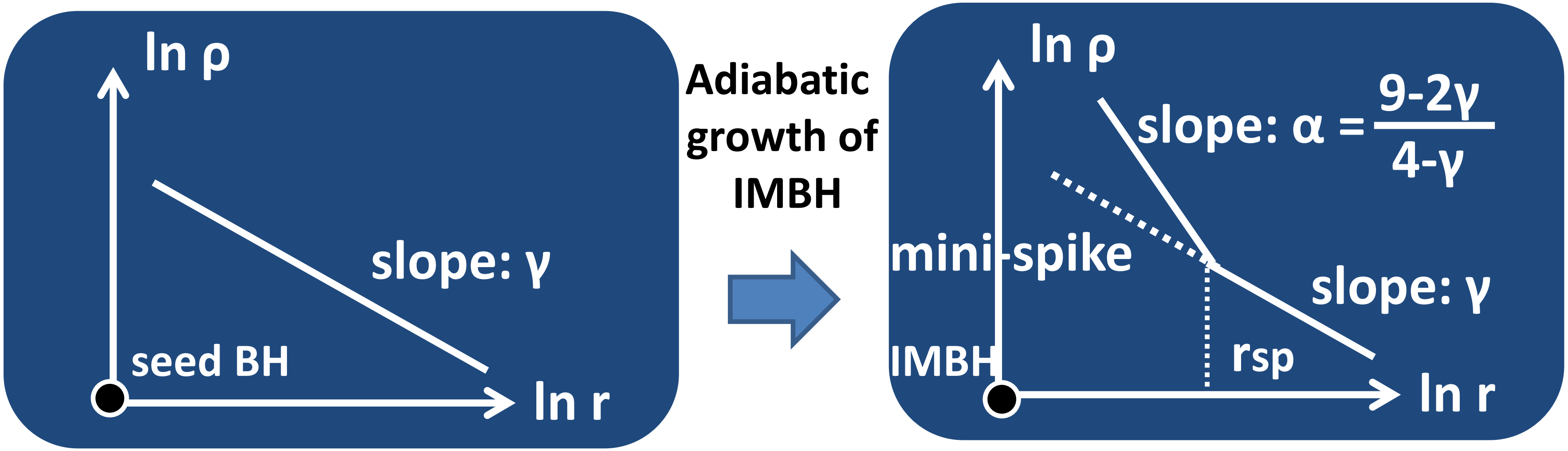


Initial profile ρi(r)　　**final profile ρf(r)**

Power-law index of DM spike

Ref. Ullio, Zhao and Kamionkowski: astro-ph/0101481

- Assumption
  - adiabatic growth of BH
  - Initial profile $\rho_i(r) \propto r^{-\gamma}$, final profile $\rho_f(r) \propto r^{-\alpha}$
- Conservation laws
  - mass conservation of DM

$$M_i^{\rm DM}(r_i) = M_f^{\rm DM}(r_f) \quad \Rightarrow \quad r_i^{3-\gamma} \propto r_f^{3-\alpha}$$

  - total angular momentum conservation

$$r_i M_i^{\rm BH+DM}(r_i) = r_f M_f^{\rm BH+DM}(r_f)$$

$$M_f^{\rm BH+DM}(r_f) \cong M_{\rm BH} \quad (r \to 0) \quad \Rightarrow \quad r_i^{4-\gamma} \propto r_f$$

$$\alpha = \frac{9-2\gamma}{4-\gamma} \; (0 \le \gamma \le 2)$$

※**rough estimate**

Power-law index of DM spike

- Assumption:
  - The growth of the central BH is adiabatic
- Adiabatic invariants:
  - Angular momentum: L
  - Radial action: Jr

$$\rho(r) \equiv \frac{4\pi}{r^2} \int_{E_{\min}}^{0} dE \int_{L_{\min}}^{L_{\max}} dL \frac{L}{v_r} f(E, L)$$

$$J_r(E, L) \equiv \frac{1}{\pi} \int_{r_{\min}}^{r_{\max}} dr v_r$$

$$f_i(E_i, L_i) = f(E_f, L_f)$$

$$L_i = L_f \equiv L$$

$$J_{r,i}(E_i, L_i) = J_{r,f} f(E_f, L_f)$$

Ref. Ullio, Zhao, Kamionkowski: astro-ph/0101481

ρi(r) ⟶ fi (E,L) ⟶ ff (E,L) ⟶ ρf(r)

Eddington's formula | **Adiabatic Invariants** | **Volume integration**

DM halo distribution

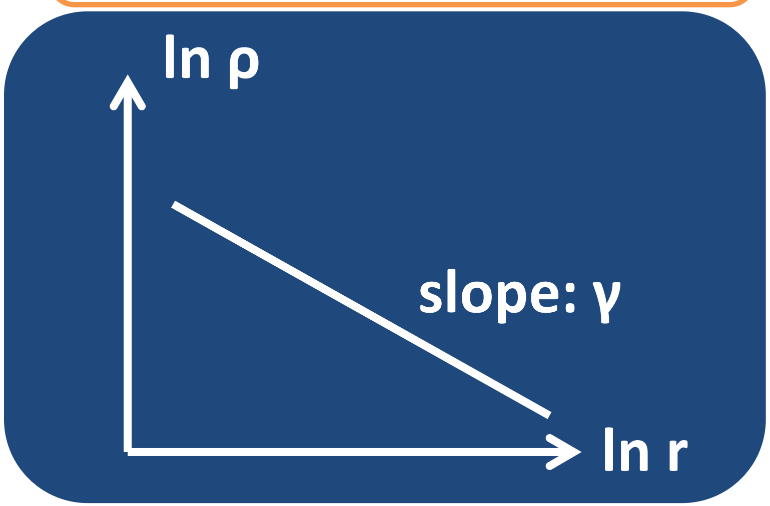
- DM mini-spike profile

$$\rho(r) = \rho_{\rm sp}\left(\frac{r_{\rm sp}}{r}\right)^{\alpha} \quad (r_{\rm ISCO} \le r \le r_{\rm sp})$$

$$r_{\rm sp} = 0.33\ {\rm pc}$$
$$\rho_{\rm sp} = 379\ M_\odot/{\rm pc}^3$$

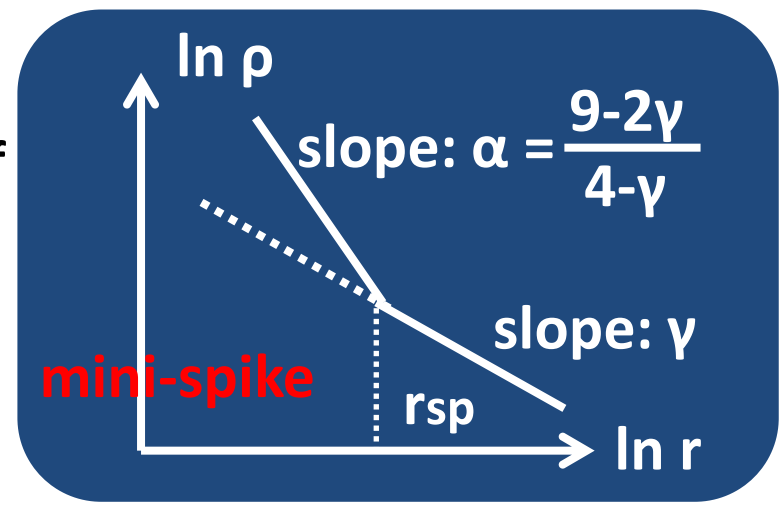
- Initial DM profile is well-approximated by Navarro-Frenk-White (NFW) profile.

$$\rho_{\rm NFW}(r) = \frac{\rho_s}{(r/r_s)(1+r/r_s)^2} \longrightarrow \begin{aligned} \gamma_{\rm NFW} &= 1 \\ \alpha_{\rm NFW} &= 7/3 \end{aligned}$$



Initial profile ρi(r)

Adiabatic growth of IMBH



final profile ρf(r)

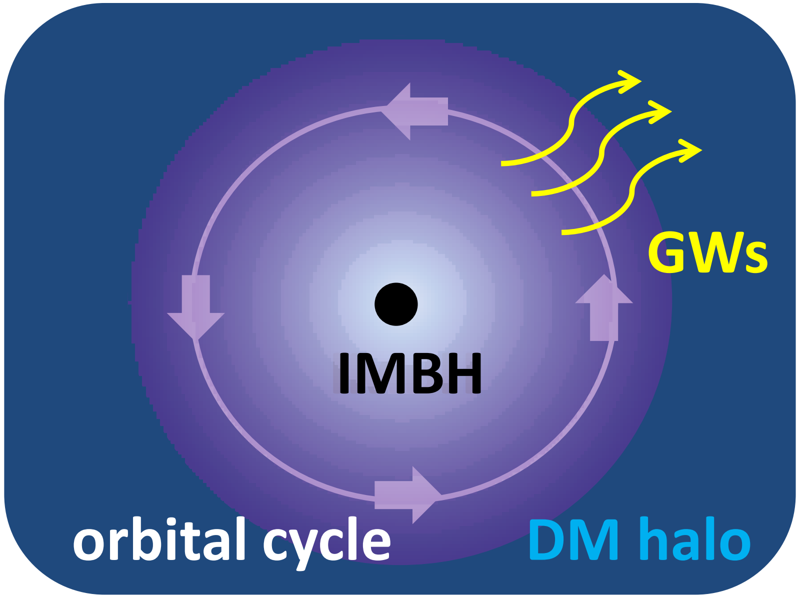
§ 3 Gravitational Wave waveform

Situation

- Consider a binary system formed of a stellar mass particle and intermediate-mass BH surrounded by dark matter (DM) halo.
- $M_{DM\ halo} \sim 10^6\ M_\odot$

  $M_{IMBH} \sim 10^3 M_\odot$

  $M_{star} \sim 1\ M_\odot$
- Assumptions
  – Circular orbit
  – Constant DM density



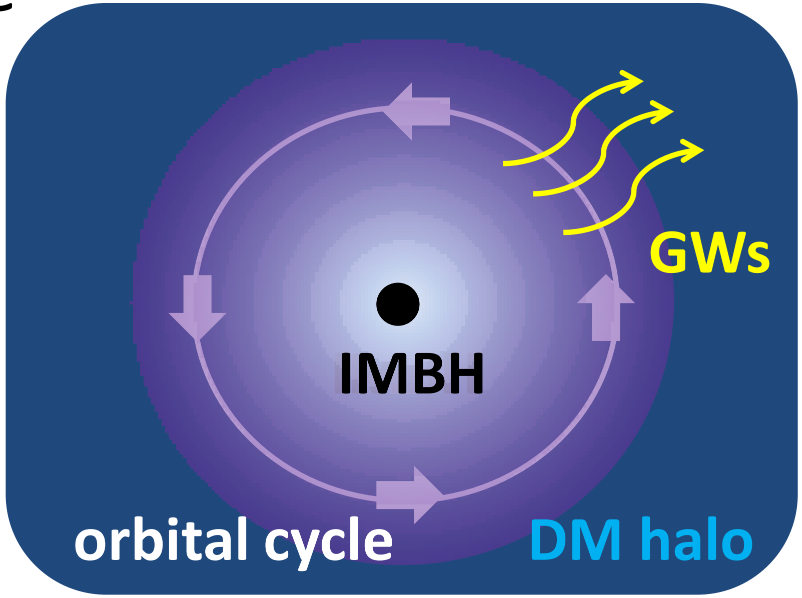
Effect of DM halo on the particle

1. Gravitational potential of the central IMBH
2. Gravitational potential of the DM halo
3. GW back-reaction force
4. Dynamical friction

   from a dence DM halo

DM mini-spike

Equation of motion (EoM)



Equation of motion for the particle

- EoM for the particle

$$\frac{d\mathbf{r}^2}{dt^2} = -\nabla\phi_{\rm IMBH} - \nabla\phi_{\rm DM} + \mathbf{f}_{\rm GW} + \mathbf{f}_{\rm DF}$$

  - 1^{st},2^{nd} terms: Gravitational potential force
  - 3^{rd} term : GW back-reaction force
  - 4^{th} term : Dynamical friction force
    - are very small effects
    - can be treat as a perturbation

GW Waveform

- GW waveform for the Newtonian circular orbit

quadrupole formula

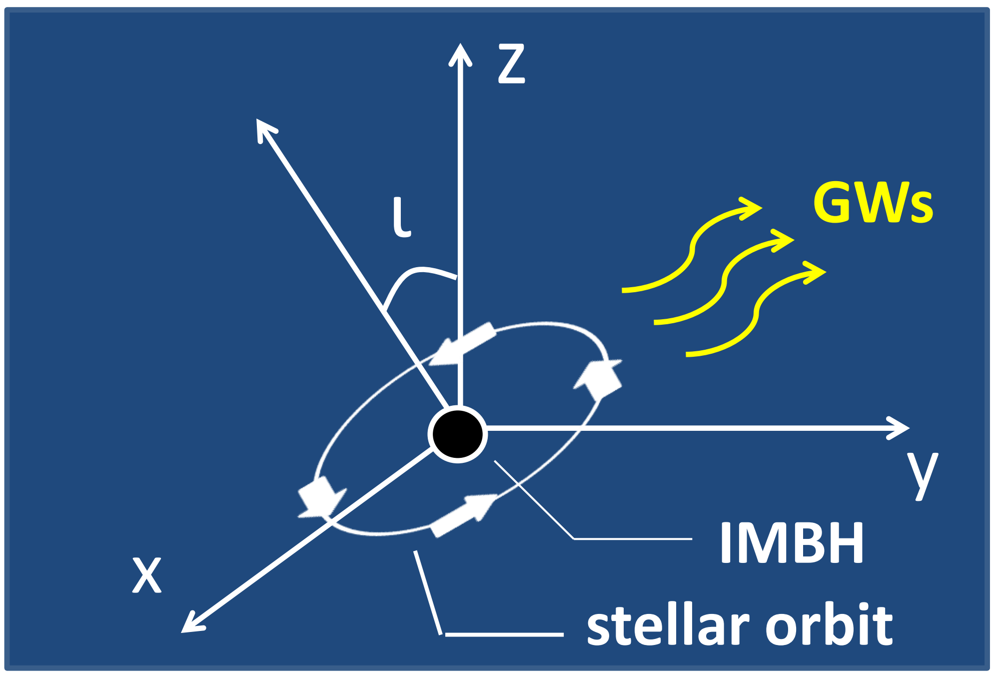
$$h^{jk}_{\rm TT} = \frac{2G}{c^4 R}\ddot{I}^{jk}_{\rm TT}$$

$$I^{jk} \equiv \int d^3x\ \rho x^j x^k$$

GW waveform

$$h_+ = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\frac{1+\cos^2\iota}{2}\cos(2\omega_s t)$$

$$h_\times = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\iota\sin(2\omega_s t)$$



- orbital radius R
- orbital frequency ω_s
- stellar mass μ
- Inclination ι
- distance to the source r

GW Waveform
including orbital time dependence

- Energy balance
  - Orbital energy E_{orbit} decrease by GW radiation loss E_{GW} and the dynamical friction loss E_{DF} .

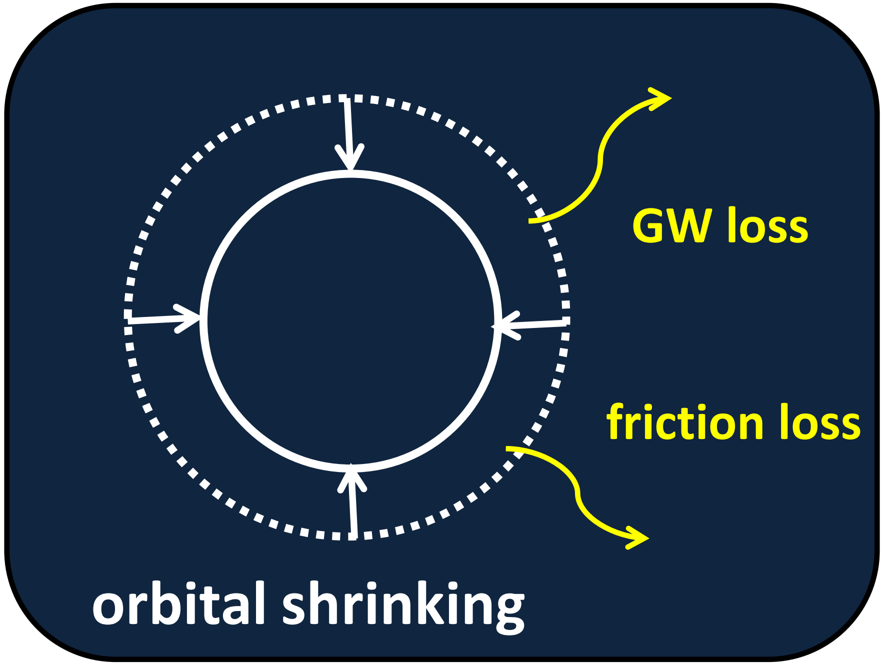
$$-\frac{dE_{\rm orbit}}{dt} = \frac{dE_{\rm GW}}{dt} + \frac{dE_{\rm DF}}{dt}$$

⇩

$$\frac{dR}{dt} = \cdots\cdots$$

⇩

$$R = R\left(t\right)$$
$$\omega_s = \omega_s\left(t\right)$$



Rewriting GW waveform

1.

$$h_+ = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\frac{1+\cos^2\iota}{2}\cos(2\omega_s t)$$

$$h_\times = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\iota\sin(2\omega_s t)$$

$R \rightarrow R(t)$

$\omega_s \rightarrow \omega_s(t)$

$\omega_{\mathrm{GW}} \rightarrow \Phi(t) \equiv \int \omega_{\mathrm{GW}}(t')\, dt'$

2.

$$h_+(t) = \frac{1}{r}\frac{4G\mu\omega_s(t)^2 R(t)^2}{c^4}\frac{1+\cos^2\iota}{2}\cos[\Phi(t)],$$

$$h_\times(t) = \frac{1}{r}\frac{4G\mu\omega_s(t)^2 R(t)^2}{c^4}\cos\iota\sin[\Phi(t)],$$

Consider only plus mode

$h(t) = h_+(t)$

$t_{\mathrm{ret}} \equiv t - r/c$

3.

$$h(t) = A(t_{\mathrm{ret}})\cos\Phi(t_{\mathrm{ret}}),$$

$$A(t) \equiv \frac{1}{r}\frac{4G\mu\omega_s(t)^2 R(t)^2}{c^4}\frac{1+\cos^2\iota}{2}$$

Fourier transform by Stationary phase method

4.

$$\tilde{h}(f) = \frac{1}{2}e^{i\Psi}A\left[\frac{2\pi}{\ddot{\Phi}}\right]^{1/2},$$

$$\Psi = 2\pi f\left(t_c + \frac{r}{c}\right) - \tilde{\Phi} - \frac{\pi}{4},$$

$$\tilde{\Phi} \equiv 2\pi f t + \Phi,$$

GW waveform: final form

- GW Waveform for plus mode in Fourier space

$$\tilde{h}(f) = \mathcal{A} f^{-7/6} e^{i\Psi(f)} M(f)^{-1/2},$$

$$\mathcal{A} = \left(\frac{5}{24}\right)^{1/2} \frac{1}{\pi^{2/3}} \frac{c}{r} \left(\frac{GM_c}{c^3}\right)^{5/6} \frac{1+\cos^2\iota}{2}$$

Mc: charp mass

$$M_c \equiv \mu^{3/5} M_{\text{eff}}^{5/2}$$

- GW Phase

$$\Psi(f) = 2\pi f \tilde{t}_c - \Phi_c - \frac{\pi}{4} - \tilde{\Phi}(f),$$

$$\tilde{\Phi}(f) = \frac{10}{3}\left(\frac{8\pi G M_c}{c^3}\right)^{-5/3}\left[-f\int_{\infty}^{f} df'\ f'^{-11/3} M^{-1}(f') + \int_{\infty}^{f} df'\ f'^{-8/3} M^{-1}(f')\right]$$

$$\tilde{\Phi} = \Phi + 2\pi i f t, \qquad \Phi = \int \omega_{\text{GW}}(t)$$

$$M(f) = 1 + 4c_\varepsilon \left(G/\pi^2 f^2\right)^{(11-2\alpha)/6}$$

$$\rho(r) = \rho_{\text{sp}}\left(\frac{r_{\text{sp}}}{r}\right)^{\alpha}$$

Two DM parameters

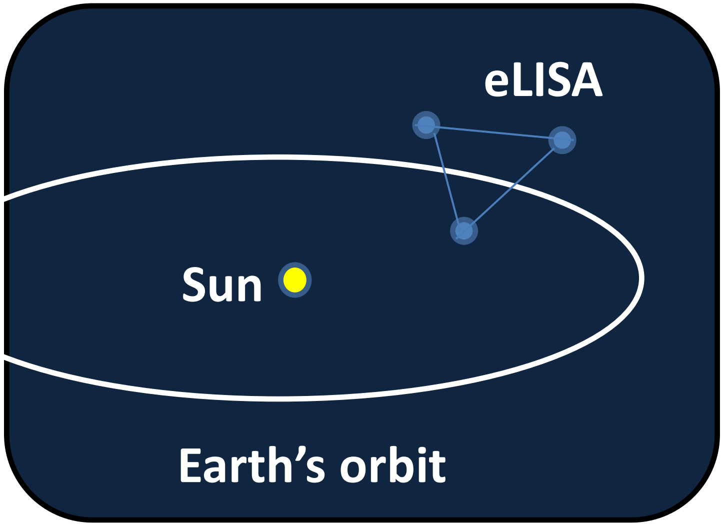
α : power-law index of DM profile

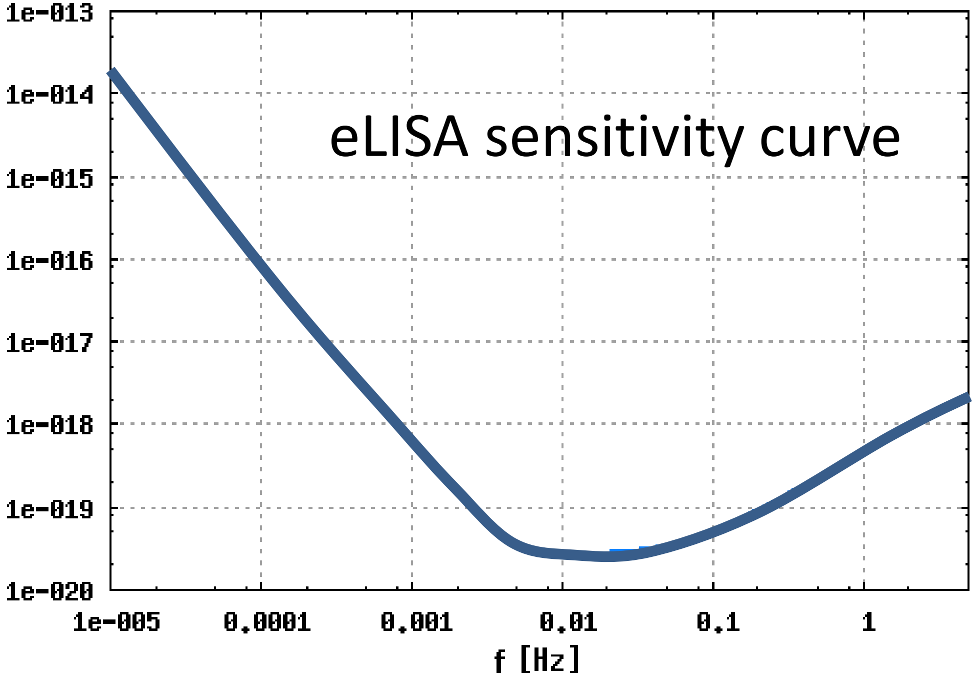
cε : the other DM parameters

§ 4 GW observations

GW observation: eLISA

- Consider eLISA observation
  - eLISA: evolved Laser Interferometer Space Antenna
  - space-based detector
  - 5 years observation until the coalescence

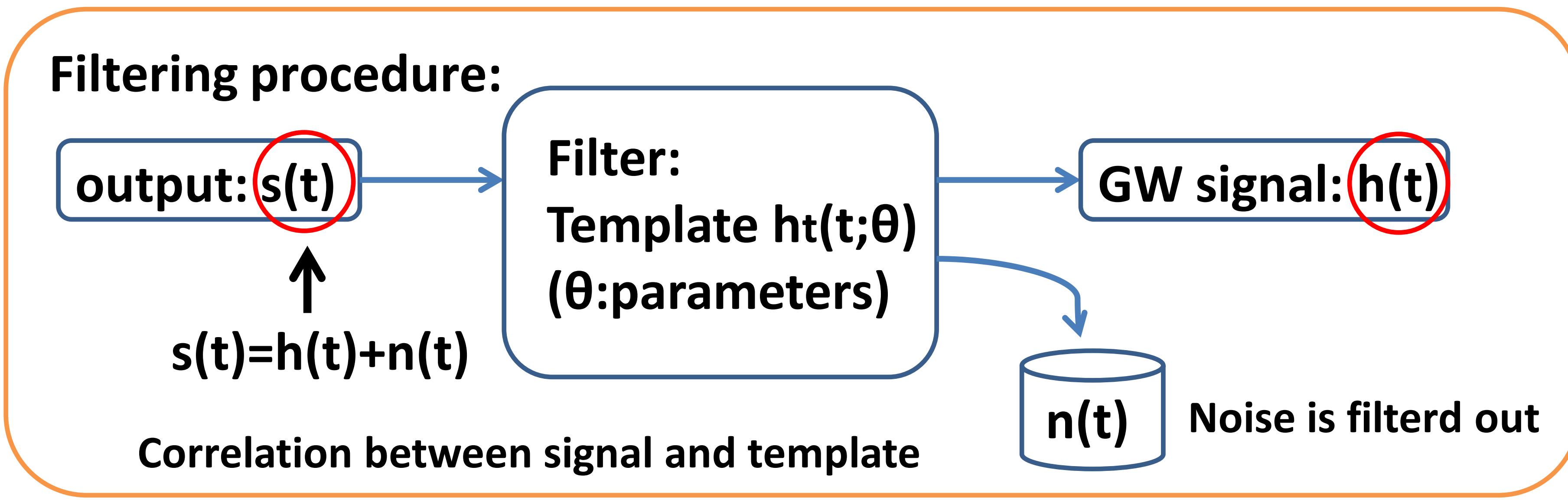




Ref. Amaro-Seoan *et al,*(2012) arXiv:1201.3621

Data Analysis: Matched filtering

- Detector output:
  - s(t) = h(t) + n(t)   (|h(t)| << |n(t)|)
    - detector output: s(t)
    - unknown GWs signal: h(t)
    - detector noise: n(t)
- We can extract the GW signal in the detector noise by mixing the output s(t) with the template ht(t)



Foundations of Parameter Estimation

- How accurate the waveform parameters are determined by GW observations?
- Detector output s(t) = h(t;θ)+n(t)
  - h(t;θ) : template, n(t) : detector noise
  - θ:waveform parameters, such as A,Mc,tc,Ψc etc.
- Assuming detector noise is stationary and Gaussian

⇒ Detector noise is a random Gaussian process

⇒ Estimator $\hat{\theta}$ have statistical errors Δθ $p\left(\Delta\theta^i\right) = \mathcal{N} \exp\left(-\frac{1}{2}\Gamma_{ij}\Delta\theta^i\Delta\theta^j\right)$

$$\Gamma_{ij} \equiv \left(\frac{\partial h}{\partial\theta^i}\middle|\frac{\partial h}{\partial\theta^j}\right)$$

$$\Delta\theta = \sqrt{\left(\Gamma^{-1}\right)_{ii}}$$

Γij : Fisher matrix

(|) : noise weighted inner product

$$\left(h_1|h_2\right) \equiv 4\mathrm{Re}\int_{f_{\mathrm{ini}}}^{\infty} \frac{\tilde{h}_1\left(f\right)\tilde{h}_2^*\left(f\right)}{S_n\left(f\right)}\, df$$

Parameter Estimation Procedure

1. Construct the theoretical waveform $h(t;\theta)$

Template

2. Derivative of $h(t;\theta)$ with respect to θ

Waveform parameters

3. Take inner product $(\partial h/\partial\theta_i | \partial h/\partial\theta_j) \equiv \Gamma_{ij}$

Fisher matrix

4. Measurements errors are the square root of the diagonal element of the inverse of Γ_{ij}

$$\Delta\theta = \sqrt{(\Gamma^{-1})_{ii}}$$

Parameter Estimation

- GW waveform

$$\tilde{h}(f) = \mathcal{A} f^{-7/6} e^{i\Psi(f)} M(f)^{-1/2},$$

$$\Psi(f) = 2\pi f \tilde{t}_c - \Phi_c - \frac{\pi}{4} - \tilde{\Phi}(f),$$

$$\tilde{\Phi}(f) = \frac{10}{3}\left(\frac{8\pi G \mathcal{M}_c}{c^3}\right)^{-5/3}\left[-f\int_{\infty}^{f} df' \; f'^{-11/3} M^{-1}(f') + \int_{\infty}^{f} df' \; f'^{-8/3} M^{-1}(f')\right]$$

$$M(f) = 1 + 4 c_\varepsilon \left(G/\pi^2 f^2\right)^{(11-2\alpha)/6}$$

- Six waveform parameters θ
  - A : overall amplitude
  - tc,ϕc : coalescence time and phase
  - Mc : charp mass
  - α, cε : dark matter parameters

$$\rho(r) = \rho_{\rm sp}\left(\frac{r_{\rm sp}}{r}\right)^{\alpha}$$

α : power-law index of DM profile, cε : the other DM parameters

Parameter Estimation

- Derivative of h(t;θ) with respect to θ

$$\frac{\partial \tilde{h}}{\partial \ln \mathcal{A}} = \tilde{h}, \qquad \frac{\partial \tilde{h}}{\partial \ln M_c} = \frac{5}{3} i \tilde{h} \tilde{\Phi}$$

$$\frac{\partial \tilde{h}}{\partial \tilde{t}_c} = 2\pi i f \tilde{h}, \qquad \frac{\partial \tilde{h}}{\partial \ln \alpha} = \alpha \tilde{h} \left(i \frac{\partial \Psi}{\partial \alpha} - \frac{1}{2} \frac{1}{M} \frac{\partial M}{\partial \alpha} \right),$$

$$\frac{\partial \tilde{h}}{\partial \Phi_c} = -i \tilde{h}, \qquad \frac{\partial \tilde{h}}{\partial \ln c_\varepsilon} = c_\varepsilon \tilde{h} \left(i \frac{\partial \Psi}{\partial c_\varepsilon} - \frac{1}{2} \frac{1}{M} \frac{\partial M}{\partial c_\varepsilon} \right)$$

$$\Gamma_{ij} \equiv \left(\frac{\partial h}{\partial \theta^i} \middle| \frac{\partial h}{\partial \theta^j} \right)$$

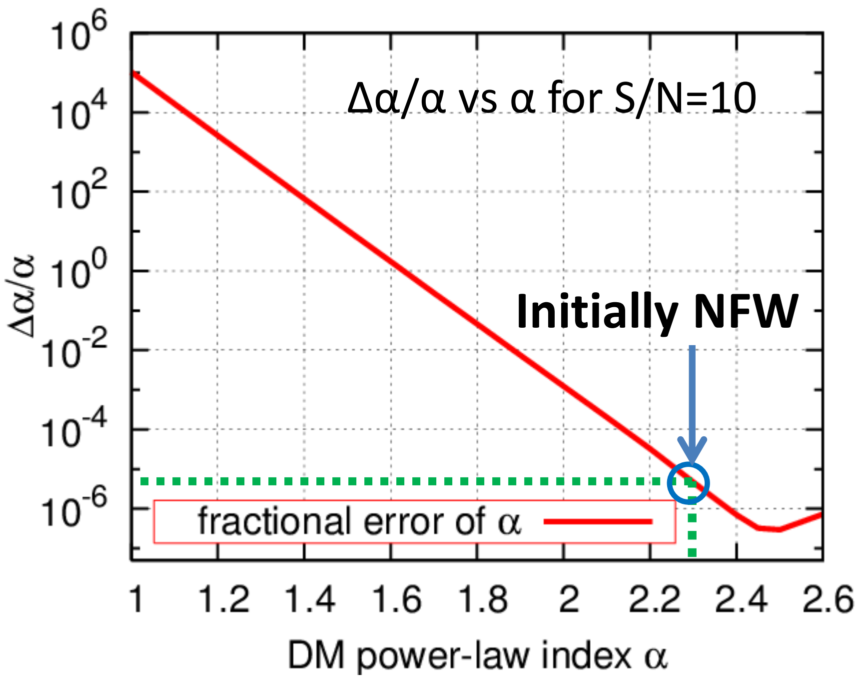
Fisher matrix

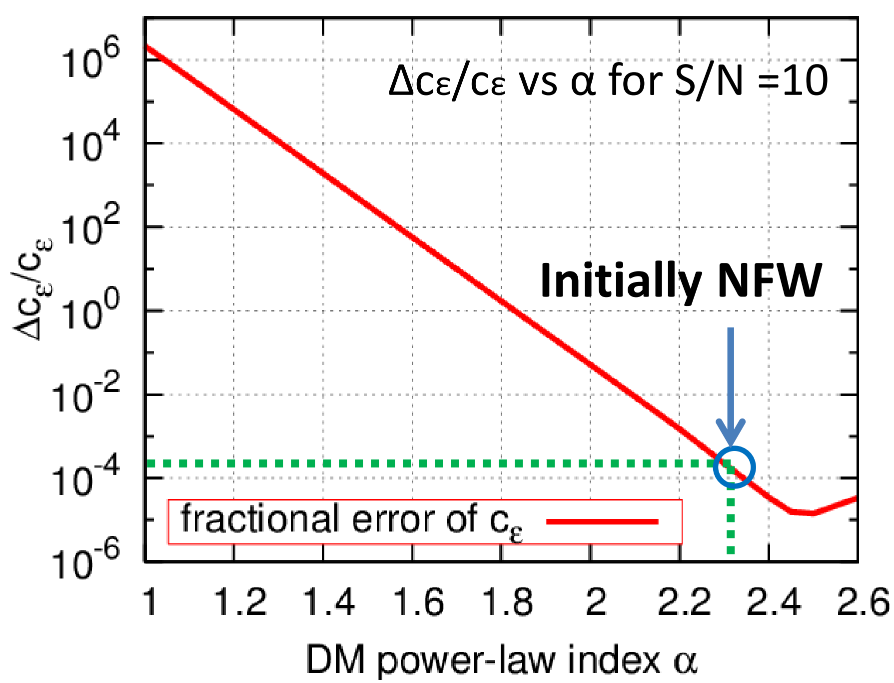
$$\Delta \theta = \sqrt{(\Gamma^{-1})_{ii}}$$

Measurements errors

Result: Error of DM parameters

- Errors of two DM parameters α, c_ε
  - For larger α, DM parameters are determined more accurately
  - α↑ ⇒ $\rho_{DM}(r\to 0)$ ↑ ⇒ effect of DM on particle ↑
  - For initially NFW profile, α=7/3
  - **DM parameters can be measurable with very good accuracy!**





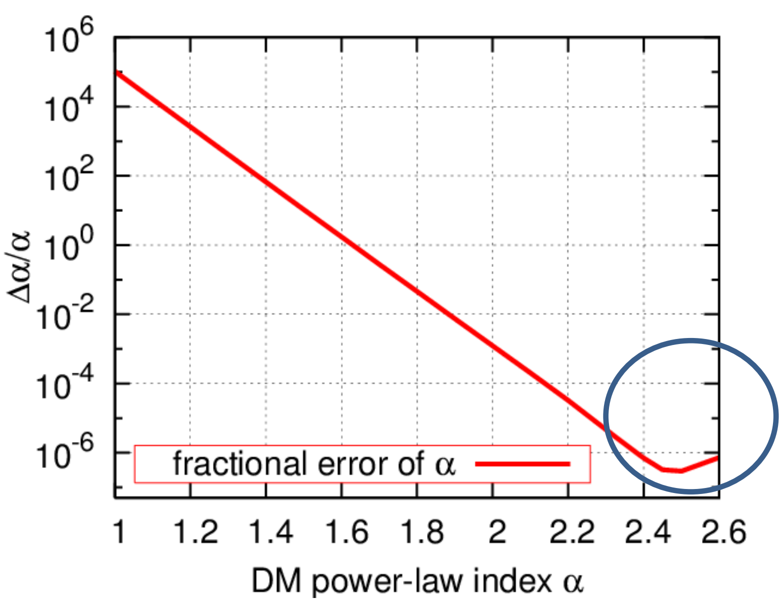
Why DM errors go up in α>2.5 ?

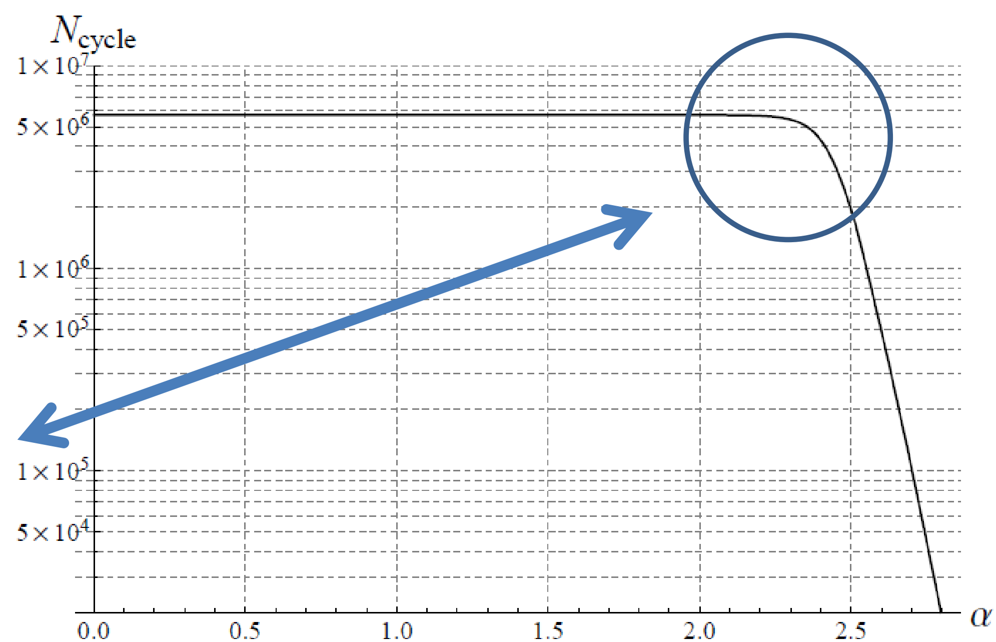
- This behavior can be explained by the number of orbital cycles in the detector band of eLISA.

$$\frac{S}{N} \sim \frac{h_0}{f_0 S_n(f_0)} N_{\rm cycle}^{1/2}$$

$$\Delta\theta \propto \frac{1}{S/N}$$

Ncycle↓ , S/N ↓, Δθ ↑





In the case of initially NFW profile

- In the case of initially NFW profile, α = 7/3
- Errors of waveform parameters are as follows

$$\frac{\Delta\mathcal{A}}{\mathcal{A}} = 0.1\left(\frac{10}{S/N}\right),$$

$$\Delta\tilde{t}_c = 1.2\left(\frac{10}{S/N}\right)\ [\mathrm{s}],$$

$$\Delta\Phi_c = 1.4\left(\frac{10}{S/N}\right)\ [\mathrm{rad}],$$

Phase parameters

$$\frac{\Delta M_c}{M_c} = 3.3\times10^{-7}\left(\frac{10}{S/N}\right),$$

$$\frac{\Delta\alpha}{\alpha} = 2.6\times10^{-6}\left(\frac{10}{S/N}\right),$$

$$\frac{\Delta c_\varepsilon}{c_\varepsilon} = 1.3\times10^{-4}\left(\frac{10}{S/N}\right).$$

Phase parameters are measurable with very good accuracy.

↓ **Why?**

For inspiral GWs,

$$\frac{S}{N} \sim \frac{h_0}{f_0 S_n(f_0)} N_{\mathrm{cycle}}^{1/2}$$

$$\Phi \equiv \int \omega_{\mathrm{GW}}\, dt \propto N_{\mathrm{cycle}}$$

↓

GW Phase are strongly related to S/N.

§ 5 Summary & Future Work

Summary

- We consider the binary composed of a stellar mass object and an IMBH surrounded by DM mini-spike.
- We research on how accurate the DM parameters contained in the GW waveform are measurable.

1. DM parameters can be determined very accurately by GW observations.
2. Observation of GWs from IMBHs will be a new tool to probe the DM distribution near the IMBH.
3. This may offer hints on the history of BHs formation.

Future work

- Including 1^{st} Post-Newtonian effect
- Non-zero eccentricity
- Accretion
- Non-spherical of DM mini-spike
- IMBH spin effect

etc……

Thank you !!